EXERCISES:
GEOMETRIC STRUCTURES AND REPRESENTATIONS OF DISCRETE GROUPS

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1. (G, X)-maps

Let G be a Lie group acting faithfully, transitively, and analytically on a manifold X.

Exercise 1. Let M be a connected (G, X)-manifold. Show that if $f_1, f_2 : M \to X$ are two (G, X)-maps, then there exists $g \in G$ such that $f_2(m) = g \cdot f_1(m)$ for all $m \in M$.

Exercise 2. Let M be a connected manifold with universal covering $\pi : \tilde{M} \to M$. Show that any for (G, X)-structure on M there is a unique (G, X)-structure on $\tilde{M}$ such that π is a (G, X)-map. Show that for this structure, for any $\gamma \in \pi_1(M, m_0)$ the deck transformation $\gamma : \tilde{M} \to \tilde{M}$ is a (G, X)-map.

2. Examples of geometric structures

Exercise 3. Let $(G, X) = (\mathrm{PGL}_2(\mathbb{R}), \mathbb{H}^2)$ (real hyperbolic geometry).

a) Draw an example of a developing map and holonomy representation for a (G, X)-structure on a closed surface of genus ≥ 2.
b) Show that a 2-dimensional torus does not admit any complete (G, X)-structure.
In fact it does not admit any (G, X)-structure at all, since such a structure would need to be complete (as a consequence of the Hopf–Rinow theorem in Riemannian geometry).

Exercise 4. Let $(G, X) = (\mathrm{PGL}_2(\mathbb{C}), \mathbb{P}^1(\mathbb{C}))$ (complex projective geometry). Show that a closed surface of genus $g \geq 1$ does not admit any complete (G, X)-structure.

Exercise 5. Let $(G, X) = (\mathrm{PGL}_3(\mathbb{R}), \mathbb{P}^2(\mathbb{R}))$, let M be a 2-dimensional torus, and let $a, b \in \pi_1(M)$ be generators of $\pi_1(M) \simeq \mathbb{Z}^2$. Show that the homomorphism $h : \pi_1(M) \to G$ defined by

$$h(a) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad h(b) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

is the holonomy of an (incomplete) (G, X)-structure on M. Draw the de-

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veloping map.

Exercise 6. For $p, q \in \mathbb{N}$ with $p+q \geq 2$, let
$$\mathbb{H}^{p,q} := \{[x] \in \mathbb{P}(\mathbb{R}^{p+q+1}) \mid x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2 < 0\}.$$

a) Show that $G = \mathrm{SO}(p, q+1)$ acts transitively on $\mathbb{H}^{p,q}$. What is the stabilizer of a point?

b) Consider the double cover
$$\hat{\mathbb{H}}^{p,q} := \{x \in \mathbb{R}^{p+q+1} \,|\, x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2 = -1\}$$
of $\mathbb{H}^{p,q}$. Check that, on any tangent space to $\hat{\mathbb{H}}^{p,q}$ in $\mathbb{R}^{p+q+1}$, the quadratic form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q+1}^2$ restricts to a quadratic form of signature (p, q).

This shows that $\hat{\mathbb{H}}^{p,q}$ has a G-invariant pseudo-Riemannian metric of signature (p, q) (i.e. on every tangent space there is a nondegenerate quadratic form of signature (p, q), and this family is G-invariant and smooth). As a consequence, $\mathbb{H}^{p,q}$ also has a G-invariant pseudo-Riemannian metric of signature (p, q). For $q = 0$, this gives the usual hyperbolic metric on the hyperbolic space $\mathbb{H}^{p,0} = \mathbb{H}^p$.

c) What is the topology of $\mathbb{H}^{p,q}$ and of its boundary in $\mathbb{P}(\mathbb{R}^{p+q+1})$?

d) Draw a picture of $\mathbb{H}^{2,1}$ (also known as AdS^3 or *anti-de Sitter* 3*-space*). Show that $\mathbb{H}^{2,1}$ identifies with $\mathrm{PSL}_2(\mathbb{R})$. What is the corresponding action of $G = \mathrm{SO}(2, 2)$?

e) Show that the group $\mathrm{U}(n, 1)$ acts properly and transitively on $\mathbb{H}^{2n,1}$. Deduce the existence of compact (G, X)-manifolds for $(G, X) = (\mathrm{SO}(2n, 2), \mathbb{H}^{2n,1})$.

f) Use a similar idea to prove the existence of compact (G, X)-manifolds for $(G, X) = (\mathrm{SO}(4n, 4), \mathbb{H}^{4n,3})$ and for $(G, X) = (\mathrm{SO}(8, 8), \mathbb{H}^{8,7})$.

An open conjecture states that these are the only values of $p, q \neq 0$ for which compact $(\mathrm{SO}(p, q+1), \mathbb{H}^{p,q})$-manifolds exist.

3. Convex projective geometry

Recall that the *cross ratio* of four distinct points $x, y, z, t \in \mathbb{P}^1(\mathbb{R})$ is defined by
$$[x, y, z, t] := \frac{(z-x)(t-y)}{(t-x)(z-y)} \in \mathbb{R}^*.$$

Exercise 7. Let Ω be a properly convex domain in $\mathbb{P}^n(\mathbb{R})$. For $x, y \in \Omega$, set
$$d(x, y) := \frac{1}{2} \log[x, y, b, a],$$
where a, b are the intersection points of $\partial\Omega$ with the projective line through x and y, with a, x, y, b in this order.

a) Show that the function $d : \Omega \times \Omega \to \mathbb{R}$ is a metric on Ω which is complete (i.e. Cauchy sequences converge) and proper (i.e. closed balls are compact). It is called the *Hilbert metric.* Check that it is invariant under the subgroup $\mathrm{Aut}(\Omega)$ of $\mathrm{PGL}_{n+1}(\mathbb{R})$ preserving Ω.

b) Show that straight lines are geodesics for d.

c) In which situation can there be more than one geodesic between two points of Ω?

Exercise 8. **a)** Show that the Hilbert metric on

$$\Omega = \mathbb{H}^n = \{[x] \in \mathbb{P}(\mathbb{R}^{n+1}) \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 < 0\}$$

coincides with the usual hyperbolic metric.
b) Show that the interior of a triangle of $\mathbb{R}^2 \subset \mathbb{P}(\mathbb{R}^3)$, endowed with its Hilbert metric, is isometric to $\mathbb{R}^2$ endowed with a norm whose unit ball is a regular hexagon.

Exercise 9. Let Ω be a properly convex domain in $\mathbb{P}^n(\mathbb{R})$ and Γ a discrete subgroup of $\mathrm{PGL}_{n+1}(\mathbb{R})$ preserving Ω. Show that Γ acts properly discontinuously on Ω. In particular, if Γ is torsion-free, then $\Gamma\backslash\Omega$ is a (G,X)-manifold for $(G,X) = (\mathrm{PGL}_{n+1}(\mathbb{R}), \mathbb{P}^n(\mathbb{R}))$.

Exercise 10. **a)** For $d \geq 2$, show that the Riemannian symmetric space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$ can be realized as a convex domain in some projective space $\mathbb{P}^n(\mathbb{R})$ (specify the dimension n). Any discrete subgroup of $\mathrm{SL}_d(\mathbb{R})$ thus gives rise to a (G,X)-manifold with $(G,X) = (\mathrm{PGL}_{n+1}(\mathbb{R}), \mathbb{P}^n(\mathbb{R}))$.
b) Similar question for $\mathrm{SL}_d(\mathbb{C})/\mathrm{SU}(d)$.