Exercises in Teichmüller dynamics

P. Hubert

Exercise 1: Compute the genus and the number of zeroes for the regular octagon (opposite sides are identified). Do the same calculation for the regular decagon and the L shaped surface (see figure 1).

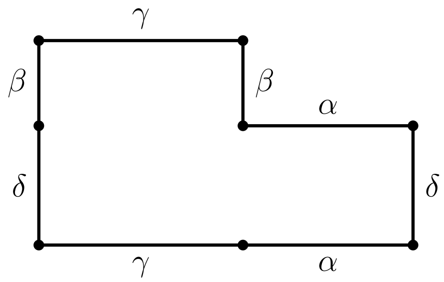

Figure 1: L-shaped surface

Exercise 2: [Polygonal Billiards]

A rational polygon is a polygon with angles commensurable to π.

Let us fix some notations. Assume that the (simply connected) polygon P has d vertices and the internal angle between the side C_i and C_{i+1} is equal to $\frac{p_i}{q_i}\pi$. And let S_i be the orthogonal symmetry with respect to the vector line associated to C_i (*not the affine line*). Let $G(P)$ be the group generated by $S_1, \dots, S_d$ and N be the $\mathrm{lcm}(q_1, \cdots, q_d)$.

1. Prove that $G(P)$ is isomorphic to the diedral group of index N (with $2N$ elements).

   Let $S(P)$ be the disjoint union gP where g belongs to G quotiented by the following equivalence relation: If $g' = S_i g$ then the side C_i in gP and $g'P$ are identified.

2. Prove that $S(P)$ is a compact Riemann surface and that it has a translation structure.

3. Make a picture of $S(P)$ when P is a square, a right angle triangle with angles $(\pi/2, \pi/8, 3\pi/8)$.

4. Prove that the genus of $S(P)$ is

$$g = 1 + \frac{N}{2} \sum_{i=1}^{d} \frac{p_i - 1}{q_i}$$

5. Deduce that the only polygons for which $S(P)$ is a torus are the square, the equilateral triangle, the right isosceles triangle and the half equilateral triangle (triangle with angles $(\pi/2, \pi/3, \pi/6)$.

Exercise 3: This exercise explains that the finiteness of the area of the translation structure is an important hypothesis. If it is not satisfied the translation structure does not always extend to the closure. Consider the complex plane with the meromorphic differential dz/z. Find the flat coordinates associated to this 1-form.

Exercise 4: We recall that a Riemann surface X is a hyperelliptic surface if it has an involution $\iota : X \to X$ such that the quotient of X by ι is the Riemann sphere $\mathbb{CP}^1$. One can check that the covering map $\pi\ X \to \mathbb{CP}^1$ is ramified over $2g+2$ points called Weierstrass points. We say that a translation surface (X, ω) is hyperelliptic if X is hyperelliptic and $\iota^*(\omega) = -\omega$. It is also true that every surface of genus 2 is hyperelliptic. We are going to understand concretely the hyperelliptic involution for a L shaped surface.

Prove that a L shaped surface is a hyperelliptic surface and that the Weierstrass points are the points A, B, C, D, E, F on figure 2

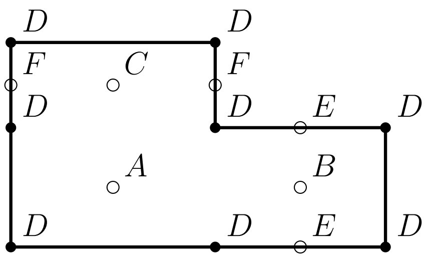

Figure 2: Weierstrass points

Describe also the hyperelliptic involution for the regular octagon.

Exercise 5: [Square tiled surfaces]

A square tiled surface is a translation surface obtained by gluing unit squares together: a right boundary of a square is glued to a left boundary, the top of square is glued a bottom of an other square (see examples on figure 3, 4).

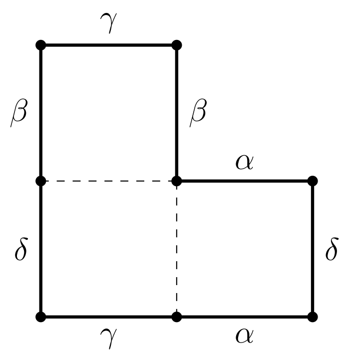

Figure 3: Example of a square tiled surface made with 3 squares

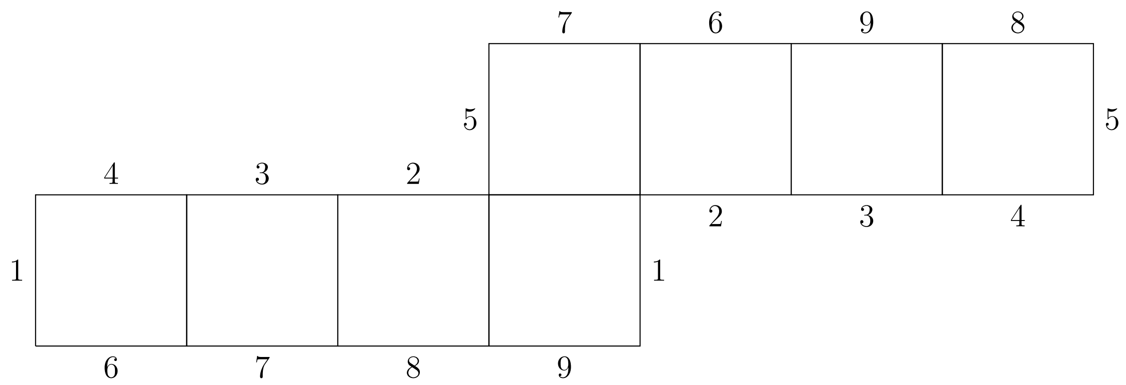

Figure 4: Wollmichsau: famous example of a square tiled surface made by 8 squares. The sides are identified with respect to the labels.

1. Prove that a square tiled surface is a ramified cover of the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ ramified at most over the origin.
2. Prove that the a square tiled surface is defined by 2 permutations: π that describes the horizontal gluings and τ the vertical gluings. Compute π and τ in both examples. Describe the conical angles at the singularities of the flat metric in terms of commutators of π and τ.
3. Prove that square tiled surfaces are the integer points in the strata (in period coordinates).
4. Prove that the group $\mathrm{SL}(2,\mathbb{Z})$ acts on square tiled surfaces with N squares. We recall that $\mathrm{SL}(2,\mathbb{Z})$ is generated by the two matrices $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is enough to prove that S and T send a square tiled surface with N squares to a surface of the same type.
5. Compute the orbit of the surface made by 3 squares.

6. A square tiled surface is reduced if it cannot be tiled by bigger squares (in other terms, the lattice of relative periods is $\mathbb{Z}^2$). We recall that the Veech group of a translation surface is its stabilizer under the $\mathrm{SL}(2,\mathbb{R})$ action. It is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ (a Fuchsian group).

   Prove that the Veech group of a reduced square tiled surface is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Also prove that it is a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. Give an interpretation of the index.

7. Prove that the Veech group of the Wollmichsau is $\mathrm{SL}(2,\mathbb{Z})$. What is the $\mathrm{SL}(2,\mathbb{Z})$-orbit of the Wollmichsau?

   The classification of $SL(2,\mathbb{Z})$-orbits of square tiled surfaces is an important open problem. It is only solved in the stratum $\mathcal{H}(2)$.

Exercise 6: [Veech groups]

Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden mean. Consider the golden L: the L shaped surface where $|\alpha| = |\beta| = 1/\phi$ and $|\gamma| = |\delta| = 1$ (see figure 1).

Prove that $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ belongs the the Veech group of the golden L.

Prove that $S = \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$ is the derivative of a multitwist associated to the horizontal decomposition into periodic cylinders and thus belongs to the Veech group of the golden L.

One can prove that the group generated by S and T is a lattice in $SL(2,\mathbb{R})$ called the Hecke group of index 5.

Prove that the horizontal multitwist is also affine in the regular octagon.

Exercise 7: [Application of the Eskin-Kontsevich-Zorich formula, Wollmichsau]

Compute the Siegel-Veech constant c_{area} for the Wollmichsau. Deduce that the non trivial Lyapunov exponents are equal to zero ($\lambda_2 = \lambda_3 = 0$).

Exercise 8: [Application of the Eskin-Kontsevich-Zorich formula, wintree model]

Let X be the genus 5 surface obtained from the windtree model (see figure 5).

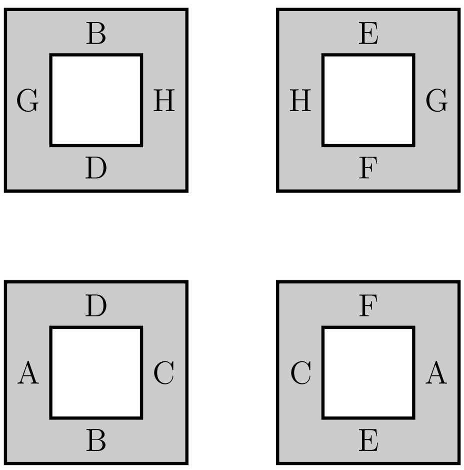

Figure 5: The genus 5 surface obtained from the windtree model.

1. Prove that X is a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ cover of a L shaped surface L (the covering group is the group generated by τ_h and τ_v the horizontal and vertical exchange of squares by *translation*).

2. Prove that the 3 intermediate covers are hyperelliptic surfaces (make a picture of each of them).

3. Applying the Eskin-Kontsevich-Zorich formula, compute all the Lyapunov exponents of Kontsevich-Zorich cocycle on the orbit closure of X.