L-SPACES, TAUT FOLIATIONS, AND ORDERABILITY

ABSTRACT. Exercises for a three-hour lecture series at the Advanced School on Geometric Group Theory and Low-Dimensional Topology at ICTP, Trieste, Italy, May 25-27, 2016.

1. Exercises for Day One.

Exercises marked by a spade (♠) are more challenging; those by a heart (♡) are staffs' picks.

(1) (♡) Prove that if Y is a closed, connected, simply-connected 3-manifold, then Y is an integer homology sphere. Your proof should invoke the names Hurewicz and Poincaré. (An earlier version mistakenly asked you to show that Y is irreducible! Argue that the Poincaré conjecture would follow from this assertion. Without the guarantee of irreducibility, revise the proof from the lecture on Day One to deduce the Poincaré conjecture from the first two conjectures made that day.)

(2) Prove that if p, q, and r are positive integers such that $qr \equiv 1 \pmod{p}$, then $L(p,q) \approx L(p,r)$.

(3) (♡) Prove that $\pi_1(Y_n) \approx \langle x, y, z \,|\, x^2 = y^3 = z^{n+2}, x = zy \rangle$.

(4) (♡) Prove that $H_1(Y_n; \mathbb{Z}) \approx \mathbb{Z}/(n-4)\mathbb{Z}$.

(5) Prove that $\pi_1(Y_3)$ is a non-trivial group by showing that it acts by symmetries of a regular icosahedron; prove that $\pi_1(Y_5)$ is a non-trivial group by showing that it acts by symmetries of a tesselation of $\mathbb{H}^2$ by $(\pi/2, \pi/3, \pi/7)$-triangles.

(6) (♠) Prove that, in fact, $\pi_1(Y_3)$ has order 120.

(7) (♡) Suppose that $\varphi : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism of a closed surface Σ_g. Show that the mapping torus $M(\varphi)$ admits a Heegaard decomposition of genus $2g$.

2. Exercises for Day Two.

(1) Devise a family of genus-2 strong Heegaard diagrams that are not connected sums of genus-1 Heegaard diagrams.

(2) (♡) Prove that if Y admits a genus-2 strong Heegaard diagram, then $\pi_1(Y)$ is not LO.

(3) (♠) Prove that the fundamental group of a strong L-space is not LO. (This is a theorem of Levine-Lewallen.)

(4) (♠) Prove that if H is a Heegaard diagram and $\widehat{CF}(H)$ has rank one, then H presents S^3.

(5) Check that the two definitions of (mod 2) grading on $\widehat{CF}(H)$ coincide: one given in terms of signs occurring in a determinant expansion, the other as an oriented intersection point in $T_\alpha \cap T_\beta \subset \mathrm{Sym}^g(\Sigma_g)$.

(6) (♡) Prove that a group G is LO iff it admits a decomposition $G = \{1\} \sqcup P \sqcup N$ such that $P \neq \emptyset$, $P \cdot P \subset P$, and $g \in P \iff g^{-1} \in N$.
(7) Prove that the fundamental group of the Weeks manifold W is not LO: $\pi_1(W) \approx \langle a, b \,|\, bababa^{-1}b^2a^{-1}, ababab^{-1}a^2b^{-1} \rangle$.
(8) (♡) Check that the ordering described on $\mathrm{Homeo}^+(\mathbb{R})$ does indeed define a left-ordering.
(9) Prove that a countable LO group G embeds into $\mathrm{Homeo}^+(\mathbb{R})$ via the following steps (or your own!). First, enumerate $G = \{g_1, g_2, \dots\}$. Define an order-preserving set map $e : G \hookrightarrow \mathbb{R}$ by first defining $e(g_1)$ arbitrarily. Having defined $e(g_1), \dots, e(g_n)$, let g_i denote the maximal element amongst $g_1, \dots, g_n$ that is less than g_{n+1} (if it exists), and let g_j denote the minimal element that is greater than g_{n+1} (again, if it exists). If g_i and g_j both exist, then define $e(g_{n+1})$ to be the average of $e(g_i)$ and $e(g_j)$. If g_i exists but not g_j, then define $e(g_{n+1}) = e(g_i) + 1$, and if g_j exists but not g_i, then define $e(g_{n+1}) = e(g_j) - 1$. Let $\Gamma = e(G)$ and let $\overline{\Gamma}$ denote the closure of Γ.
   (a) Prove that if (a, b) is a maximal connected component of $\mathbb{R} - \overline{\Gamma}$, then $a, b \in \Gamma$.
   (b) Let G act on Γ by the rule $g \cdot e(h) = e(gh)$, $\forall g, h \in G$. Prove that the action of G on Γ is continuous.
   (c) Prove that the action of G on $\overline{\Gamma}$ is continuous.
   (d) Show how to extend the action of G on $\overline{\Gamma}$ to an action on $\mathbb{R}$ by orientation-preserving homeomorphisms.
   (e) To see the necessity of careful choice of embedding e described above, show that there exists an order-preserving set map $e : \mathbb{Z}^2 \to \mathbb{R}$, where $\mathbb{Z}^2$ carries the lexicographical ordering, with the property that the induced action of $\mathbb{Z}^2$ on $\overline{e(\mathbb{Z}^2)}$ is discontinuous.
(10) Using the Boyer-Rolfsen-Wiest theorem, prove that if $b_1(Y) > 0$, then $\pi_1(Y)$ is LO.

3. Exercises for Day Three.

(1) (♡) Prove that a manifold with a co-orientable taut foliation does not contain a Reeb component. Is the hypothesis on co-orientability necessary?
(2) For those who know some group cohomology: prove that if Y is an irreducible integer homology sphere with fundamental group π, then $H^2(\pi; \mathbb{Z}) = 0$. (If it helps, the cohomology of a group π can be defined as the cohomology of an Eilenberg-Maclane space $K(\pi, 1)$.)
(3) (♡) Prove that a non-compact leaf in a foliation of a compact 3-manifold meets a closed transversal.
(4) Prove that in a taut foliation of a compact 3-manifold, there exists a single closed transversal meeting all leaves.

4. Further reading.

(1) "A survey of Heegaard Floer homology" by András Juhász.
(2) "Ordered groups and topology" by Adam Clay and Dale Rolfsen.
(3) Chapters 2 and 4 of "Foliations and the geometry of 3-manifolds" by Danny Calegari.
(4) "Groups acting on the circle" by Étienne Ghys.