Brownian motion, evolving geometries and entropy formulas

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Outline

1. Stochastic Calculus on manifolds (stochastic flows)
2. Analysis of evolving manifolds
3. Heat equations under Ricci flow and functional inequalities
4. Geometric flows and entropy formulas

I. Stochastic flows

M differentiable manifold, $\dim M = n$ and

$$TM \xrightarrow{\pi} M$$

its *tangent bundle*. The space of *vector fields* on M is denoted by

$$\begin{aligned}\Gamma(TM) &= \{A\colon M \to TM \text{ smooth} \mid \pi \circ A = \mathrm{id}_M\} \\ &= \{A\colon M \to TM \text{ smooth} \mid A(x) \in T_xM \text{ for all } x \in M\}\end{aligned}$$

Identify $\Gamma(TM)$ and $\mathbb{R}$-derivations on $C^\infty(M)$,

$$\Gamma(TM) \equiv \big\{A\colon C^\infty(M) \to C^\infty(M)\ \mathbb{R}\text{-linear} \mid A(fg) = fA(g) + gA(f)\big\},$$

via

$$A(f)(x) := df_x\, A(x) \in \mathbb{R}, \quad x \in M.$$

Flow to a vector field

To $A \in \Gamma(TM)$ consider the smooth curve $t \mapsto x(t) \in M$ s.th.

$$x(0) = x \quad \text{and} \quad \dot{x}(t) = A(x(t)).$$

Write $\phi_t(x) := x(t)$. In this way, we get the *flow to A*:

$$\begin{cases} \frac{d}{dt}\phi_t = A(\phi_t), \\ \phi_0 = \mathrm{id}_M. \end{cases}$$

This means, for any $f \in C_c^\infty(M)$:

$$\tfrac{d}{dt}(f \circ \phi_t) = A(f) \circ \phi_t, \quad f \circ \phi_0 = f,$$

or in integrated form,

$$f(\phi_t(x)) - f(x) - \int_0^t A(f)(\phi_s(x))\, ds = 0, \quad t \geq 0,\ x \in M.$$

The curve $\phi_{\bullet}(x)\colon\ t \mapsto \phi_t(x)$ is the *flow curve* (or *integral curve*) to A starting at x. Let $P_t f := f \circ \phi_t$, then $\frac{d}{dt} P_t f = P_t(A(f))$, and

$$\tfrac{d}{dt}\Big|_{t=0} P_t f = A(f).$$

Flow to a second order differential operator

Let L be a second order PDO on M, e.g.

$$L = A_0 + \sum_{i=1}^{r} A_i^2,$$

where $A_0, A_1, \ldots, A_r \in \Gamma(TM)$ for some $r \in \mathbb{N}$.

Question Is there a notion of a flow to L?

Definition Let $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space. An adapted continuous M-valued process

$$X_{\bullet}(x) \equiv (X_t(x))_{t\geq 0}$$

is called *flow process* to L (or *L-diffusion*) with starting point x if $X_0(x) = x$ and if, for all $f \in C_c^\infty(M)$, the process

$$N_t^f(x) := f(X_t(x)) - f(x) - \int_0^t (Lf)(X_s(x))\, ds, \quad t \geq 0,$$

is a martingale, i.e.

$$\mathbb{E}^{\mathscr{F}_s} \underbrace{\left[f(X_t(x)) - f(X_s(x)) - \int_s^t (Lf)(X_r(x))\, dr \right]}_{= N_t^f(x) - N_s^f(x)} = 0, \quad \text{for all } s \leq t.$$

Since $N_0^f(x) = 0$, the martingale property implies

$$\mathbb{E}\big[N_t^f(x)\big] = \mathbb{E}\big[N_0^f(x)\big] = 0.$$

Hence, defining $P_t f(x) := \mathbb{E}\left[f(X_t(x))\right]$, we observe that

and thus $$P_t f(x) = f(x) + \int_0^t \mathbb{E}\left[(Lf)(X_s(x))\right] ds,$$

$$\frac{d}{dt} P_t f(x) = \mathbb{E}\left[(Lf)(X_t(x))\right] = P_t(Lf)(x), \quad \text{and}$$

$$\frac{d}{dt}\Big|_{t=0} \mathbb{E}\left[f(X_t(x))\right] \equiv \frac{d}{dt}\Big|_{t=0} P_t f(x) = Lf(x).$$

The last formula shows that as for deterministic flows we can recover the operator L from its stochastic flow process.

Remark As for deterministic flows, stochastic flows may explode in finite times. Then $X_{\boldsymbol{.}}(x)|[0, \zeta(x)[$ with a stopping times $\zeta(x)$.

Example (Euclidean Brownian motion)

Let $M = \mathbb{R}^n$ and $L = \Delta$ where Δ is the Laplacian on $\mathbb{R}^n$. Let X_t be standard Brownian motion on $\mathbb{R}^n$ (speeded up by the factor 2). By Itô's formula, for $f \in C^\infty(\mathbb{R}^n)$,

$$\begin{aligned} d(f \circ X_t) &= \sum_{i=1}^n \partial_i f(X_t)\, dX_t^i + \sum_{i,j=1}^n \partial_i \partial_j f(X_t)\, dX_t^i dX_t^j \\ &= \langle (\nabla f)(X_t), dX_t \rangle + (\Delta f)(X_t)\, dt. \end{aligned}$$

Thus,

$$f(X_t) - f(X_0) - \int_0^t (\Delta f)(X_s)\, ds, \quad t \geq 0,$$

is a martingale. This means that

$$X_t(x) := x + X_t$$

is an L-diffusion to Δ.

What are L-diffusions good for?

a. (*Dirichlet problem*)

Let $\varnothing \neq D \subsetneq M$ open, connected, rel. compact, $\varphi \in C(\partial D)$.

Dirichlet problem (DP): Find $u \in C(\bar{D}) \cap C^2(D)$ s.th.

$$\text{(DP)} \qquad \begin{cases} Lu = 0 \text{ on } D \\ u|_{\partial D} = \varphi. \end{cases}$$

Assume existence of a stochastic flow $(X_t(x))_{t \geq 0}$ to L. Choose a sequence of open domains $D_n \uparrow D$ such that $\bar{D}_n \subset D$, and let

$$\tau_n(x) = \inf\{t \geq 0,\ X_t(x) \notin D_n\}.$$

Then

$$\tau_n(x) \uparrow \tau(x) = \inf\{t \geq 0,\ X_t(x) \notin D\}$$

where $\tau(x)$ is the *first exit time* of D when starting at x.

Given a solution u to (DP), choose $u_n \in C_c^\infty(M)$ such that $u_n|D_n = u|D_n$ and $\operatorname{supp} u_n \subset D$. Then

$$u_n(X_t(x)) - u_n(x) - \int_0^t (Lu_n)(X_r(x))\,dr$$

is a martingale, as well as

$$u_n(X_{t\wedge\tau_n(x)}(x)) - u_n(x) - \int_0^{t\wedge\tau_n(x)} \underbrace{(Lu_n)(X_r(x))}_{=0}\,dr.$$

Thus, if $x \in D_n$, we obtain

$$u(x) = \mathbb{E}\left[u(X_{t\wedge\tau_n(x)}(x))\right],$$

and by dominated convergence,

$$u(x) = \lim_{n\uparrow\infty} \mathbb{E}\left[u(X_{t\wedge\tau_n(x)}(x))\right] = \mathbb{E}\left[u(X_{t\wedge\tau(x)}(x))\right].$$

Hypothesis $\boxed{\tau(x) < \infty \text{ a.s.}}$ (the process exits D in finite time). Then

$$u(x) = \mathbb{E}\left[\lim_{t\to\infty} u(X_{t\wedge\tau(x)}(x))\right] = \mathbb{E}\left[u(X_{\tau(x)}(x))\right] = \mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right].$$

In other words,

$$u(x) = \mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] = \int_{\partial D} \varphi(z)\,\mu_x(dz),$$

where the exit measure is given by

$$\mu_x(A) = \mathbb{P}\left\{X_{\tau(x)}(x) \in A\right\}, \quad A \subset \partial D \text{ measurable.}$$

Moral:

(i) (Uniqueness) Under the hypothesis

$$\textbf{(A)} \qquad \boxed{\tau(x) < \infty \text{ a.s.} \quad \text{for all } x \in D}$$

uniqueness of solutions to the Dirichlet problem (DP) holds. Hypothesis **(A)** concerns non-degeneracy of the operator L.

(ii) (Existence) Under the hypothesis

$$\textbf{(B)} \qquad \boxed{\tau(x) \to 0 \ \text{ in probability } \ \text{if } \ D \ni x \to a \in \partial D}$$

we have

$$\mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] \to \varphi(a), \qquad \text{if } D \ni x \to a \in \partial D.$$

Hypothesis **(B)** concerns regularity of the boundary ∂D. Then one may define $u(x) := \mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right]$.

b. (*Heat equation*)

Let L be a 2nd order PDO on M and $f \in C(M)$. Want to find $u = u(t,x)$ defined on $\mathbb{R}_+ \times M$ s.th.

$$\text{(HE)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = Lu \quad \text{on }]0,\infty[\times M, \\ u|_{t=0} = f. \end{cases}$$

Fix $T > 0$. Then if X_t is a L-diffusion, the "time-space process" $(X_t(x), T - t)$ will be a diffusion on $M \times [0, T]$ for the parabolic operator

$$L - \frac{\partial}{\partial t}$$

with starting point (x, T).

Hypothesis $\zeta(x) = +\infty$ a.s. for all $x \in M$ (non-explosion)

Let u be a *bounded* solution of (HE). Then, for $0 \le t < T$,

$$u(X_t(x), T-t) - u(x, T) - \int_0^t \left[(L - \partial_t)\, u(\cdot, T-r)\right](X_r(x))\, dr,$$

is a martingale.

As a consequence, we obtain

$$u(x, T) = \mathbb{E}\left[u(X_t(x), T-t)\right] \to \mathbb{E}\left[u(X_T(x), 0)\right] = \mathbb{E}\left[f(X_T(x))\right]$$

where for the limit $t \uparrow T$ we used dominated convergence (u is bounded).

Conclusion. Under the hypothesis $\zeta(x) = +\infty$ for $x \in M$, we have uniqueness of (bounded) solutions to the heat equation (HE). Solutions are necessarily of the form

$$\boxed{u(x, t) = \mathbb{E}\left[f(X_t(x))\right].}$$

II. How to construct stochastic flows?

Stochastic differential equations (SDEs) on manifolds

Definition

Let M be a differentiable manifold, $\pi\colon TM \to M$ its tangent bundle and E a finite dimensional vector space (e.g. $E = \mathbb{R}^r$).

An SDE on M is a pair (A, Z) where

(1) Z is a semimartingale taking values in E;

(2) $A\colon M \times E \to TM$ a homomorphism of vector bundles$/M$, i.e.

$$(x, e) \longmapsto A(x)e := A(x, e)$$

$$\begin{array}{ccc} M \times E & \xrightarrow[A]{} & TM \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[\mathrm{id}]{} & M \end{array}$$

Formally $A \in \Gamma(E^* \otimes TM)$. In particular,

$$\begin{cases} \forall x \in M \text{ fixed}, & A(x) \in \mathrm{Hom}(E, T_xM), \\ \forall e \in E \text{ fixed}, & A(\cdot)e \in \Gamma(TM). \end{cases}$$

For the SDE (A, Z) we also write

$$\boxed{dX = A(X) \circ dZ}$$

or

$$\boxed{dX = \sum_{i=1}^{r} A_i(X) \circ dZ^i}$$

where $A_i = A(\cdot)e_i \in \Gamma(TM)$ and $e_1, \dots, e_r$ is a basis of E.

Definition

Let (A, Z) be an SDE on M. A continuous semimartingale X_t taking values in M, is called solution to the SDE

$$dX = A(X) \circ dZ$$

with initial condition $X_0 = x_0$, if for all $f \in C_c^\infty(M)$:

$$f(X_t) = f(x_0) + \int_0^t (df)_{X_s} A(X_s) \circ dZ_s.$$

Here:

$$E \xrightarrow{A(x)} T_x M \xrightarrow{(df)_x} \mathbb{R}, \quad x \in M.$$

Example

Let $E = \mathbb{R}^{r+1}$ and $Z = (t, Z^1, \ldots, Z^r)$ where $Z = (Z^1, \ldots, Z^r)$ is a Brownian motion on $\mathbb{R}^r$. Denote the standard basis of $\mathbb{R}^{r+1}$ by $(e_0, e_1, \ldots, e_r)$. To the homomorphism of vector bundles

$$A\colon M \times E \to TM$$

over M associate the vector fields

$$A_i := A(\cdot)e_i \in \Gamma(TM), \quad i = 0, 1, \ldots, r.$$

Then the SDE

$$\boxed{dX = A(X) \circ dZ}$$

writes as

$$\boxed{dX = A_0(X)\,dt + \sum_{i=1}^{r} A_i(X) \circ dZ^i}.$$

- For $f \in C_c^\infty(M)$, we find

$$
\begin{aligned}
d(f \circ X) &= \sum_{i=0}^{r} (A_i f)(X) \circ dZ^i \\
&= (A_0 f)(X)\, dt + \sum_{i=1}^{r} (A_i f)(X) \circ dZ^i \\
&= (A_0 f)(X)\, dt + \sum_{i=1}^{r} (A_i^2 f)(X)\, dt + \sum_{i=1}^{r} (A_i f)(X)\, dZ^i \\
&= (Lf)(X)\, dt + \sum_{i=1}^{r} (A_i f)(X)\, dZ^i.
\end{aligned}
$$

- Thus,

$$
f(X_t) - f(X_0) - \int_0^t (Lf)(X_s)\, ds, \quad t \geq 0,
$$

is a martingale where

$$
L = A_0 + \sum_{i=1}^{r} A_i^2.
$$

Corollary

Let Z be a Brownian motion on $\mathbb{R}^r$. Then solutions X to the SDE

$$\boxed{dX = A_0(X)\,dt + \sum_{i=1}^{r} A_i(X) \circ dZ^i}$$

are L-diffusions to the operator

$$L = A_0 + \sum_{i=1}^{r} A_i^2.$$

Theorem (SDE: Existence and uniqueness of solutions)

Let (A, Z) be an SDE *on M and let x_0 be an $\mathscr{F}_0$-measurable random variable taking values in M. There exists a unique maximal solution $X|[0, \zeta[$ (where $\zeta > 0$ a.s.) of the* SDE

$$dX = A(X) \circ dZ$$

with initial condition $X_0 = x_0$. Uniqueness holds in the sense that if $Y|[0, \xi[$ is another solution with $Y_0 = x_0$, then

$$\xi \leq \zeta \text{ a.s. and } X|[0, \xi[= Y \text{ a.s.}$$

Brownian motions and moving frames

Brownian motions on M are L-diffusions (stochastic flows) to the Laplace-Beltrami operator Δ on M.

Good news. We have a method to construct Brownian motions.

Bad news. There is no canonical way to write Δ in Hörmander form as a sum of squares.

Definition

Let $\pi\colon P \to M$ be the G-principal bundle of orthonormal frames with $G = \mathrm{O}(n;\mathbb{R})$. The fibre P_x consists of the linear isometries $u\colon \mathbb{R}^n \to T_xM$ where $u \in P_x$ is identified with the $\mathbb{R}$-basis

$$(u_1, \ldots, u_n) := (ue_1, \ldots, ue_n).$$

The Levi-Civita connection in TM induces canonically a *G-connection* in P given as a G-invariant differentiable splitting h of the following exact sequence of vector over P:

$$\begin{tikzcd} 0 \arrow[r] & \ker d\pi \arrow[r] & TP \arrow[r, "d\pi"] & \pi^* TM \arrow[r] \arrow[l, bend left, dashed, "h"] & 0. \end{tikzcd}$$

The splitting gives a decomposition of TP:

$$TP = V \oplus H := \ker d\pi \oplus h(\pi^* TM).$$

For $u \in P$, we call H_u the *horizontal space at u* and $V_u = \{v \in T_u P\colon\ (d\pi)v = 0\}$ the *vertical space at u*.

The bundle isomorphism

$$h\colon\ \pi^* TM \xrightarrow{\sim} H \hookrightarrow TP$$

is called *horizontal lift* of the G-connection; fibrewise it reads as

$$h_u\colon\ T_{\pi(u)} M \xrightarrow{\sim} H_u.$$

- The orthonormal frame bundle $P = \mathrm{O}(TM)$, considered as a manifold, is parallelizable.
- The horizontal subbundle H is trivialized by the *standard-horizontal vector fields* $H_1, \dots, H_n$ in $\Gamma(TP)$ defined by

$$H_i(u) := h_u(ue_i).$$

- The canonical second order partial differential operator on $\mathrm{O}(TM)$,

$$\Delta^{\mathrm{hor}} := \sum_{i=1}^{n} H_i^2,$$

is called Bochner's *horizontal Laplacian*.

(a) Let Z be a semimartingale on $\mathbb{R}^n$. Solve the following SDE on the frame bundle $P = \mathrm{O}(TM)$:

$$dU = \sum_{i=1}^{n} H_i(U) \circ dZ^i, \quad U_0 = u_0.$$

(b) Project U onto the manifold M:

$$X = \pi \circ U$$

(c) From X we can recover again Z via $Z = \int_U \vartheta$ where U is the unique horizontal lift of X to P with $U_0 = u_0$ and

$$\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n), \quad \vartheta_u(X_u) := u^{-1}(d\pi X_u), \quad u \in P,$$

the canonical 1-form.

We call X on M *stochastic development* of Z. The frame U moves along X by stochastic parallel transport.

Theorem (Stochastic development)

The following three conditions are equivalent:

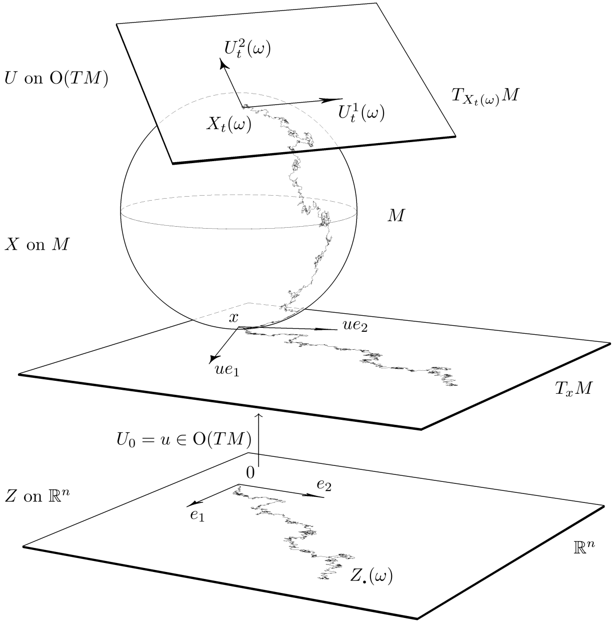
- *Z is a Brownian motion on $\mathbb{R}^n$ (diffusion with generator $\Delta_{\mathbb{R}^n}$).*
- *U is an L-diffusion on $P = \mathrm{O}(TM)$ to*

$$L = \Delta^{\mathrm{hor}} = \sum_{i=1}^{n} H_i^2.$$

- *X is a Brownian motion M (diffusion with generator the Laplace-Beltrami operator Δ on M).*

Indeed: Observe that

$$\Delta^{\mathrm{hor}}(f \circ \pi) = (\Delta f) \circ \pi$$


U on $\mathrm{O}(TM)$
$U_t^2(\omega)$
$T_{X_t(\omega)}M$
$X_t(\omega)$
$U_t^1(\omega)$
M
X on M
x
ue_2
ue_1
T_xM
$U_0 = u \in \mathrm{O}(TM)$
0
e_2
Z on $\mathbb{R}^n$
e_1
$\mathbb{R}^n$
$Z_\bullet(\omega)$

Definition (Parallel transport along a semimartingale)

For $0 \le s \le t$, consider

$$\begin{tikzcd}
T_{X_s}M \arrow[rr, "\sim"'] & & T_{X_t}M \\
& \mathbb{R}^n \arrow[ul, "U_s"] \arrow[ur, "U_t"'] &
\end{tikzcd}$$

The isometries

$$//_{s,t} := U_t \circ U_s^{-1} \colon T_{X_s}M \to T_{X_t}M$$

are called *stochastic parallel transport along* X.