Brownian motion, evolving geometries and entropy formulas

Talk 2

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ICTP, Trieste
July 4 to 15, 2016

Outline

1. Stochastic Calculus on manifolds (stochastic flows)
2. Analysis of evolving manifolds
3. Heat equations under Ricci flow and functional inequalities
4. Geometric flows and entropy formulas

I. Geometries evolving in time: Deformation of Riemannian metrics $g(t)$ under certain evolution equations

Eminent example Ricci flow (R. Hamilton, 1982)

- Start with a given metric g_0 on M and let it evolve under

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}, \quad g(0) = g_0$$

- Idea behind Ricci flow: Ricci flow works as heat equation on the space of Riemannian metrics.
- For instance, in terms of local coordinates x_i, if $\Delta x_i = 0$, then

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\,\Delta g_{ij} \;+ \text{lower order terms.}$$

- The scalar curvature $\mathrm{R} := \operatorname{trace}\mathrm{Ric}$ satisfies the reaction-diffusion equation

$$\frac{\partial}{\partial t}\mathrm{R} = \Delta\mathrm{R} + 2|\mathrm{Ric}|^2.$$

Depending on the sign $\pm$ in

$$\frac{\partial}{\partial t} g(t) = \pm 2 \,\mathrm{Ric}_{g(t)}, \quad g(0) = g_0$$

we talk about backward/forward Ricci flow.

Brownian motion with respect to a time varying metric

Let $g(t)$ be a C^1 family of Riemannian metrics on M.

- A continuous adapted process X is called Brownian motion with respect to $g(t)$ if

  $\forall\, f \in C_c^\infty(M)$,

$$d(f(X_t)) - \Delta_{g(t)} f\,(X_t)\,dt = 0 \quad \text{(mod mart)}$$

- We call X shortly a g(t)-Brownian motion on M.

Evolution equation for densities

Let $X_t(x,s)$ be a $g(t)$-Brownian motion starting from x at time s.

- Consider the smooth density

  $$(x,s,y,t) \mapsto p(x,s;y,t), \quad 0 \le s < t,\ x,y \in M,$$

  defined by

  $$\mathbb{P}\{X_t(x,s) \in dy\} = p(x,s;y,t)\,\mathrm{vol}_t(dy), \quad s < t\,,$$

  where $\mathrm{vol}_t(dy)$ is the Riemannian volume on $(M, g(t))$.
- For $p_t := p(x,s;\cdot,t)$ we have

  $$\begin{cases} \dfrac{d}{dt} p_t + \dfrac{1}{2}\left(\mathrm{trace}\,\dot{g}(t)\right) p_t = \Delta_{g(t)} p_t, \\ p_t(y)\,\mathrm{vol}_t(dy) \to \delta_x \ \text{ in law as } t \downarrow s. \end{cases}$$

Corollary

For $t > s$ let

$$p_t = p(x, s; \cdot, t).$$

For the forward Ricci flow, we have:

$$\frac{d}{dt} p_t = \Delta_{g(t)} p_t + \mathrm{R}(\cdot, t)\, p_t.$$

For the backward Ricci flow, we have:

$$\frac{d}{dt} p_t = \Delta_{g(t)} p_t - \mathrm{R}(\cdot, t)\, p_t.$$

Here $\mathrm{R}(y, t) := \mathrm{trace}\, \mathrm{Ric}_{g(t)}(y)$ denotes the scalar curvature at the point $y \in M$ for the metric $g(t)$.

Heat equation with respect to moving Riemannian metrics

- Study the heat equation under Ricci flow
- Consider positive solutions u to the heat equation:

$$\begin{cases} \dfrac{\partial}{\partial t} u - \Delta_{g(t)} u = 0 \\ \dfrac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

or to the conjugate heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u + \Delta_{g(t)} u - \mathrm{R}(t,\cdot) u = 0 \\ \dfrac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

Motivation comes from Perelman's work

II. Perelman's modification of Hamilton's Ricci flow

Perelman's $\mathcal{F}$-functional

Let M be a smooth compact manifold without boundary and let $\mathcal{M}$ be the set of Riemannian metrics on M.
Consider

$$\mathcal{F}\colon \mathcal{M} \times C^\infty(M) \to \mathbb{R},$$
$$\mathcal{F}(g,f) := \int_M (R + |\nabla f|^2)\, e^{-f}\, d\mathrm{vol}$$

where $R = \mathrm{trace}\,\mathrm{Ric}$ denotes the scalar curvature of (M,g).

Gradient flow to Perelman's $\mathcal{F}$-functional

The gradient flow of $\mathcal{F}$ on $\mathcal{M} \times C^\infty(M)$, under the constraint that

$$e^{-f}\, d\mathrm{vol} \equiv \text{static measure},$$

is given by the Modified Ricci Flow

(MRF)
$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,(\mathrm{Ric} + \mathrm{Hess}\, f), \\ \dfrac{\partial}{\partial t} f = -\Delta f - R. \end{cases}$$

If g and f evolve according to MRF, then

$$\frac{d}{dt}\mathcal{F}(g,f) = 2\int_M |\mathrm{Ric}_g + \mathrm{Hess}_g f|_g^2\; e^{-f}\, d\mathrm{vol}_g.$$

MRF modulo time dependent diffeomorphisms

Modulo diffeomorphisms the evolution of the metric is Ricci flow. More precisely, let ϕ_t be the flow generated by the (time-dependent) vector field ∇f, and let

$$g^*(t) := \phi_t^* g(t), \quad f^*(t) := \phi_t^* f(t) \equiv f(t) \circ \phi_t.$$

Then

$$\begin{cases} \dfrac{\partial}{\partial t} g^* = -2\,\mathrm{Ric}_{g^*} \\ \dfrac{\partial}{\partial t} f^* = -\Delta^* f^* - R^* + |\nabla^* f^*|_g^2 \end{cases}$$

where R^* and Δ^* are taken with respect to the metric $g(t)$. Perelman's $\mathcal{F}$-functional is invariant under diffeomorphisms, hence

$$\mathcal{F}(g(t), f(t)) = \mathcal{F}(g^*(t), f^*(t)).$$

In other words If g and f evolve according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2, \end{cases}$$

then

$$\frac{d}{dt}\mathcal{F}(g,f) = 2\int_M |\mathrm{Ric} + \mathrm{Hess}\, f|^2\, e^{-f}\, d\mathrm{vol}_g.$$

In particular, $\mathcal{F}(g(t), f(t))$ is non-decreasing in time and monotonicity is strict unless

$$\mathrm{Ric} + \mathrm{Hess}\, f = 0 \quad \text{(steady Ricci soliton)}.$$

Ricci flow under conjugate backward heat equation

Set

$$u := e^{-f}.$$

Then g and u evolve according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} u = -\Delta u + Ru. \end{cases}$$

For $\mathcal{F}(g,u) = \int_M (R + |\nabla \log u|^2)\, u\, d\mathrm{vol}_g$ we have

$$\frac{d}{dt}\mathcal{F}(g,u) = 2\int_M |\mathrm{Ric} - \mathrm{Hess}\log u|^2\, u\, d\mathrm{vol}_g.$$

The measure $u(t,y)\,\mathrm{vol}_{g(t)}(dy)$ stays constant under the flow.

Theorem (Boltzmann-Shannon entropy) Let

$$\mu_t(dy) := u(t,y)\,\mathrm{vol}_{g(t)}(dy).$$

be the measure on M with density $u(t,\cdot)$ with respect to the volume measure to $g(t)$ as reference measure.
Let $\mathcal{E}(t)$ be the Boltzmann-Shannon entropy of μ_t,

$$\mathcal{E}(t) = \int_M (u\log u)(t,y)\,\mathrm{vol}_{g(t)}(dy).$$

Then the first two derivatives of $\mathcal{E}(t)$ are given by

$$\mathcal{E}'(t) = \int_M (R + |\nabla \log u|^2)\, u\, d\mathrm{vol}_g \equiv \mathcal{F}(g,u)$$

$$\mathcal{E}''(t) = 2\int_M |\mathrm{Ric} - \mathrm{Hess}\log u|^2\, u\, d\mathrm{vol}_g$$

III. Stochastic Analysis of evolving manifolds

- Let $(M, g_t)_{t \in I}$ be a smooth family of Riemannian manifolds, indexed by $I = [0, T]$. We call $(M, g_t)_{t \in I}$ an evolving manifold. Let $\mathbb{M} := M \times I$ be space time and consider the tangent bundle TM over $\mathbb{M}$:

$$TM \xrightarrow{\pi} \mathbb{M}, \quad \pi \text{ projection.}$$

- There is a natural *space-time connection* ∇ on TM, considered as bundle over space-time $\mathbb{M}$, defined by

$$\nabla_X Y = \nabla_X^{g_t} Y \quad \text{and} \quad \nabla_{\partial_t} Y = \partial_t Y + \frac{1}{2}(\partial_t g_t)(Y, \cdot)^{\sharp g_t}$$

- This connection is compatible with the metric, i.e.

$$\frac{d}{dt}|Y|_{g_t}^2 = 2\langle Y, \nabla_{\partial_t} Y\rangle_{g_t}$$

- The connection allows to define parallel transport along curves, but curves in space-time.

- Typically, we consider curves in $\mathbb{M}$ of the form

$$\gamma_t = (x_t, \rho_t), \quad t \in [0, T]$$

where ρ_t is a monotone differentiable transformation of $[0, T]$.

- Our examples here are:

$$\rho_t = t \quad \text{and} \quad \rho_t = T - t.$$

- Let $G = \mathrm{O}(n)$ and

$$\mathcal{F} \xrightarrow{\pi} \mathbb{M}$$

the G-principal bundle of orthonormal frames with fibres

$$\mathcal{F}_{(x,t)} = \{u \colon \mathbb{R}^n \to (T_xM, g_t) \mid u \text{ isometry}\}$$

and

$$T\mathcal{F} = V \oplus H := \ker d\pi \oplus h(\pi^* T\mathbb{M}).$$

the induced splitting of $T\mathcal{F}$.

- In terms of the *horizontal lift* of the G-connection,

$$h_u\colon\ T_{\pi(u)}\mathbb{M} \xrightarrow{\sim} H_u, \quad u \in \mathcal{F},$$

we get to each $\alpha X + \beta\partial_t \in T_{(x,t)}\mathbb{M}$ and each frame $u \in \mathcal{F}_{(x,t)}$, a unique "horizontal lift" $\alpha X^* + \beta D_t \in H_u$ of $\alpha X + \beta\partial_t$ such that

$$\pi_*(\alpha X^* + \beta D_t) = \alpha X + \beta\partial_t.$$

- In terms of the standard-horizontal vector fields on $\mathcal{F}$T,

$$H_i \in \Gamma(T\mathcal{F}), \quad H_i(u) = (ue_i)^* \equiv h_u(ue_i), \quad i = 1, \ldots, n,$$

we define Bochner's horizontal Laplacian on $\mathcal{F}$:

$$\Delta_{\text{hor}} = \sum_{i=1}^{n} H_i^2.$$

- Let $(M, g_t)_{t \in I}$ where $[0, T] \subset I \subset \mathbb{R}_+$. Recall that

$$\pi \colon \mathcal{F} \to \mathbb{M} := M \times I \quad \text{where } \pi(u) = (x, t) \text{ if } u \in \mathcal{F}_{(x,t)}.$$

- Let $\rho_t \colon [0, T] \to [0, T]$ be monotonic; here

$$\rho_t = t \quad \text{or} \quad \rho_t = T - t.$$

Finally let $D_t^\rho := \dot{\rho}(t)\, D_t = \pm D_t$.

- Consider the following Stratonovich SDE on $\mathcal{F}$:

$$dU = \pm D_t(U)\, dt + \sum_{i=1}^n H_i(U) \circ dZ^i, \quad U_0 = u,$$

where Z is a continuous semimartingale taking values in $\mathbb{R}^n$.

- If U solves the SDE then

$$\pi(U_t) = (X_t, \rho_t)$$

for some process X on M, the *stochastic development* of Z.

- Modulo choice of initial conditions each of the three processes X, U, Z determines the two others.
  (1) We call (X_t, ρ_t) a (space-time) Brownian motion if Z is a Brownian motion on $\mathbb{R}^n$.
  (2) We call (X_t, ρ_t) a (space-time) martingale if Z is a local martingale on $\mathbb{R}^n$.
- Let

$$//_{r,s} := U_s \circ U_r^{-1} \colon (T_{x_r}M, g_{\rho_r}) \to (T_{x_s}M, g_{\rho_s}), \quad 0 \le r \le s \le T,$$

  be the parallel transport along X_t (which by construction consists of isometries!). For the sake of brevity $//_s := //_{0,s}$.
- In the special case $\rho_t = t$, resp. $\rho_t = T - t$, we call (X_t, t), resp. $(X_t, T - t)$ a Brownian motion on $\mathbb{M}$ based at $(x, 0)$, resp. based at (x, T), if $X_0 = x$. In the same way, we talk about martingales on $\mathbb{M}$ based at $(x, 0)$, resp. (x, T).

III. An application: gradient-entropy estimate

- Assume that all manifolds are (M, g_t) are complete $(t \in I)$. Let $u\colon \mathbb{M} \to \mathbb{R}$ be a positive solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta_{g(t)} u.$$

- It is straight-forward to check:

$$\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right)(u \log u) = \frac{|\nabla u|^2}{u},$$

$$\left(\Delta_{g(t)} - \frac{\partial}{\partial t}\right)\frac{|\nabla u|^2}{u} = u\left(2|\nabla\nabla \log u|^2 + \left(2\mathrm{Ric} + \frac{\partial g}{\partial t}\right)\left(\frac{\nabla u}{u}, \frac{\nabla u}{u}\right)\right)$$

- Now assume that

$$\frac{\partial g}{\partial t} \geq -2\mathrm{Ric},$$

i.e. (g_t) is a supersolution to the Ricci flow.

Then, if $(X_t, T-t)$ is a Brownian motion based at (x, T) where $T \in I$, it is trivial to check that the process

$$N_t := (T-t)\,\frac{|\nabla u|^2}{u}(X_t, T-t) + \big(u \log u\big)(X_t, T-t), \quad 0 \le t \le T,$$

is a local submartingale. Hence assuming that N_t is a true submartingale, we obtain that $\mathbb{E}[N_0] \le \mathbb{E}[N_T]$ which gives

$$T\,\frac{|\nabla u|^2}{u}(x, T) + \big(u \log u\big)(x, T) \le \mathbb{E}\left[\big(u \log u\big)(X_T, 0)\right].$$

Theorem

Keeping assumptions as above. For each positive solution $u : [0, T] \times M \to \mathbb{R}_+$ to the time-dependent heat equation, we have

$$\left|\frac{\nabla u}{u}\right|^2 (x, T) \leq \frac{1}{T}\,\mathbb{E}\left[\frac{u(X_T, 0)}{u(x, T)} \log \frac{u(X_T, 0)}{u(x, T)}\right].$$

In particular,

(1) *Then, for any $\delta > 0$,*

$$\left|\frac{\nabla u}{u}\right| (x, T) \leq \frac{\delta}{2T} + \frac{1}{2\delta}\,\mathbb{E}\left[\frac{u(X_T, 0)}{u(x, T)} \log \frac{u(X_T, 0)}{u(x, T)}\right]$$

(2) *(Hamilton's gradient estimate in global form)*
If $m_T := \sup_{M \times [0,T]} u$, then

$$\frac{|\nabla u|}{u}(x, T) \leq \frac{1}{T^{1/2}} \sqrt{\log \frac{m_T}{u(x, T)}}\,.$$