Brownian motion, evolving geometries and entropy formulas

Talk 3

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Outline

1. Stochastic Calculus on manifolds (stochastic flows)
2. Analysis of evolving manifolds
3. Heat equations under Ricci flow and functional inequalities
4. Geometric flows and entropy formulas

I. Heat equation on a Riemannian manifold

Harnack inequalities and gradient estimates

- Let u be a positive solution of the heat equation on a Riemannian manifold (M, g):

$$\frac{\partial}{\partial t} u = \Delta u$$

- (Gradient estimate) What can be said about

$$|\nabla u| \quad \text{or} \quad \frac{|\nabla u|}{u}\,?$$

- (Harnack inequalities) For $s \leq t$, how to compare

$$u(x, s) \quad \text{and} \quad u(y, t)?$$

- Gradient estimates $\equiv$ infinitesimal versions of Harnack inequalities; Harnack inequalities $\equiv$ integrated versions of gradient estimates

Stationary solutions = harmonic functions

Let u be harmonic on some domain D in a Riemannian manifold:

$$\Delta u = 0$$

- **Cheng-Yau**
  Let M be complete and $D \subset M$ be an open, relatively compact domain. Let u be harmonic on D and strictly positive. Then

$$\frac{|\nabla u|}{u}(x) \leq c(n) \left[\sqrt{K} + \frac{1}{r(x)}\right]$$

  if $\mathrm{Ric}|D \geq -K$, $K \geq 0$ (where $r(x) = \mathrm{dist}(x, \partial D)$ and $n = \dim M$).

For a probabilistic proof see Arnaudon, Driver, Th. (2007).

By integrating Cheng-Yau along geodesic curves we obtain as Corollary:

Elliptic Harnack inequality

Let u be harmonic on $B_R(x) \subset M$ where M is complete. Then

$$\sup_{B_{R/2}(x)} u \leq C(n, R, K) \inf_{B_{R/2}(x)} u$$

Back to the parabolic case

- Let M be a complete Riem. manifold and u be a solution of

$$\frac{\partial}{\partial t} u = \Delta u \quad \text{on} \quad M \times \mathbb{R}_+$$

- There is an exact formula for $(\nabla u)(\cdot, t)_x$ in terms of Brownian motion X_t starting from x:

$$X_t = X_t(x)$$

- Recall: A Brownian motion X_t on M is characterized by the property that for each $f \in C^\infty(M)$,

$$d(f(X_t)) - \Delta f(X_t)\, dt = 0 \quad \text{(mod differentials of loc mart.)}$$

- For $x \in M$ define a linear transformation

$$Q_t\colon\ T_xM \to T_xM$$

as solution to the pathwise ODE

$$\begin{cases} dQ_t = -\mathrm{Ric}_{//_t}\, Q_t\, dt \\ Q_0 = \mathrm{id}_{T_xM} \end{cases}$$

where

$$\mathrm{Ric}_{//_t} := //_t^{-1} \circ \mathrm{Ric}_{X_t} \circ //_t \in \mathrm{End}(T_xM)$$

and $//_t\colon\ T_xM \to T_{X_t}M$ is parallel transport along $X_t = X_t(x)$:

$$\begin{array}{ccc} T_xM & \overset{}{\underset{\mathrm{Ric}_{//_t}}{\dashrightarrow}} & T_xM \\ {\scriptstyle //_t}\downarrow & & \uparrow{\scriptstyle //_t^{-1}} \\ T_{X_t}M & \underset{\mathrm{Ric}_{X_t}}{\longrightarrow} & T_{X_t}M \end{array}$$

By convention $\mathrm{Ric}_x(v) = \mathrm{Ric}_x(v,\cdot)^{\sharp g}$ for $v \in T_xM$.

Let M be stochastically complete (BM has infinite lifetime). Let u be the solution to the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \quad u|_{t=0} = f \in C_b(M).$$

1. Writing $u(x,t) = (P_t f)(x)$ we have

$$(P_t f)(x) = \mathbb{E}[f(X_t(x)], \quad f \in C_b(M).$$

Indeed: For fixed $t > 0$,

$$n_s = (P_{t-s} f)\,(X_s(x)), \quad 0 \le s \le t,$$

is a martingale starting at $P_t f(x)$; thus $P_t f(x) = \mathbb{E}[n_t]$.

2. In terms of the $\mathrm{Aut}(T_x M)$-valued process Q_t from above,

$$(dP_t f)_x = \mathbb{E}\left[Q_t^* //_t^{-1} (df)_{X_t(x)}\right], \quad f \in C_c^\infty(M)$$

Indeed: It is enough to check that

$$Q_s^* //_s^{-1} (dP_{t-s} f)_{X_s(x)}, \quad 0 \le s \le t,$$

is a martingale in $T_x^* M$, starting at $(dP_t f)_x$.

Remark It is straight-forward to derive from the representation

$$(dP_tf)_x = \mathbb{E}\left[Q_t^* //_t^{-1} (df)_{X_t(x)}\right], \quad f \in C_c^\infty(M)$$

functional inequalities:

For instance, let $K \in \mathbb{R}$. Assume that

$$\mathrm{Ric} \geq K.$$

Then

$$|\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad t \geq 0.$$

Theorem (Gradient formula)

Fixing $x \in M$, let D be a relatively compact neighbourhood of x and let $\tau_D(x)$ be the first exit time of $X_t(x)$ from D.

Let u be a bounded solution of the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \quad u|_{t=0} = f \in C_b^\infty(M).$$

Then, for each $v \in T_x M$,

$$\langle \nabla u(\cdot, t)_x, v \rangle = -\mathbb{E}\left[f(X_t(x)) \int_0^\tau \langle Q_s \dot{\ell}_s, dZ_s \rangle \right]$$

where

- $\tau = \tau_D(x) \wedge t$
- *Z is a Brownian motion in $T_x M$*
- *ℓ_s is any adapted process in $T_x M$ with absolutely continuous paths such that (some $\varepsilon > 0$)*

$$\ell_0 = v, \quad \ell_\tau = 0 \quad \text{and} \quad \left(\int_0^\tau |\dot{\ell}_s|^2 \, ds \right)^{1/2} \in L^{1+\varepsilon}.$$

II. Heat equation with respect to moving Riemannian metrics

- Study the heat equation under Ricci flow
- Consider positive solutions u to the heat equation:

$$\begin{cases} \dfrac{\partial}{\partial t} u - \Delta_{g(t)} u = 0 \\ \dfrac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

- Later we shall deal with the conjugate heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u + \Delta_{g(t)} u - \mathrm{R}(t,\cdot) u = 0 \\ \dfrac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

- Let $\mathbb{M} := M \times I$ be space-time and let

$$(X_r, T - r)$$

be Brownian motion on $\mathbb{M}$ based at (x, T). Thus time runs backwards. By construction, X_r is a $g_-(r)$-Brownian motion with $g_-(r) := g(T - r)$.

- Let $(M, g_t)_{t \in I}$ be a smooth family of Riemannian metrics. We consider the heat equation on $(M, g_t)_{t \in I}$:

$$\frac{\partial}{\partial t} u = \Delta_{g_t} u, \quad u|_{t=s} = f \in C(M).$$

- If u is a bounded solution, then

$$u(X_r(x, 0), T - r), \quad 0 \leq r \leq T - s,$$

is a martingale, and by taking expectations we get the formula

$$u(x, T) = \mathbb{E}[u(X_{T-s}(x, 0), s)] = \mathbb{E}[f(X_{T-s}^{(x,0)}], \quad 0 \leq s \leq T \text{ in } I.$$

- There is also a stochastic representation of $(\nabla u)(\cdot, T)_x$.
- For $x \in M$ the linear transformation

$$Q_t \colon T_xM \to T_xM$$

needs to be redefined as solution to the pathwise ODE

$$\begin{cases} dQ_t = -//_t^{-1} \left(\mathrm{Ric}_g - \frac{1}{2}\,\partial_t g\right)_{(X_t, T-t)} //_t\, Q_t\, dt \\ Q_0 = \mathrm{id}_{T_xM}. \end{cases}$$

- We see that $Q_t = \mathrm{id}$ if and only if the metric evolves by forward Ricci flow.
- This explains why Riemannian manifolds evolving under Ricci flow share many properties of Ricci flat static manifolds.

III. Characterization of Ricci flow by functional inequalities

Recall first again the case of a static Riemannian manifold.

- Let M be a complete and stochastically complete and $K \in \mathbb{R}$. Denote by

$$u(x,t) = (P_t f)(x)$$

the solution to the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \quad u|_{t=0} = f \in C_c^\infty(M).$$

- **Characterisation of "Ricci bounded below"**
  The following conditions are well-known to be equivalent:
  - $\mathrm{Ric} \geq K$;
  - $|\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|$;
  - $|\nabla P_t f|^2 \leq e^{-2Kt} P_t |\nabla f|^2$;
  - $P_t(f^2 \log f^2) - (P_t f^2) \log(P_t f^2) \leq \frac{2(1-e^{-2Kt})}{K} P_t |\nabla f|^2$;
  - $P_t(f^2) - (P_t f)^2 \leq \frac{1-e^{-2Kt}}{K} P_t |\nabla f|^2$.

- Now let $(M, g_t)_{t\in I}$ be a smooth family of Riemannian metrics and consider the heat equation on $(M, g_t)_{t\in I}$:

$$\frac{\partial}{\partial t}u = \Delta_{g_t} u, \quad u|_{t=s} = f \in C_b(M).$$

  Denote

$$u(x, T) = (P_{s,T} f)(x), \quad 0 \le s \le T \text{ in } I.$$

- Analogous to the case of a static manifold we can characterize supersolutions to the Ricci flow by functional equations.

Characterization of supersolutions to the Ricci flow

For a smooth family $(M, g(t))_{t\in I}$ of Riemannian metrics are equivalent:

- $(M, g(t))_{t\in I}$ is a supersolution to the Ricci flow, i.e.

$$\frac{\partial}{\partial t} g(t) \geq -2\,\mathrm{Ric}_{g(t)}.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t\in I}$ satisfies

$$|\nabla P_{s,T} f|_{g(T)} \leq P_{s,T} |\nabla f|_{g(s)}.$$

- For each $f \in C_c^\infty(M)$ the heat flow on $(M, g(t))_{t\in I}$ satisfies

$$|\nabla P_{s,T} f|^2_{g(T)} \leq P_{s,T} |\nabla f|^2_{g(s)}.$$

- Denote by $\mathcal{P}_{(x,T)}M$ the space of continuous paths

$$\gamma_t = (x_t, T - t)$$

based at (x, T) and $\mathbb{P}_{(x,T)}$ the probability measure on $\mathcal{P}_{(x,T)}M$ induced by the space-time BM $(X_t, T - t)$.

- For $\sigma = (0 \le \sigma_1 < \ldots < \sigma_k \le T)$ consider the evaluation map

$$e_\sigma(\gamma) = (x_{\sigma_1}, \ldots, x_{\sigma_k}).$$

- Let $F\colon \mathcal{P}_{(x,T)}M \to \mathbb{R}$ be a cylindrical function, i.e.

$$F = u \circ e_\sigma$$

where $u\colon M^k \to \mathbb{R}$ is smooth and of compact support.

- Consider the "parallel gradient":

$$\nabla^{\text{par}} F := e_\sigma^* \left(\sum_{i=1}^{k} //_{\sigma_i}^{-1} \nabla^{(i)}_{g(T-\sigma_i)} u \right).$$

Characterization of solutions to the Ricci flow:

For a smooth family $(M, g(t))_{t\in I}$ of Riemannian metrics are equivalent (R. Haslhofer and A. Naber):

- $(M, g(t))_{t\in I}$ is a solution to the Ricci flow, i.e.

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)}.$$

- For each cylindrical function $F\colon \mathcal{P}_{(x,T)}M \to \mathbb{R}$,

$$|\nabla_x \mathbb{E}_{(x,T)} F| \le \mathbb{E}_{(x,T)} |\nabla^{\mathrm{par}} F|.$$

- For each cylindrical function $F\colon \mathcal{P}_{(x,T)}M \to \mathbb{R}$,

$$|\nabla_x \mathbb{E}_{(x,T)} F|^2 \le \mathbb{E}_{(x,T)} |\nabla^{\mathrm{par}} F|^2.$$