Brownian motion, evolving geometries and entropy formulas

Talk 4

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Outline

1. Stochastic Calculus on manifolds (stochastic flows)
2. Analysis of evolving manifolds
3. Heat equations under Ricci flow and functional inequalities
4. Geometric flows and entropy formulas

I. Entropy under Ricci flow

- Consider positive solutions to

$$\begin{cases} \dfrac{\partial}{\partial t}u - \Delta_{g(t)}u = 0 \\ \dfrac{\partial}{\partial t}g_t = -2\,\mathrm{Ric}_{g(t)} \end{cases}$$

  It is convenient to let time run backwards in both equation.

- Then: Backward heat equation under backward Ricci flow
  Thus

$$\begin{cases} \dfrac{\partial}{\partial t}u + \Delta u = 0 \\ \dfrac{\partial}{\partial t}g = 2\,\mathrm{Ric} \end{cases}$$

- Let $(X_t(x), t)$ the space-time Brownian motion starting at $(x, 0)$. Then $X_t(x)$ is a $g(t)$-Brownian motion on M.
  For simplicity always start at time $s = 0$.

- Let $X_t(x)$ be a $g(t)$-Brownian motion on M. Consider the heat kernel measure

$$m_t(dy) := \mathbb{P}\{X_t(x) \in dy\}.$$

- We are interested in the entropy of

$$\mu_t := u(\cdot, t)\, dm_t \equiv u(X_t(x), t)\, d\mathbb{P}$$

- The quantity

$$\int_M u(y,t)\, m_t(dy) = \mathbb{E}[u(X_t(x), t)]$$

stays constant along the flow, since $u(X_t(x), t)$ is a martingale.

Theorem

Denote by

$$\mathcal{E}(t) = \mathbb{E}[(u \log u)(X_t(x), t)]$$
$$= \int_M (u \log u)(y, t)\, m_t(dy)$$

the entropy of $\mu_t = u(\cdot, t)\, dm_t \equiv u(X_t(x), t)\, d\mathbb{P}$.

The first two derivatives of $\mathcal{E}(t)$ *are given by*

$$\mathcal{E}'(t) = \mathbb{E}\left[\frac{|\nabla u|^2}{u}(X_t(x), t)\right]$$
$$\mathcal{E}''(t) = 2\,\mathbb{E}\left[\left(u\,|\mathrm{Hess}\log u|^2\right)(X_t(x), t)\right].$$

Applications to the classification of ancient solutions to the heat equation. (Hongxin Guo, Robert Philipowski, A.Th. 2015)

- With the substitution $\tau := -t$, solutions to the backward equation above defined for all $t \geq 0$ correspond to ancient solutions of the (forward) heat equation, $\tau \leq 0$, under forward Ricci flow.
- Let
$$\theta := \lim_{t \to \infty} \mathcal{E}'(t) \in [0, +\infty].$$

Example Consider $u(t, y) = e^{y-t}$ on $\mathbb{R}$ with the standard metric. Then

$$\mathcal{E}(t) = t \quad \text{and} \quad \theta = 1.$$

Proposition

Assume that $\frac{\partial g}{\partial t} = 2\mathrm{Ric}$ (or $\frac{\partial g}{\partial t} \leq 2\mathrm{Ric}$) and let u be a positive solution of the backward heat equation.

- Then u is constant if and only if $\theta = 0$.
- If the entropy $\mathcal{E}(t)$ grows sublinearly, i.e.

$$\lim_{t \to \infty} \mathcal{E}(t)/t = 0,$$

  then $\theta = 0$ and u is constant.

Consider

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric} \\ \dfrac{\partial}{\partial t} u + \Delta u = Ru. \end{cases}$$

Now

$$\mathbb{E}\left[\exp\left(-\int_0^t R(X_s(x),s)\,ds\right)\,u(X_t(x),t)\right] = u(x,0) \quad \text{indep. of } t.$$

Take

$$\mathbb{P}_{x,t} := \exp\left(-\int_0^t R(X_s(x),s)\,ds\right) d\mathbb{P}$$

as reference measure.

- Consider the entropy of the measure

$$\mu_{x,t} := u(X_t(x), t)\, d\mathbb{P}_{x,t}$$

defined as

$$\mathcal{E}(t) = \mathbb{E}_{x,t}\left[(u \log u)(X_t(x), t)\right]$$

where $\mathbb{E}_{x,t}$ denotes expectation w/r to $\mathbb{P}_{x,t}$.

- The derivative of $\mathcal{E}(t)$ is given by

$$\mathcal{E}'(t) = \mathbb{E}_{x,t}\left[\left((R + \left|\nabla \log u\right|^2)u\right)(X_t(x), t)\right].$$

Theorem

Consider the following entropy functional

$$\mathrm{Ent}(g,u,t) := \mathbb{E}_{x,t}\big[(u\log u)(X_t(x),t)\big] - 2\int_0^t \mathbb{E}_{x,s}\big[\Delta u(X_s(x),s)\big]\,ds.$$

Then

$$\frac{d}{dt}\,\mathrm{Ent}(g,u,t) = \mathbb{E}_{x,t}\left[\left(\frac{\left|\nabla u\right|^2}{u} - 2\Delta u + Ru\right)(X_t(x),t)\right],$$

$$\frac{d^2}{dt^2}\,\mathrm{Ent}(g,u,t) = 2\,\mathbb{E}_{x,t}\left[\left(\left|\mathrm{Ric} - \mathrm{Hess}\log u\right|^2 u\right)(X_t(x),t)\right].$$

We observe that

$$\mathcal{F}(g, u, t) := \frac{d}{dt}\,\mathrm{Ent}(g, u, t)$$

is non-decreasing in time and monotonicity is strict unless

$$\mathrm{Ric} + \mathrm{Hess}\, f = 0 \quad \text{(steady Ricci soliton)}$$

where $f = \log u$.

III. Ricci solitons

A complete Riemannian manifold (M, g) is said to be a gradient Ricci soliton if there exists $f \in C^\infty(M; \mathbb{R})$ such that

$$\mathrm{Ric} + \mathrm{Hess}(f) = \rho\, g$$

for some $\rho \in \mathbb{R}$. The function f is called a potential function of the Ricci soliton.

- $\rho = 0$: steady soliton;
- $\rho > 0$: shrinking soliton;
- $\rho < 0$: expanding soliton.

Note that if $f = \mathrm{const}$, then (M, g) is Einstein.

Ricci solitons are special solutions to the Ricci flow

- If (M, g) is Einstein with

$$\mathrm{Ric} = \rho\, g,$$

then

$$g(t) := (1 - 2\rho t)\, g$$

solves the Ricci flow equation.

- Likewise, if (M, g, f) is a gradient Ricci soliton with

$$\mathrm{Ric} + \mathrm{Hess}(f) = \rho\, g,$$

then

$$g(t) := (1 - 2\rho t)\, \varphi_t^* g$$

solves the Ricci flow equation. Here φ_t is the 1-parameter family of diffeomorphisms generated by $\nabla f/(1 - 2\rho t)$.

Let M again be a compact manifold. To study shrinking solitons, Perelman introduced the so-called $\mathcal{W}$-functional.
Instead of the $\mathcal{F}$-functional one considers

$$\mathcal{W}\colon\ \mathcal{M}\times C^\infty(M)\times\mathbb{R}_+^*\to\mathbb{R},$$
$$\mathcal{W}(g,f,\tau):=\int_M\Big[\tau\left(R+|\nabla f|^2\right)+f-n\Big]\,\frac{e^{-f}}{(4\pi\tau)^{n/2}}\,d\mathrm{vol}_g$$

One studies the gradient flow of $\mathcal{W}(g,f,\tau)$. This leads to evolutions $g(t)$, $f(t)$ and $\tau(t)$ where τ is then a strictly positive smooth function $\tau(t)$.

Theorem (Perelman 2002)

Let $g(t)$, $f(t)$ and $\tau(t)$ develop according to

$$\begin{cases} \frac{\partial}{\partial t} g = -2\,\mathrm{Ric} \\ \frac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2 + \frac{n}{2\tau} \\ \frac{\partial}{\partial t} \tau = -1. \end{cases}$$

Then

$$\frac{d}{dt}\mathcal{W}(g,f,\tau) = 2\tau \int_M \left|\mathrm{Ric} + \mathrm{Hess}\, f - \frac{g}{2\tau}\right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}}\, d\mathrm{vol}_g.$$

In particular, $\mathcal{W}(g,f,\tau)$ is non-decreasing in time and monotonicity is strict unless (M,g) satisfies

$$\mathrm{Ric} + \mathrm{Hess}\, f = \frac{g}{2\tau} \quad \text{(shrinking Ricci soliton).}$$

Let

$$u := \frac{e^{-f}}{(4\pi\tau)^{n/2}} \quad \text{or} \quad f = -\left(\log u + \frac{n}{2}\log(4\pi\tau)\right).$$

Then $g(t)$, $u(t)$ and $\tau(t)$ evolve according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} u + \Delta u = Ru, \\ \dfrac{\partial}{\partial t} \tau = -1. \end{cases}$$

Let

$$\mathcal{W}(g,u,\tau) = \int_M \left[\tau\left(R + |\nabla \log u|^2\right) - \log u - \frac{n}{2}\log(4\pi\tau) - n\right] u\, d\mathrm{vol}_g\,.$$

Then

$$\frac{d}{dt}\mathcal{W}(g,u,\tau) = 2\tau \int_M \left|\mathrm{Ric} - \mathrm{Hess}\log u - \frac{g}{2\tau}\right|^2 u\, d\mathrm{vol}_g.$$

In particular, $\mathcal{W}(g,u,\tau)$ is non-decreasing in time and monotonicity is strict unless

$$\mathrm{Ric} - \mathrm{Hess}\log u = \frac{g}{2\tau}.$$

Entropy of the Gaussian measure on $\mathbb{R}^n$

- Let

$$d\mu_t(y) = (4\pi t)^{-n/2}\, e^{-|y|^2/4t}\, dy =: \gamma_t(y)\, dy$$

be the standard Gaussian measure on $\mathbb{R}^n$.

- The Boltzmann-Shannon entropy of μ_τ is given as

$$\mathcal{E}_0(t) := \int_{\mathbb{R}^n} (\gamma_\tau \log \gamma_\tau)(y)\, dy = -\frac{n}{2}\big[1 + \log(4\pi\tau)\big].$$

Relative entropy

Let $g(t)$, $u(t)$ and $\tau(t)$ evolve according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric}, \\ \dfrac{\partial}{\partial t} u + \Delta u = Ru, \\ \dfrac{\partial}{\partial t} \tau = -1. \end{cases}$$

We normalize u such that

$$\int_M u(t)\, d\mathrm{vol}_{g(t)} \equiv 1.$$

Theorem (Relative entropy)

Let

$$H(g,u,t) := \mathcal{E}(t) - \mathcal{E}_0(t)$$
$$\equiv \int_M u \log u \, d\mathrm{vol}_g - \Big(-\frac{n}{2}\big[1 + \log(4\pi\tau)\big]\Big).$$

Then

$$\frac{d}{dt} H(g,u,t) = \int_M \Big[R + |\nabla \log u|^2 - \frac{n}{2\tau}\Big] \, u \, d\mathrm{vol}_g$$

and $$\frac{d}{dt} \tau \, H(g,u,t) = \mathcal{W}(g,u,\tau).$$

Excursion Lei Ni's entropy formula for positive solutions of the heat equation on a static Riemannian manifold.

Lei Ni (2004) Let $u > 0$ be a positive solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$$

on a compact static Riemannian manifold (M, g). Let

$$H(u,t) := \int_M u \log u \, d\mathrm{vol} - \left(-\frac{n}{2}\big[1 + \log(4\pi t)\big]\right)$$

be the difference between the Boltzmann entropy of the measure $u(x)\,\mathrm{vol}(dx)$ on M (normalized to be a probability measure) and the Boltzmann entropy of the standard Gaussian measure $\mu(dy)$ on $\mathbb{R}^n$.

Then

$$\frac{d}{dt}H(u,t) = \int_M \left(\Delta \log u + \frac{n}{2t}\right) u \, d\mathrm{vol}.$$

Observation Suppose that $\mathrm{Ric} \geq 0$.

Then, by the differential Harnack inequality,

$$|\nabla \log u|^2 - \frac{\Delta u}{u} \leq \frac{n}{2t},$$

equivalently

$$\Delta \log u + \frac{n}{2t} \geq 0.$$

In this case $H(u,t)$ non-decreasing as function of t.

V. Relative entropies and W-functionals

Let $g(t)$, $u(t)$ and $\tau(t)$ evolve according to

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric} \\ \dfrac{\partial}{\partial t} u + \Delta u = Ru \\ \dfrac{\partial}{\partial t} \tau = -1. \end{cases}$$

For simplicity $\tau(t) = T - t$.

Consider again on M the entropy functional

$$\mathrm{Ent}(g,u,t) := \mathbb{E}_{t,x}\big[(u\log u)(t,X_t(x))\big] - 2\int_0^t \mathbb{E}_{s,x}\big[\Delta u(s,X_s(x))\big]\,ds,$$

and the corresponding expression on $\mathbb{R}^n$,

$$\mathrm{Ent}_0(t) = \mathbb{E}\big[(\gamma_{\tau(t)}\log\gamma_{\tau(t)})(B_t)\big] - 2\int_0^t \mathbb{E}\left[\Delta\gamma_{\tau(s)}(B_s)\right] ds$$

where γ_t is the standard Gaussian kernel and B_t standard Brownian motion on $\mathbb{R}^n$ starting at 0.

Recall that the standard Gaussian measure on $\mathbb{R}^n$ is given by

$$d\mu_t(y) = (4\pi t)^{-n/2}\, e^{-|y|^2/4t}\, dy =: \gamma_t(y)\, dy.$$

A straightforward manipulation shows (with $\tau(t) = T - t$)

$$\begin{aligned}
\mathrm{Ent}_0(t) &= \int_{\mathbb{R}^n} \big(\gamma_\tau(y) \log \gamma_\tau(y)\big)\, \gamma_t(y)\, dy - 2\, t \Delta \gamma_T(0) \\
&= -\frac{1}{2}\, \frac{n}{(4\pi T)^{n/2}} \left(\frac{t}{T} + \log(4\pi\tau)\right) + \frac{1}{2}\frac{n}{(4\pi T)^{n/2}}\, \frac{t}{T} \\
&= -\frac{1}{2}\, \frac{n}{(4\pi T)^{n/2}}\, \log(4\pi\tau).
\end{aligned}$$

Normalize u such that

$$\mathbb{E}_{t,x}\big[u(t, X_t(x))\big] \equiv \frac{1}{(4\pi T)^{n/2}}$$

and consider the relative entropy

$$\mathbb{H}(g, u, t) := \mathrm{Ent}(g, u, t) - \mathrm{Ent}_0(t).$$

Theorem (Relative entropy; W-functional)

Let $g(t)$, $u(t)$ and $\tau(t)$ as above. Let

$$\mathbb{H}(t) \equiv \mathbb{H}(g,u,t) := \mathrm{Ent}(g,u,t) - \mathrm{Ent}_0(t) \quad \text{and}$$
$$\mathbb{W}(t) \equiv \mathbb{W}(g,u,t) := (\tau\mathbb{H}(t))'$$

Then

$$\frac{d}{dt}\mathbb{H}(t) = \mathbb{E}^*\left[\left(|\nabla \log u|^2 - 2\,\frac{\Delta u}{u} + R - \frac{n}{2\tau}\right)(t, X_t(x))\right],$$
$$\frac{d}{dt}\mathbb{W}(t) = 2\tau\,\mathbb{E}^*\left[\left|\mathrm{Ric} - \mathrm{Hess}\log u - \frac{g}{2\tau}\right|^2(t, X_t(x))\right].$$

Important observations

- *The relative entropy* $\mathbb{H}(t)$ *is non-increasing in time.*

  Indeed: The right-hand-side of $\frac{d}{dt}\mathbb{H}(t)$ is non-positive due to the Li-Yau inequality for solutions of the conjugate heat equation under Ricci flow:

  If $R \geq 0$ then

$$|\nabla \log u|^2 - 2\,\frac{\Delta u}{u} + R - \frac{n}{2\tau} \leq 0.$$

- *The W-functional* $\mathbb{W}(t)$ *is non-decreasing in time and monotonicity is strict unless* (M, g) *satisfies*

$$\mathrm{Ric} + \mathrm{Hess}\, f = \frac{g}{2\tau} \quad \text{(shrinking Ricci soliton)}$$

  where $f = \log u$.

The case of a surface ($\dim M = 2$)

- For a compact surface $(M, g(t))$ of *positive* curvature $R(t,\cdot)$ *Hamilton's surface entropy* (1988) is defined as

$$\mathrm{Ent}(t) := \int_M R(t,y)\log R(t,y)\,\mathrm{vol}_t(dy).$$

- He showed that $\mathrm{Ent}(t)$ is *non-increasing* along the normalized (forward) Ricci flow.

The case of a surface $(\dim M = 2)$

On a surface of positive curvature things simplify:

Instead of

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2\,\mathrm{Ric} \\ \dfrac{\partial}{\partial t} u + \Delta u = R u \end{cases}$$

we may consider

$$\begin{cases} \dfrac{\partial}{\partial t} g = -R\, g \\ \left(\dfrac{\partial}{\partial t} - \Delta - R\right) R = 0. \end{cases}$$

Now R itself solves the conjugate heat equation.

No breather theorems for non-compact manifolds

- A breather of a geometric flow is a periodic solution changing only by diffeomorphisms and rescaling.
- More precisely, a solution $(M, g(t))$ is a breather if there is a diffeomorphism $\varphi\colon M \to M$, a constant $c > 0$ and times $t_1 < t_2$ such that

$$g(t_2) = c\,\varphi^* g(t_1).$$

- According to $c < 1$, $c = 1$ or $c > 1$, the breather is called shrinking, steady or expanding, respectively.
- One wants to rule out non-trivial breathers, e.g. no steady or expanding breather theorems, like
  every steady breather is Ricci-flat,
  every expanding breather is a gradient soliton, etc
- The above formulas are suited to non-compact manifolds, since all measures are probability measures.

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