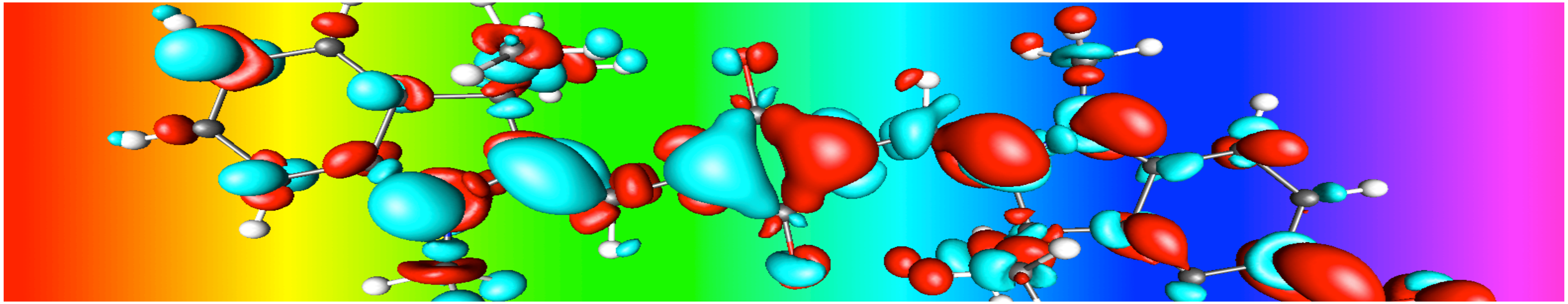
Advanced Workshop on High-Performance & High-Throughput Materials Simulations using Quantum ESPRESSO
ICTP, Trieste, Italy, January 16 to 27, 2017

Time-dependent density functional theory (TDDFT)



Ralph Gebauer

The Abdus Salam
International Centre
for Theoretical Physics

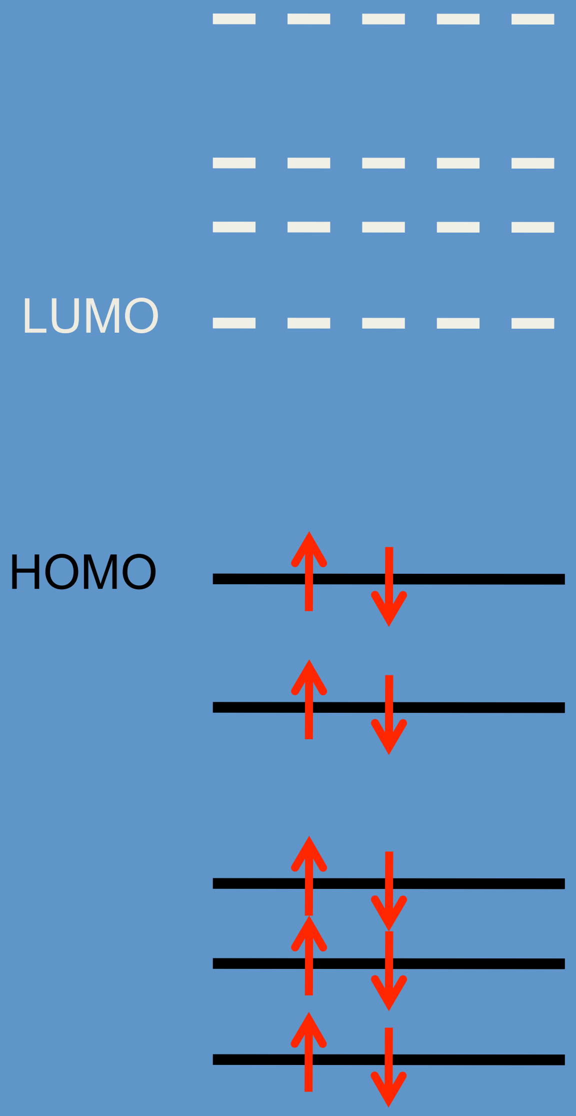
Monday, January 23rd, 2017

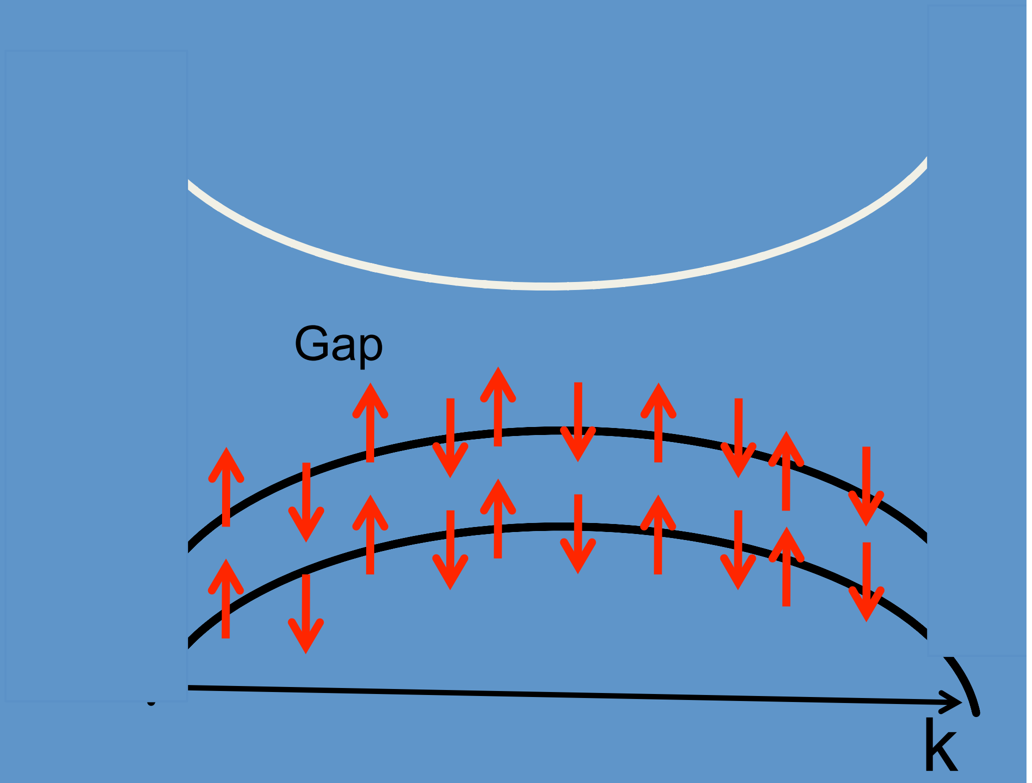
Electronic excitations … what's that?

```
 End of self-consistent calculation

          k = 0.0000 0.0000 0.0000 (  8440 PWs)   bands (ev):

  -29.5187 -13.9322 -11.7782 -11.7782  -8.8699  -1.8882  -1.8882  -0.2057
    0.9409   1.0554

    highest occupied, lowest unoccupied level (ev):    -8.8699   -1.8882

!    total energy              =     -43.17760726 Ry
```

Why single-particle states?

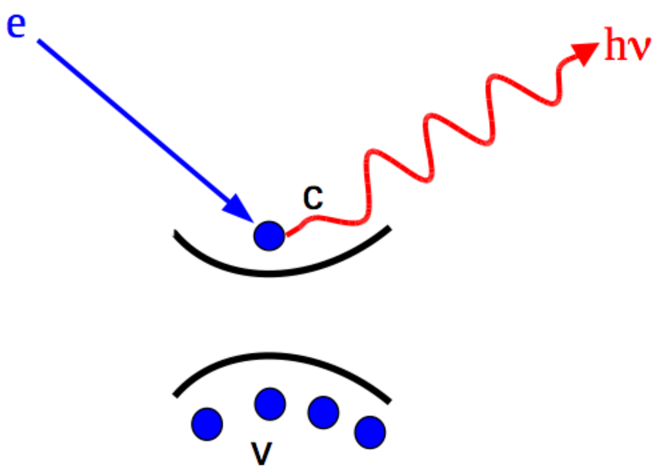
➔ concept of quasi-particles





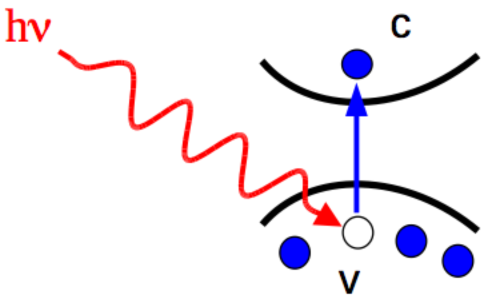
Excitations: Charged vs Neutral

Charged Excitations
N -> N+1 (or N-1)
(Photoemission Spectroscopy)



Inverse Photoemission

Neutral Excitations
N -> N
(Optical and Dielectric Spectroscopy)

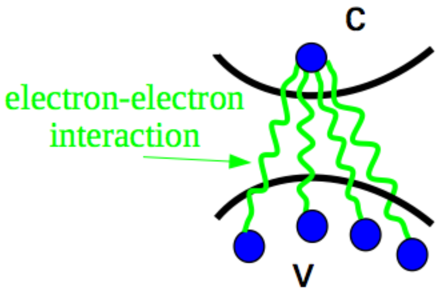


Optical Absorption

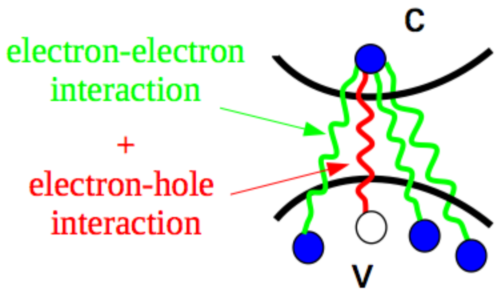
Excitations: Charged vs Neutral

Charged Excitations
N -> N+1 (or N-1)

Neutral Excitations
N -> N

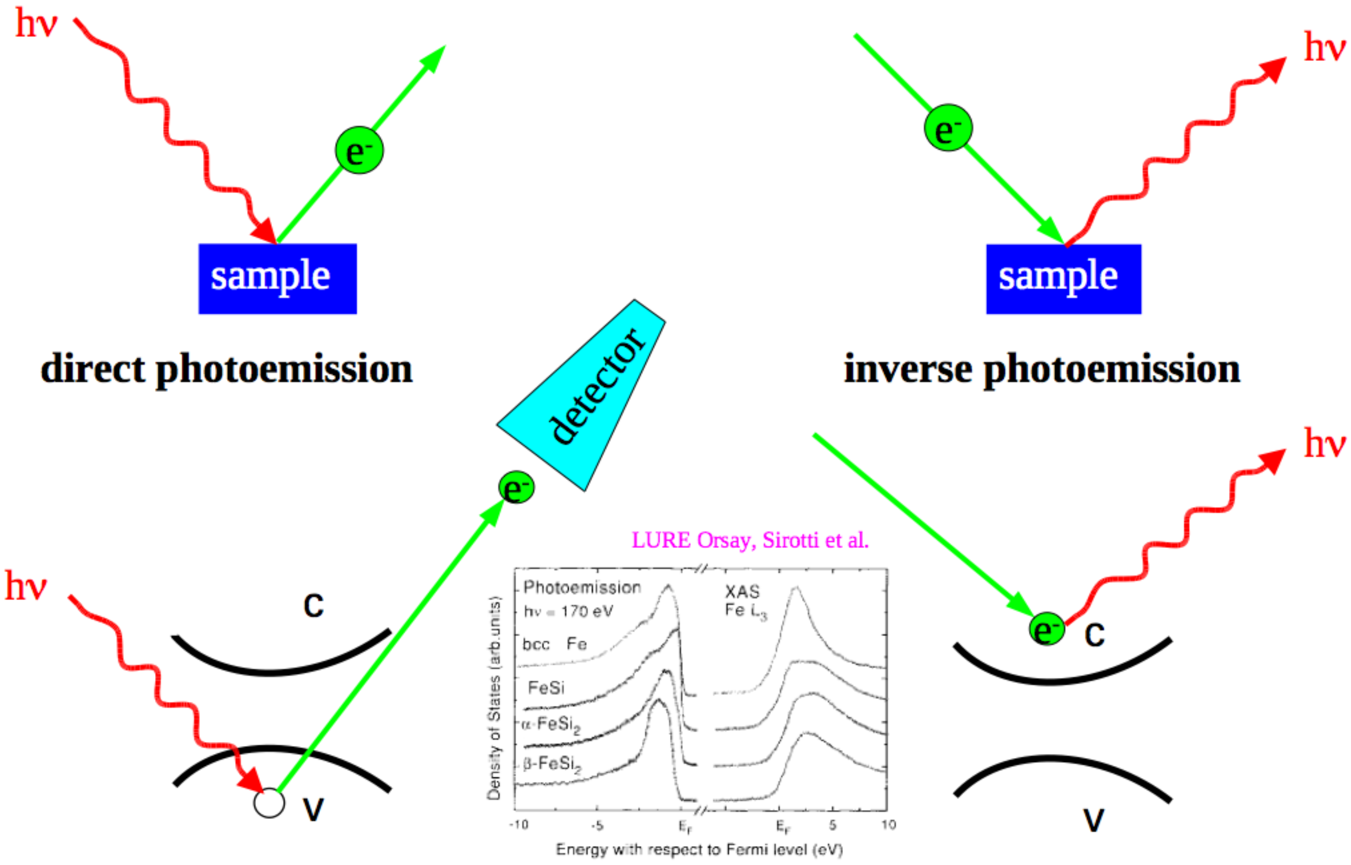


Inverse Photoemission



Optical Absorption

Photoemission Spectroscopy

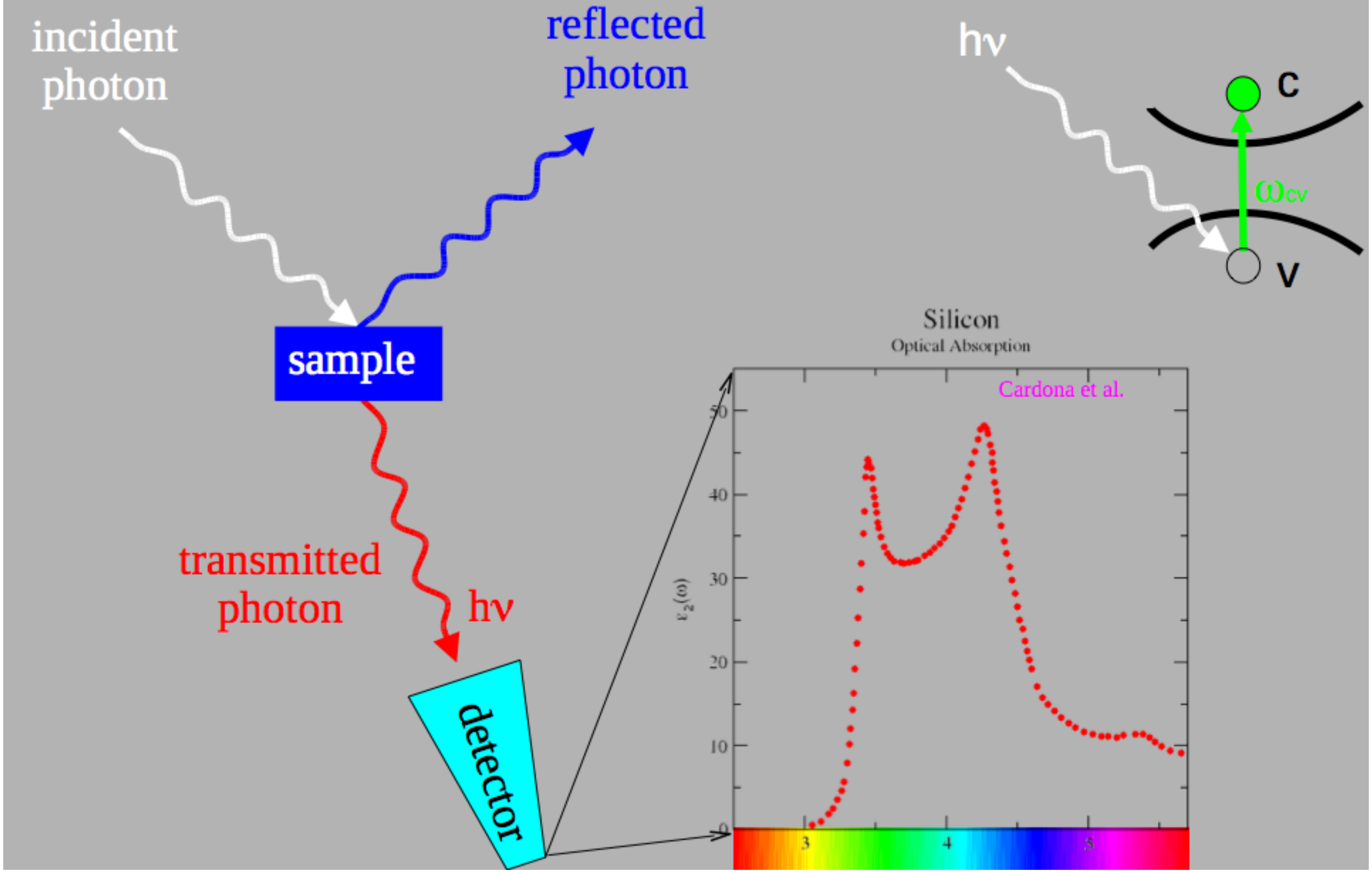


direct photoemission

inverse photoemission

LURE Orsay, Sirotti et al.

Optical Spectroscopy



Ab-initio approaches to excited states:

TDDFT (Time-dependent density functional theory): Neutral excitations

Many-body perturbation theory:

- ➔ GW (charged excitations)
- ➔ BSE (Bethe Salpeter equation) (neutral excitations)

Rules of thumb for using TDDFT

		RPA	ALDA
Isolated	Energy Loss	ok	ok
	Optical Prop	ok but..	ok but..
Solids	Energy Loss	ok	ok
	Optical Prop	no	no

Framework: What is TDDFT all about?

1964: Hohenberg and Kohn: Density Functional Theory (DFT)

work in terms of electron density (instead of many-particle wavefunctions)

DFT is a **ground state** theory

1984: Runge and Gross: **Time-Dependent** Density Functional Theory (TDDFT)

$$\text{Given }\ |\Psi(t=0)\rangle : V(\boldsymbol{r},t) \Leftrightarrow n(\boldsymbol{r},t)$$

like DFT, TDDFT is **formally exact**

Recall: Basic ground-state DFT

For practical calculations: Kohn-Sham framework

$$n_\sigma(\boldsymbol{r}) = \sum_i^{N_\sigma} |\phi_{i\sigma}(\boldsymbol{r})|^2$$

The density is written in terms of Kohn-Sham orbitals which satisfy

$$\left[-\frac{\nabla^2}{2} + v_\sigma^{KS}[n_\uparrow, n_\downarrow](\boldsymbol{r})\right]\phi_{i\sigma}(\boldsymbol{r}) = \epsilon_{i\sigma}\phi_{i\sigma}(\boldsymbol{r})$$

$$\boxed{\begin{aligned} F[n_\uparrow, n_\downarrow] &= T_s[n_\uparrow, n_\downarrow] + E_{Hartree} + E_{xc}[n_\uparrow, n_\downarrow] \\ E_{Hartree} &= \frac{1}{2}\int d^3r \int d^3r' \frac{n(\boldsymbol{r})\, n(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \\ E_{xc}^{LDA} &= \int d^3r\, n(\boldsymbol{r})\epsilon_{xc}^{unif}(n_\uparrow(\boldsymbol{r}), n_\downarrow(\boldsymbol{r})) \end{aligned}}$$

The Runge-Gross Theorem

Generalizing the HK theorem to time-dependent systems

There exists a one-to-one correspondence between the external v(r,t) and the electron density n(r,t), for systems evolving from a fixed many-body state.

Proof:

$$\begin{aligned} |\Psi(t_0)\rangle &= |\Psi'(t_0)\rangle \equiv |\Psi_0\rangle \\ n(\boldsymbol{r},t_0) &= n'(\boldsymbol{r},t_0) \equiv n^0(\boldsymbol{r}) \\ \boldsymbol{j}(\boldsymbol{r},t_0) &= \boldsymbol{j}'(\boldsymbol{r},t_0) \equiv \boldsymbol{j}^0(\boldsymbol{r}) \end{aligned}$$

Step 1: Different potentials v and v' yield different current densities j and j'

Step 2: Different current densities j and j' yield different densities n and n'

$$v(\boldsymbol{r},t) \neq v'(\boldsymbol{r},t) + c(t) \qquad \Rightarrow \qquad n(\boldsymbol{r},t) \neq n'(\boldsymbol{r},t)$$

Using TDDFT in practice

Finding an equivalent of the Kohn-Sham formalism

$$\mathrm{i}\frac{\partial}{\partial t}\varphi_{i\sigma}(\boldsymbol{r},t) = \hat{H}_{\sigma}^{\mathrm{KS}}(\boldsymbol{r},t)\varphi_{i\sigma}(\boldsymbol{r},t)$$

With a time-dependent Hamiltonian:

$$\hat{H}_{\sigma}^{\mathrm{KS}}(\boldsymbol{r},t) = -\frac{\nabla^2}{2} + v_{\sigma}^{\mathrm{KS}}[n_\uparrow, n_\downarrow](\boldsymbol{r},t)$$

Density and potentials are now defined like:

$$n_{\sigma}(\boldsymbol{r},t) = \sum_{i}^{N_\sigma} |\varphi_{i\sigma}(\boldsymbol{r},t)|^2$$

$$v_{\sigma}^{\mathrm{KS}}[n_\uparrow, n_\downarrow](\boldsymbol{r},t) = v_{\sigma}(\boldsymbol{r},t) + \int \mathrm{d}^3 r' \, \frac{n(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|} + v_{\sigma}^{\mathrm{xc}}[n_\uparrow, n_\downarrow](\boldsymbol{r},t)$$

Which functional to use ?

The easiest and probably most widely used functional is the

Adiabatic Local Density Approximation (ALDA)

$$v_\sigma^{\mathrm{xc\ ALDA}}(\boldsymbol{r},t) = \frac{\partial}{\partial n_\sigma}\left[n\,\varepsilon_{\mathrm{xc}}^{\mathrm{unif}}(n_\uparrow,n_\downarrow)\right]\Big|_{n_\alpha=n_\alpha(\boldsymbol{r},t)}$$

TDDFT in real time:

(1996:Bertsch; 2001: Octopus code)

- Consider a general time-dependent perturbation:

$$V_{pert}(\boldsymbol{r}, t)$$

- Obtain orbitals, charge density, and potentials by solving the Schrödinger equation explicitly in real time:

$$\psi_j(t+\Delta) = \exp\left(-iH(t+\frac{\Delta}{2})\Delta\right)\psi_j(t)$$

(*Nonlinear* TD Schrödinger equation)

- Can be used for *linear response* calculations, or for general TD *non-linear* problems.

A first application: Photochemistry

- Recent experimental progress made it possible to produce ultra-short intense laser pulses (few fs)
- This allows one to probe bond breaking/formation, charge transfer, etc. on the relevant time scales

- Nonlinear real-time TDDFT calculations can be a valuable tool to understand the physics of this kind of probe.

- Visualizing chemical bonds: Electron localization function

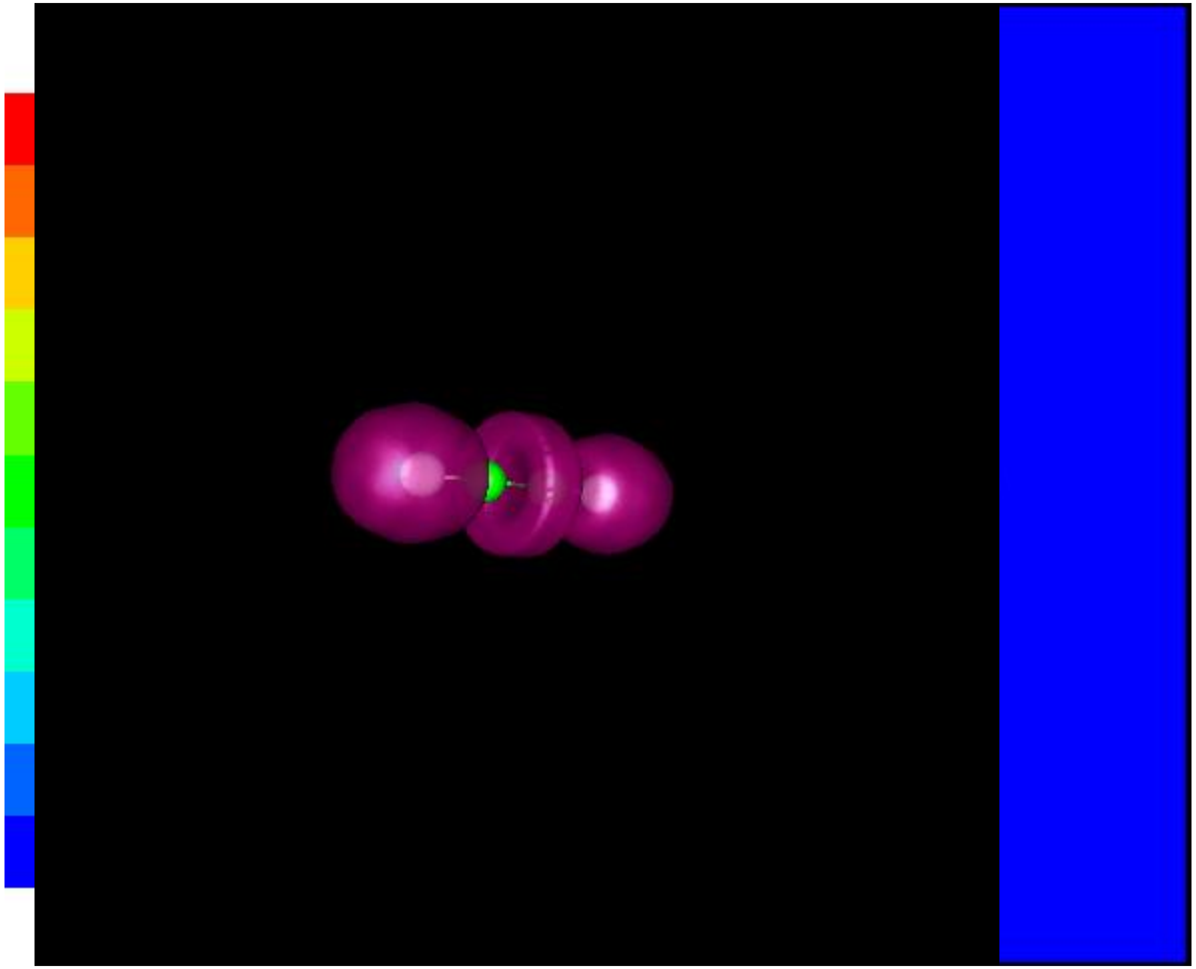
Nonlinear optical response

- Electron localization function:

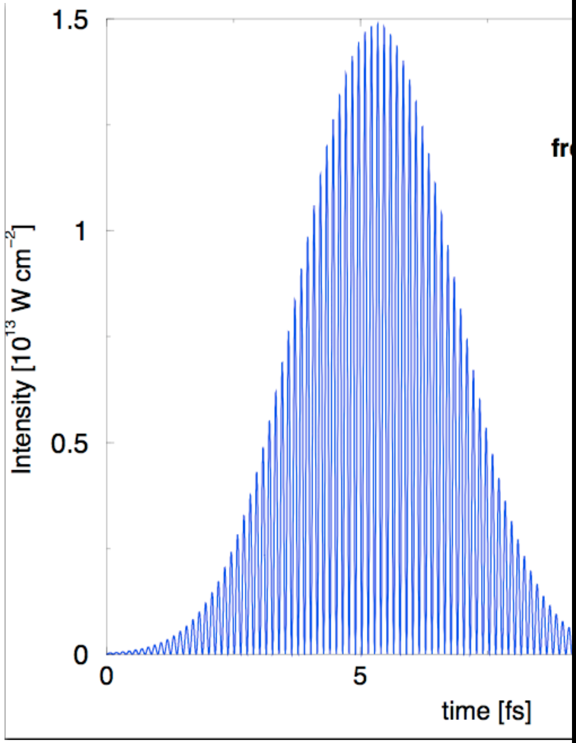
$$ELF(\boldsymbol{r},t) = \frac{1}{1+[D_\sigma(\boldsymbol{r},t)/D_\sigma^0(\boldsymbol{r},t)]^2}$$

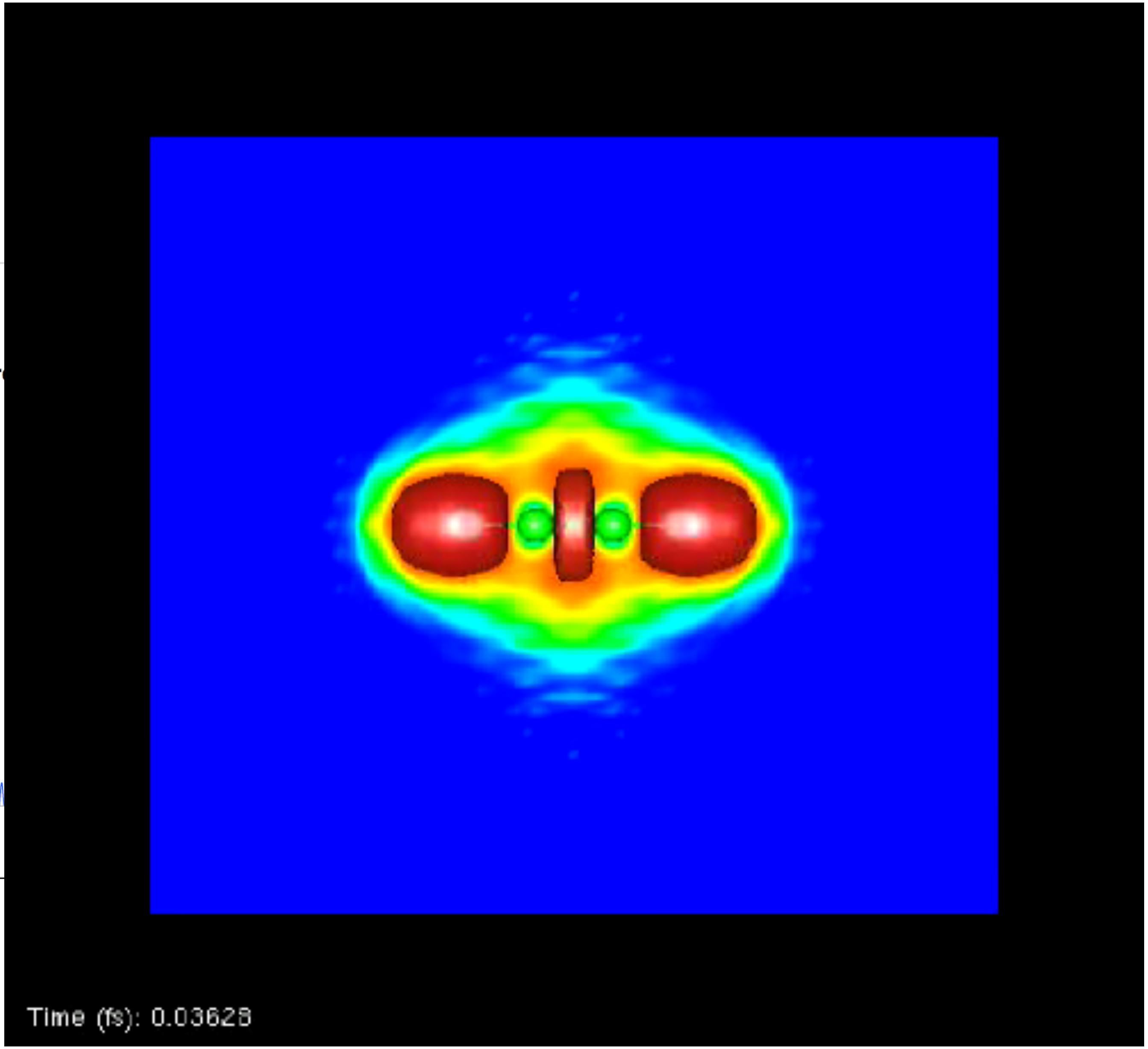
$$D_\sigma(\boldsymbol{r},t) = \tau_\sigma(\boldsymbol{r},t) - \frac{1}{4}\frac{[\nabla n_\sigma(\boldsymbol{r},t)]^2}{n_\sigma(\boldsymbol{r},t)} - \frac{j_\sigma^2(\boldsymbol{r},t)}{n_\sigma(\boldsymbol{r},t)}$$

Example: Ethyne C_2H_2



Example: Ethyne C_2H_2





How can we calculate optical spectra?

Consider a perturbation δV applied to the ground-state system:

$$\delta n(r,t) = \int d^3r'dt'\chi(r,r';t-t')\,\delta V(r',t')$$

The induced dipole is given by the induced charge density:

$$d(t) = \int d^3r\,\delta n(r,t)\hat{r}$$

Consider the perturbation due to an electric field:

$$\delta V(r,t) = -eE_{ext}(t)\cdot\hat{r}$$

How can we calculate optical spectra?

The dipole susceptibility is then given by:

$$d(t) = \int dt' \, \alpha(t - t') \, E_{ext}(t')$$

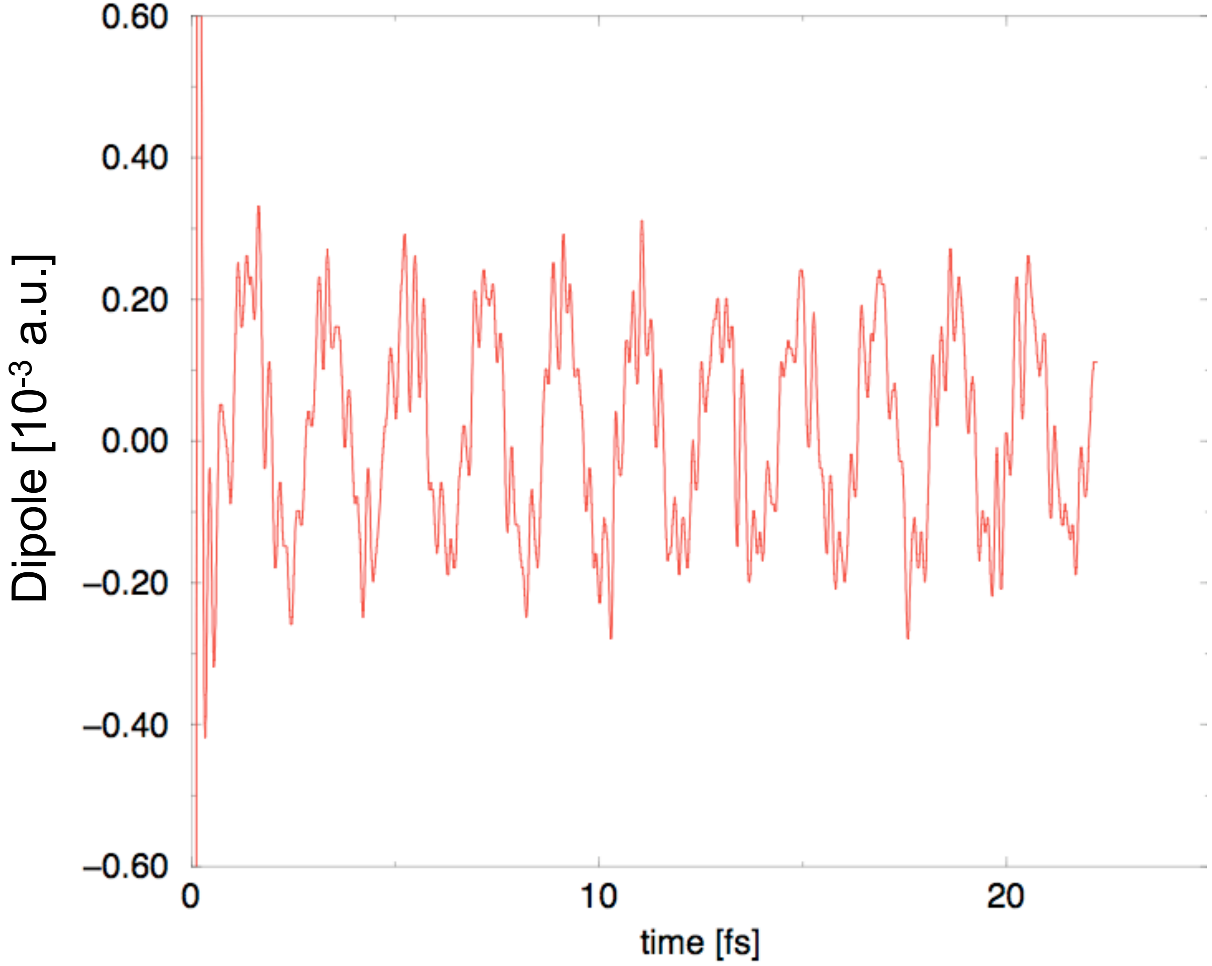
The experimentally measured strength function S is related to the Fourier transform of α:

$$S(\omega) = \frac{2m}{\pi e^2 \hbar} \, \omega \, \mathrm{Im} \, \alpha(\omega)$$

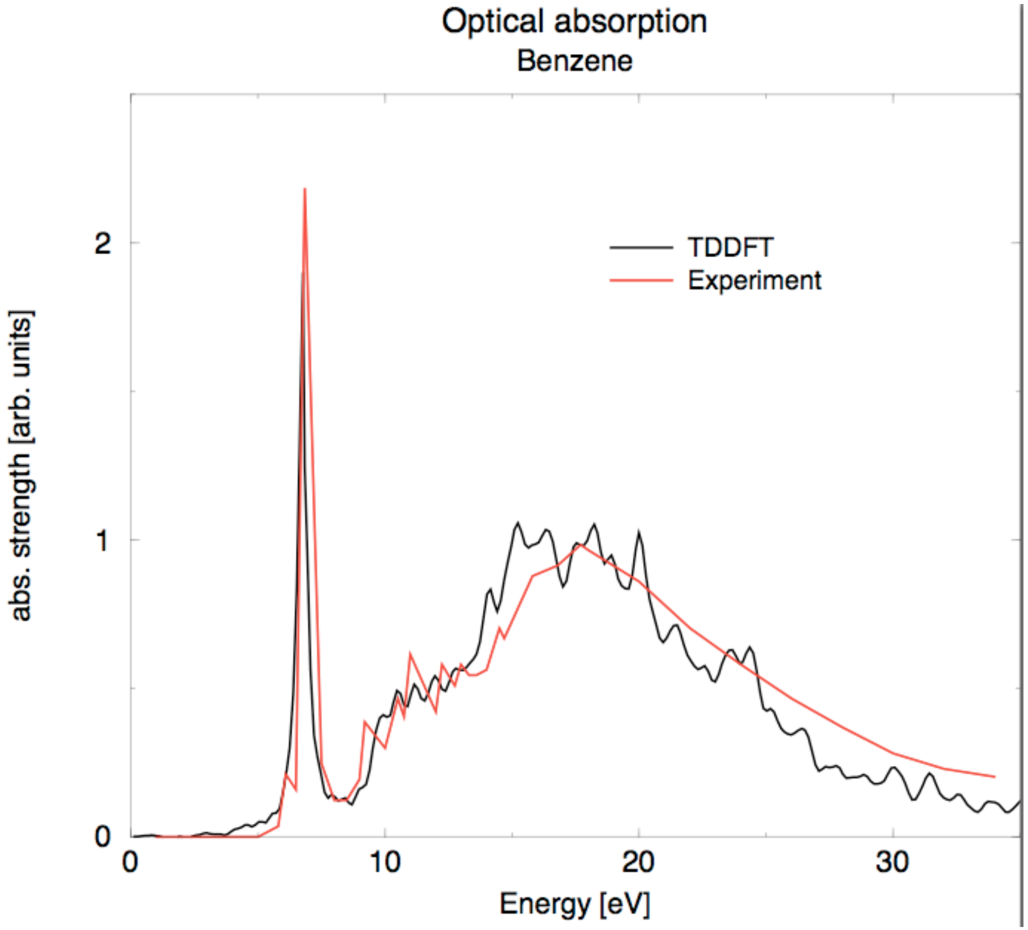
In practice: We take an E-field pulse $E_{ext} = E_0 \, \delta(t)$, calculate d(t), and obtain the spectrum $S(\omega)$ by calculating

$$d(\omega) = \int_0^\infty dt \, e^{i\omega t - \delta t} \, d(t)$$

A typical dipole-function d(t) …



… and the resulting spectrum



Linear response formalism in TDDFT:

- Calculate the system’s ground state using DFT
- Consider a monochromatic perturbation:

$$V_{pert}(\boldsymbol{r},t) = V_0(\boldsymbol{r})\left(\exp(i\omega t) + \exp(-i\omega t)\right)$$

- Linear response: assume the time-dependent response:

$$\psi_j(t) = e^{-i\epsilon_j t}\left(\psi_j^0 + \delta\psi_j^+ e^{i\omega t} + \delta\psi_j^- e^{-i\omega t}\right)$$

$$\delta n(\boldsymbol{r},t) = \delta n^+(\boldsymbol{r})e^{i\omega t} + \delta n^-(\boldsymbol{r})e^{-i\omega t}$$

$$\delta V(\boldsymbol{r},t) = V_{pert}(\boldsymbol{r},t) + \delta V_{SCF}^+(\boldsymbol{r})e^{i\omega t} + \delta V_{SCF}^-(\boldsymbol{r})e^{-i\omega t}$$

- Put these expressions into the TD Schrödinger equation

Linear response formalism in TDDFT:

$$\begin{aligned}
\omega\delta\psi_v^+(\boldsymbol{r}) &= \left(H_{KS}^0-\epsilon_v^0\right)\delta\psi_v^+ + \hat{P}_{\mathbf{c}}\left(\delta V_{SCF}^+(\boldsymbol{r})+V_{pert}(\boldsymbol{r})\right)\psi_v^0(\boldsymbol{r}) \\
-\omega\delta\psi_v^-(\boldsymbol{r}) &= \left(H_{KS}^0-\epsilon_v^0\right)\delta\psi_v^- + \hat{P}_{\mathbf{c}}\left(\delta V_{SCF}^-(\boldsymbol{r})+V_{pert}(\boldsymbol{r})\right)\psi_v^0(\boldsymbol{r})
\end{aligned}$$

Now define the following linear combinations:

$$\begin{aligned}
x_v(\boldsymbol{r}) &= \frac{1}{2}\left(\delta\psi_v^+(\boldsymbol{r})+\delta\psi_v^-(\boldsymbol{r})\right) \\
y_v(\boldsymbol{r}) &= \frac{1}{2}\left(\delta\psi_v^+(\boldsymbol{r})-\delta\psi_v^-(\boldsymbol{r})\right)
\end{aligned}$$

$$(\omega - \boldsymbol{\mathcal{L}})\left|\boldsymbol{x}, \boldsymbol{y}\right\rangle = \left|\boldsymbol{0}, \boldsymbol{v}\right\rangle \qquad \boldsymbol{\mathcal{L}} = \begin{pmatrix} 0 & D \\ K & 0 \end{pmatrix}$$

With the following definitions:

$$\begin{aligned} D\left|\boldsymbol{x}\right\rangle &= \left\{\left(H^0_{KS} - \epsilon_i\right) x_i(\boldsymbol{r})\right\} \\ K\left|\boldsymbol{x}\right\rangle &= \left\{\left(H^0_{KS} - \epsilon_i\right) x_i(\boldsymbol{r}) + \psi^0_i(\boldsymbol{r}) \sum_j \int d\boldsymbol{r}' \; f_{Hxc}(\boldsymbol{r}, \boldsymbol{r}') \psi^0_{i'}(\boldsymbol{r}') x_{i'}(\boldsymbol{r}')\right\} \end{aligned}$$

$$\begin{pmatrix} 0 & D \\ K & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \omega \begin{pmatrix} x \\ y \end{pmatrix}$$

Linear response TD-DFT essentially means solving a non-hermitean eigenvalue equation of dimension $2\ N_v \times N_c$.

Standard way to proceed (Casida's equations):

- Solve the time-independent problem to completely diagonalize the ground-state Hamiltonian.

  [Some computer time can be saved by limiting the diagonalization to the lower part of the spectrum]

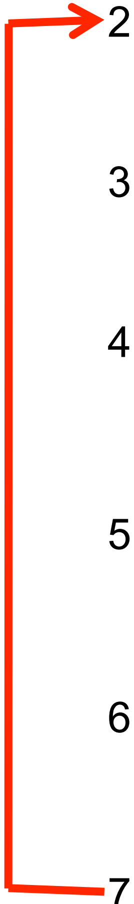
- Obtain as many eigenstates/frequencies of the TD-DFT problem as needed (or as possible).

  [Some computer time can be saved by transforming the non-hermitean problem to a hermitean one (e.g. Tamm-Dancoff approx.)]

Eigenstates of very large matrices: Davidson methods

Let H be a hermitean matrix, or large dimension, and we look for few low-lying eigenstates.

1. Select a set of trial eigenvectors $\{\mathbf{b}_i\}$ (typically 2x the number of desired eigenstates)



2. Calculate the representation of H in the space of trial vectors:

$$G_{ij} = \langle \mathbf{b}_i | H | \mathbf{b}_j \rangle$$

3. Diagonalize G (M is the number of desired eigenstates):

$$\mathbf{G}\alpha^k = \lambda^k \alpha^k, \qquad k = 1, 2, \cdots, M$$

4. Create new trial vectors **c**:

$$\mathbf{c}^k = \sum_i \alpha_i^k \mathbf{b}_i$$

5. Calculate the residue **r**:

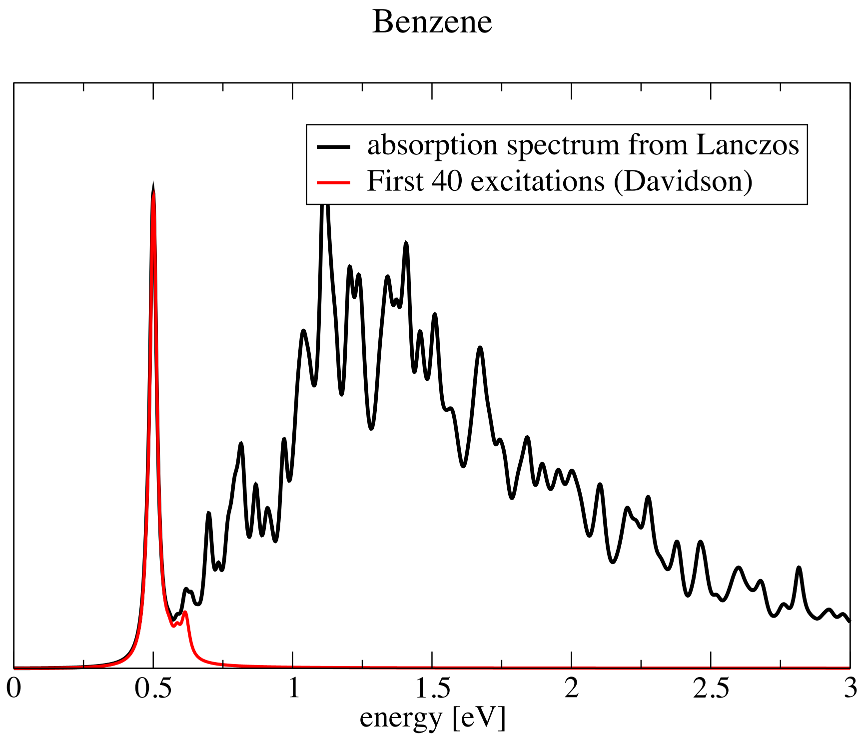
$$\mathbf{r}^k = \sum_i \left(\mathbf{H} - \lambda^k\right) \mathbf{c}^k$$

6. Using an approximation $\tilde{\mathbf{H}}$ for $\mathbf{H}$, calculate the correction vectors :

$$\delta^k = \left(\lambda^k - \tilde{\mathbf{H}}\right)^{-1} \mathbf{r}^k$$

7. Orthoganalize the $\{\delta^k\}$ to the $\{\mathbf{b}_i\}$ and get new trial eigenvectors.

Example: Benzene molecule



```
#       Energy(Ry)          total
      0.38112073E+00     0.28954952E-06
      0.41924668E+00     0.24532963E-08
      0.41936205E+00     0.91804138E-08
      0.43614131E+00     0.14279507E-04
      0.47779248E+00     0.45835218E-01
      0.47796122E+00     0.69172881E-05
      0.47839553E+00     0.30424303E-02
      0.47973541E+00     0.41971527E-07
      0.49171128E+00     0.56778070E-08
      0.49213150E+00     0.26186798E-08
      0.50060722E+00     0.35194127E+00
      0.50062231E+00     0.35154654E+00
      0.50216495E+00     0.20407694E-07
      0.50225774E+00     0.85588290E-07
      0.50474444E+00     0.14963819E-08
      0.51163438E+00     0.69570326E-05
      0.51165089E+00     0.20331996E-06
      0.51361736E+00     0.46846540E-02
```

Advantages:

One obtains not only the frequency (and oscillator strength), but the fulll eigenvector of each elementary excitation.

[Info can be used for spectroscopic assignments, to calculate forces, etc]

Disadvantages:

One obtains not only the frequency (and oscillator strength), but the fulll eigenvector of each elementary excitation.

[Info is often not needed, all the information is immediately destroyed after computation]

Computationally extremely demanding (large matrices to be diagonalized)

Time-dependent density functional perturbation theory (TDDFPT)

Remember: The photoabsorption is linked to the dipole polarizability $\alpha(\omega)$

$$d(t) = \int dt' \alpha(t - t') \mathcal{E}(t')$$

If we choose $\mathcal{E}(t') = \mathcal{E}_0 \delta(t')$, then knowing d(t) gives us $\alpha(t)$ and thus $\alpha(\omega)$.

Therefore, we need a way to calculate the observable d(t), given the electric field perturbation $\mathcal{E}_0 \delta(t)$.

Consider an observable A:

$$A(t) = \sum_i \left(\left\langle \delta\psi_i(t) \middle| \hat{A} \middle| \psi_i^0 \right\rangle + \left\langle \psi_i^0 \middle| \hat{A} \middle| \delta\psi_i(t) \right\rangle \right)$$

Its Fourier transform is:

$$\begin{aligned}
\tilde{A}(\omega) &= \sum_i \left(\left\langle \psi_i^0 \middle| \hat{A} \middle| \delta\psi_i^-(\omega) \right\rangle + \left\langle \psi_i^0 \middle| \hat{A} \middle| \delta\psi_i^+(\omega) \right\rangle \right) \\
&= 2 \sum_i \left\langle \psi_i^0 \middle| \hat{A} \middle| x_i(\omega) \right\rangle \\
&= 2 \left\langle \boldsymbol{a}, 0 \middle| \boldsymbol{x}, \boldsymbol{y} \right\rangle
\end{aligned}$$

$$\begin{aligned}\tilde{A}(\omega) &= \sum_i \left(\left\langle \psi_i^0 \middle| \hat{A} \middle| \delta\psi_i^-(\omega) \right\rangle + \left\langle \psi_i^0 \middle| \hat{A} \middle| \delta\psi_i^+(\omega) \right\rangle \right) \\ &= 2\sum_i \left\langle \psi_i^0 \middle| \hat{A} \middle| x_i(\omega) \right\rangle \\ &= 2\left\langle \boldsymbol{a}, 0 \middle| \boldsymbol{x}, \boldsymbol{y} \right\rangle\end{aligned}$$

Recall: $(\omega - \mathcal{L}) \left| \boldsymbol{x}, \boldsymbol{y} \right\rangle = \left| \mathbf{0}, \boldsymbol{v} \right\rangle$

Therefore:

$$\tilde{A}(\omega) = 2\left\langle \boldsymbol{a}, \mathbf{0} \middle| (\omega - \mathcal{L})^{-1} \middle| \mathbf{0}, \boldsymbol{v} \right\rangle$$

Thus in order to calculate the spectrum, we need to calculate one given matrix element of $(\omega - \mathcal{L})^{-1}$.

In order to understand the method, look at the hermitian problem:

$$\langle v| \left(\omega - H\right)^{-1} |v\rangle$$

Build a Lanczos recursion chain:

$$\begin{aligned}
\phi_{-1} &= 0 \\
\phi_0 &= |v\rangle \\
b_{n+1}\phi_{n+1} &= (H - a_n)\phi_n - b_n\phi_{n-1} \\
\langle\phi_{n+1}|\phi_{n+1}\rangle &= 1 \\
a_n &= \langle\phi_n|H|\phi_n\rangle
\end{aligned}$$

$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & 0 & \vdots \\ 0 & b_2 & a_2 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}$$

$$H = \begin{pmatrix} a_0 & b_1 & 0 & \cdots & 0 \\ b_1 & a_1 & b_2 & 0 & \vdots \\ 0 & b_2 & a_2 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & b_n \\ 0 & \cdots & 0 & b_n & a_n \end{pmatrix}$$

$$(\omega - H) = \begin{pmatrix} \omega - a_0 & -b_1 & 0 & \cdots & 0 \\ -b_1 & \omega - a_1 & -b_2 & 0 & \vdots \\ 0 & -b_2 & \omega - a_2 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & -b_n \\ 0 & \cdots & 0 & -b_n & \omega - a_n \end{pmatrix}$$

$$(\omega - H) = \begin{pmatrix} \boxed{\omega - a_0} & -b_1 & 0 & \cdots & 0 \\ -b_1 & \boxed{\omega - a_1} & -b_2 & 0 & \vdots \\ 0 & -b_2 & \boxed{\omega - a_2} & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & -b_n \\ 0 & \cdots & 0 & -b_n & \boxed{\omega - a_n} \end{pmatrix}$$

$$(\omega - H)^{-1} = \cfrac{1}{\omega - \boxed{a_0} + \cfrac{\boxed{b_1^2}}{\omega - \boxed{a_1} + \cfrac{\boxed{b_2^2}}{\omega - \boxed{a_2} + \cdots}}}$$

Back to the calculation of spectra:

Recall: $(\omega - \mathcal{L})\,|\boldsymbol{x}, \boldsymbol{y}\rangle = |\mathbf{0}, \boldsymbol{v}\rangle$

Therefore:

$$\tilde{A}(\omega) = 2\left\langle \boldsymbol{a}, \mathbf{0} \middle| (\omega - \mathcal{L})^{-1} \middle| \mathbf{0}, \boldsymbol{v} \right\rangle$$

Use a recursion to represent L as a tridiagonal matrix:

$$\mathcal{L} = \begin{pmatrix} a_1 & b_1 & 0 & & & & \\ c_1 & a_2 & b_2 & & & & \\ 0 & c_2 & a_3 & b_3 & & & \\ & & & \ddots & \ddots & \ddots & b_{N-1} \\ & & & & & c_{N-1} & a_N \end{pmatrix}$$

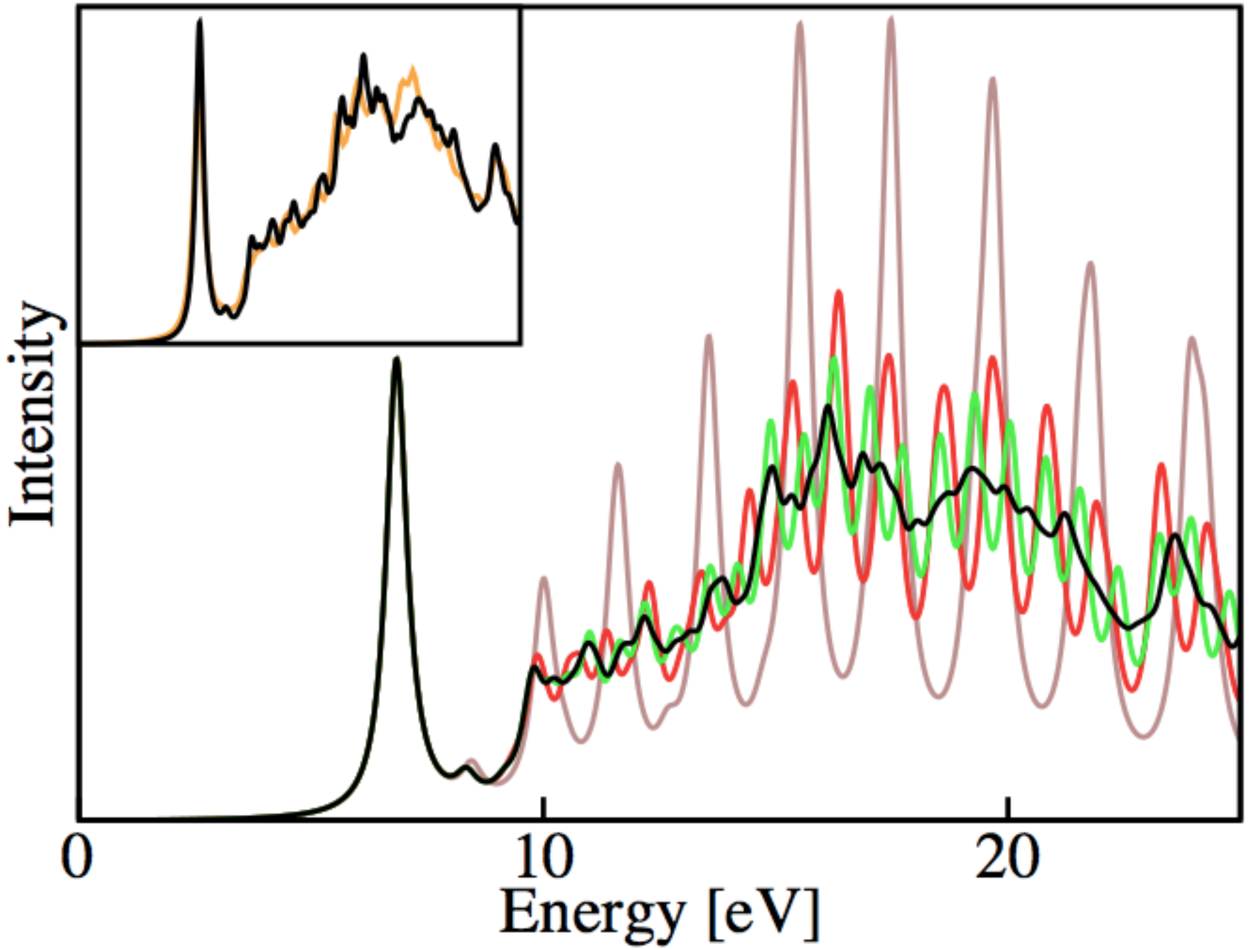
And the response can be written as a continued fraction!

$$\tilde{A}(\omega) = 2\left\langle \boldsymbol{a}, \boldsymbol{0} \middle| (\omega - \mathcal{L})^{-1} \middle| \boldsymbol{0}, \boldsymbol{v} \right\rangle$$

$$= \frac{1}{\omega - a_1 + b_2 \frac{1}{\omega - a_2 + \cdots} c_2}$$

How does it work?

Benzene spectrum



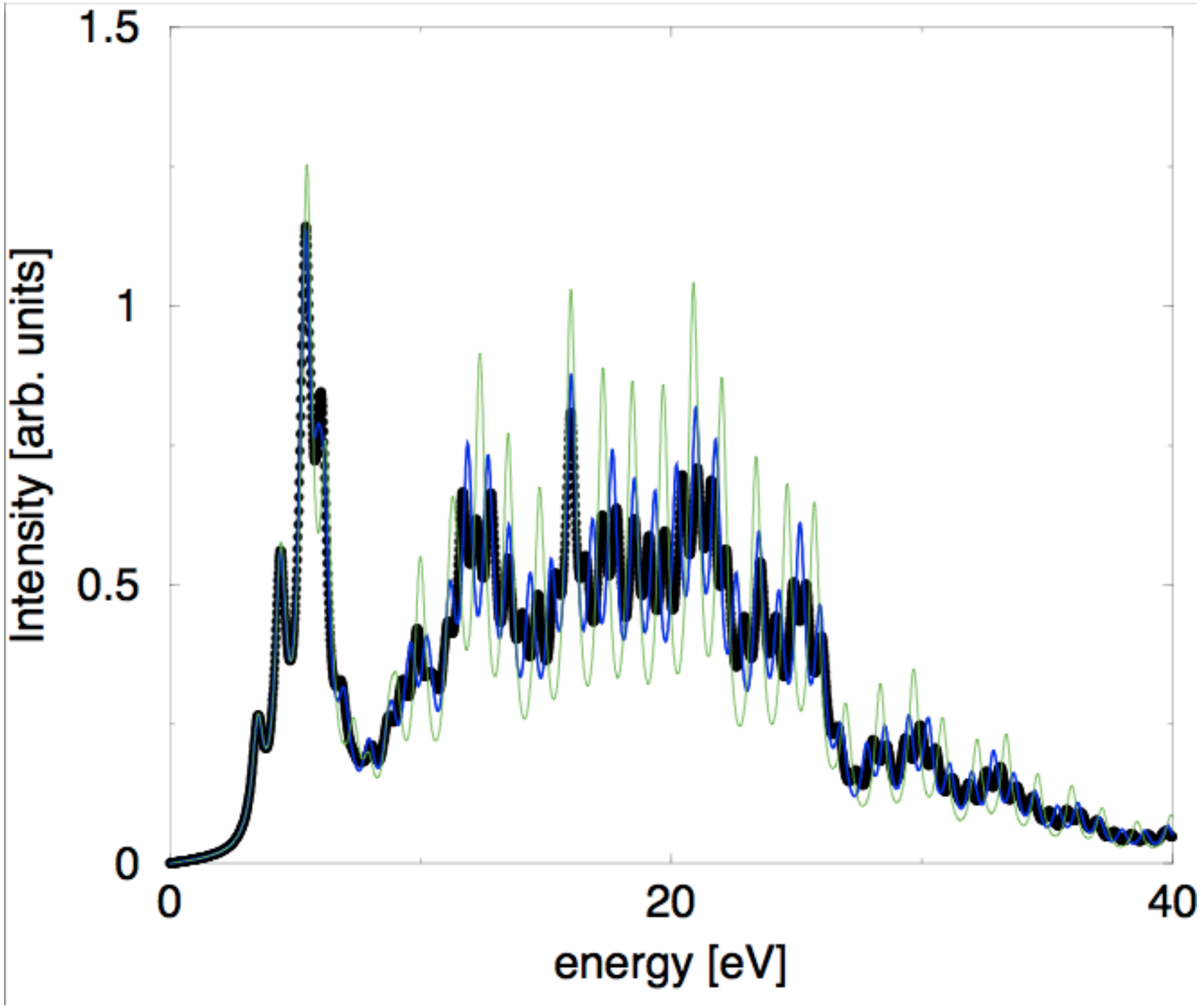
Plum: 1000

Red: 2000

Green: 3000

Black: 6000

Spectrum of C_{60}

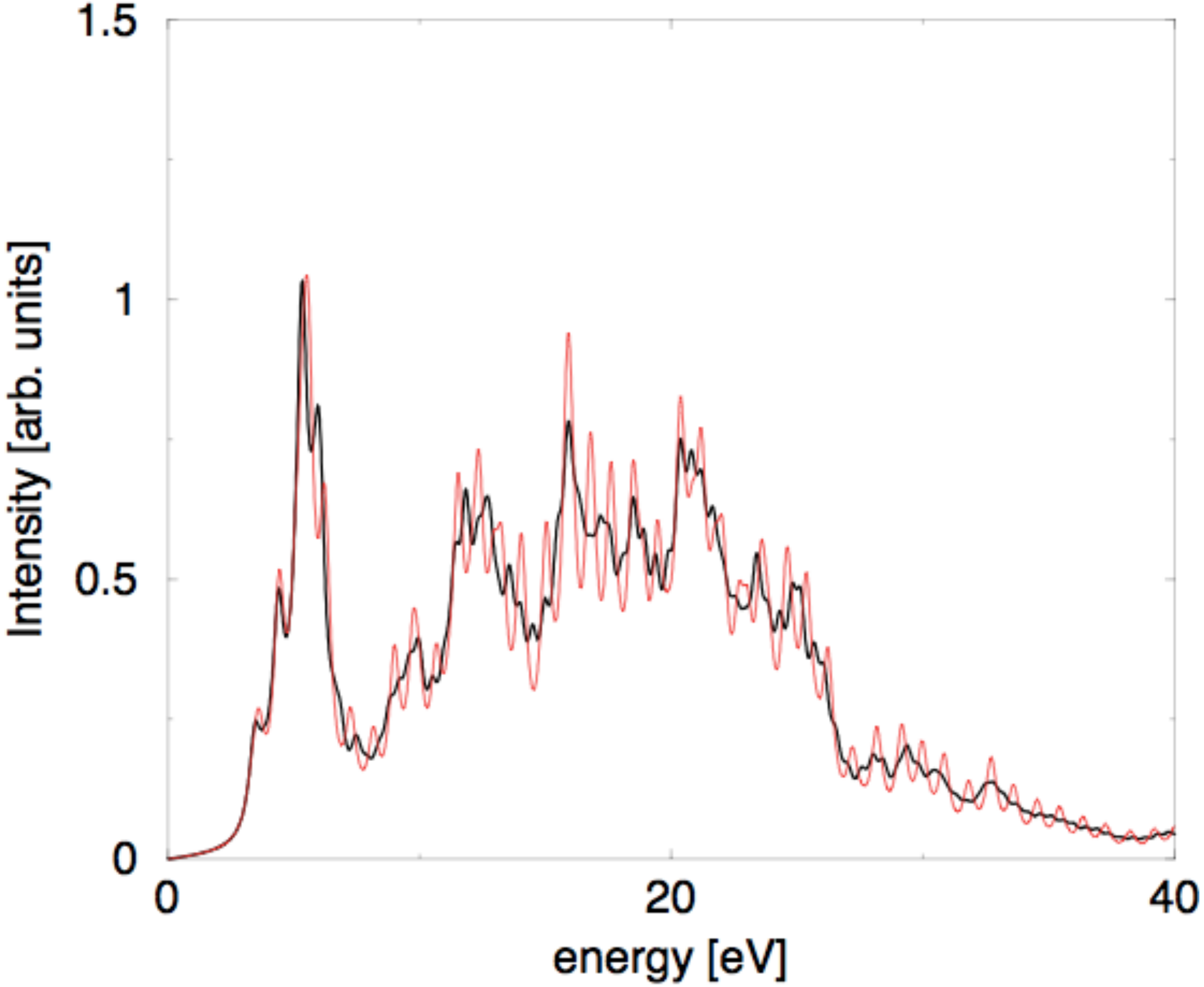


Black: 4000

Blue: 3000

Green: 2000

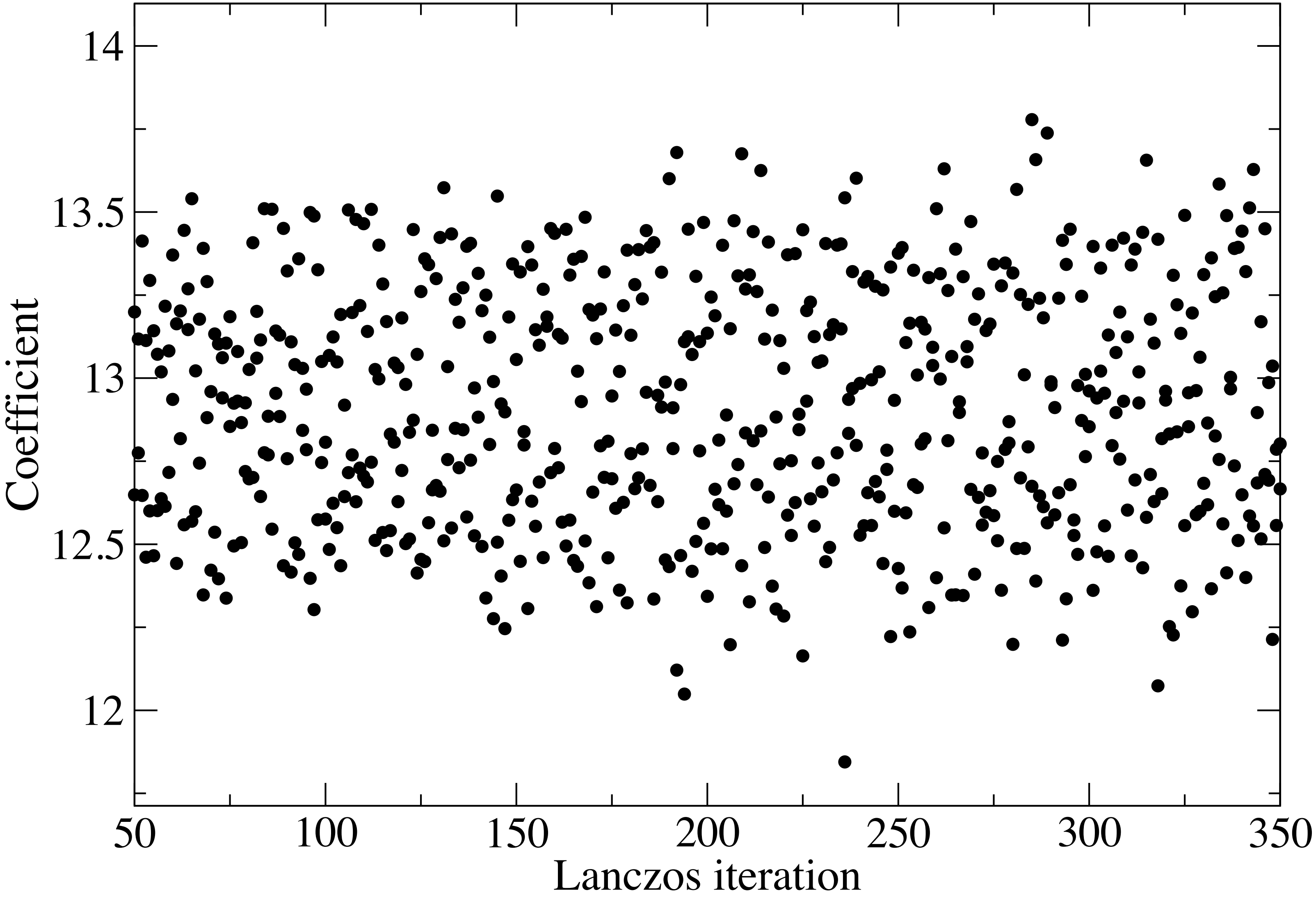
Spectrum of C_{60}: Ultrasoft pseudopotenitals



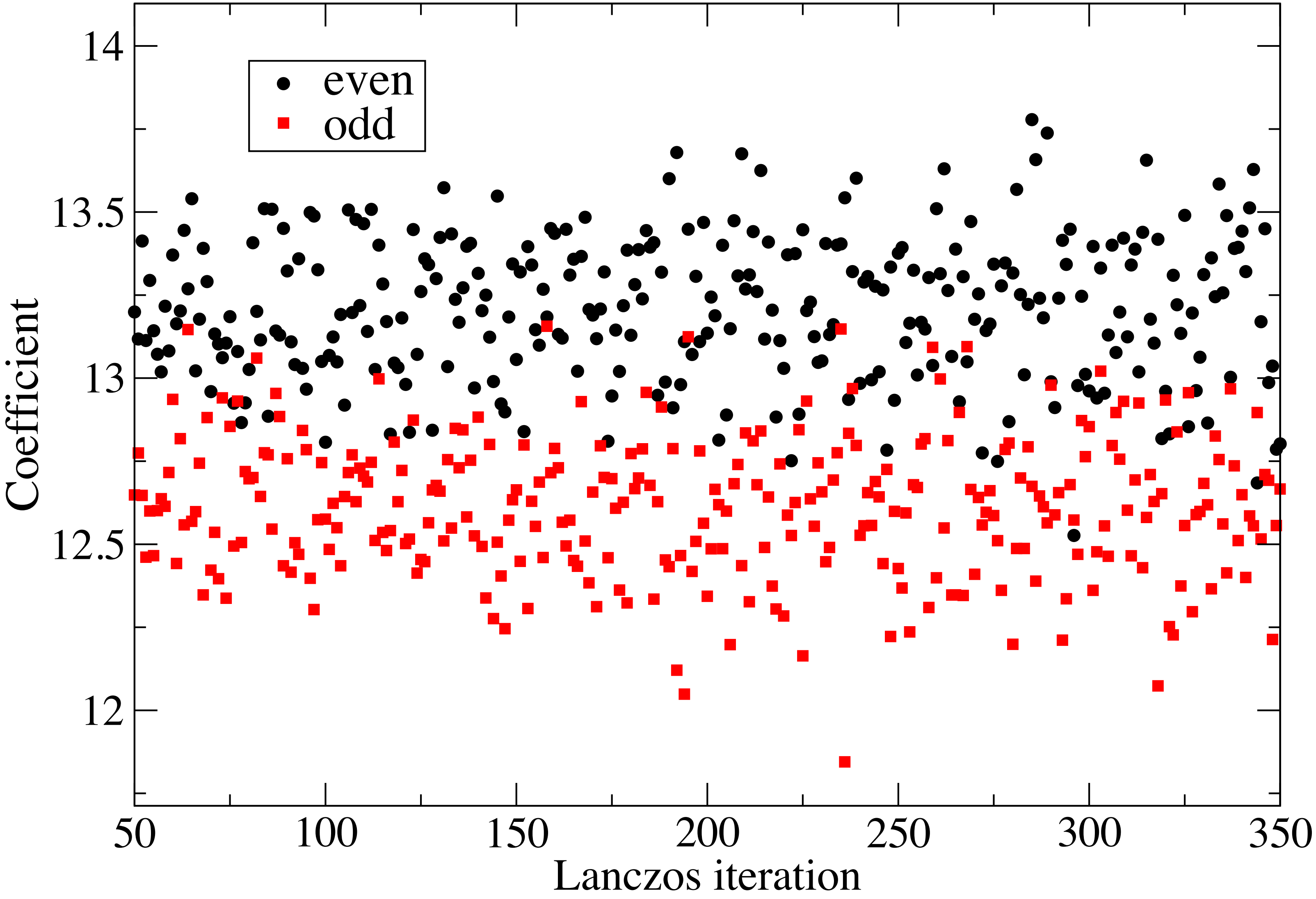
Black: 2000

Red: 1000

Speeding up convergence:
Looking at the Lanczos coefficients

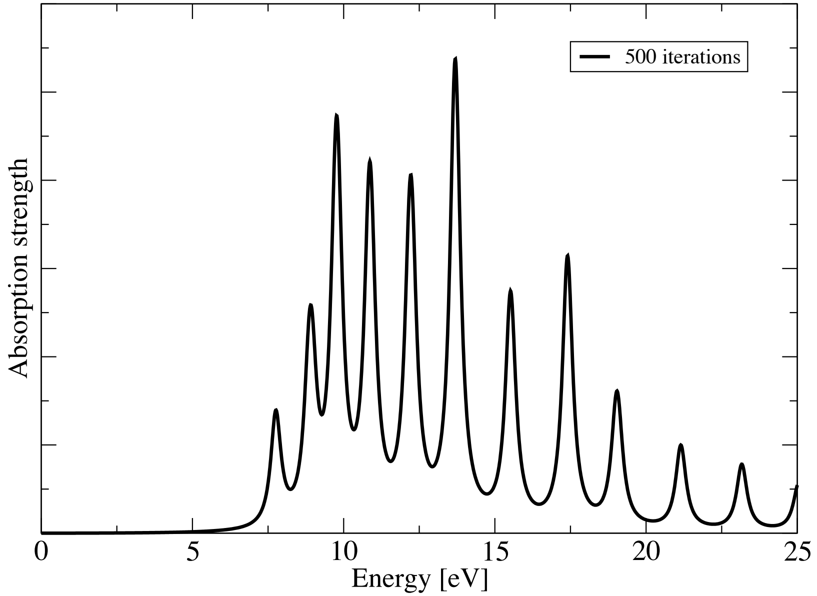


Speeding up convergence:
Looking at the Lanczos coefficients



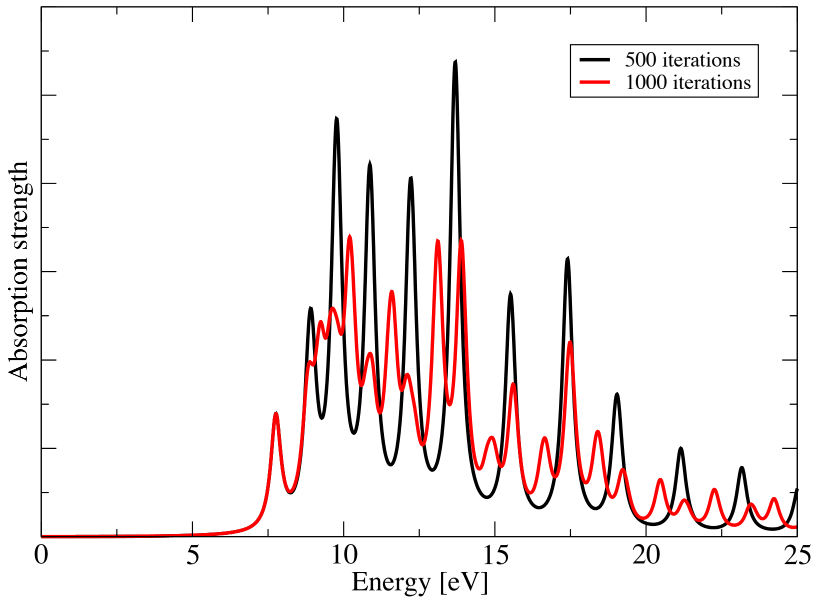
Effect of the terminator:

No terminator:



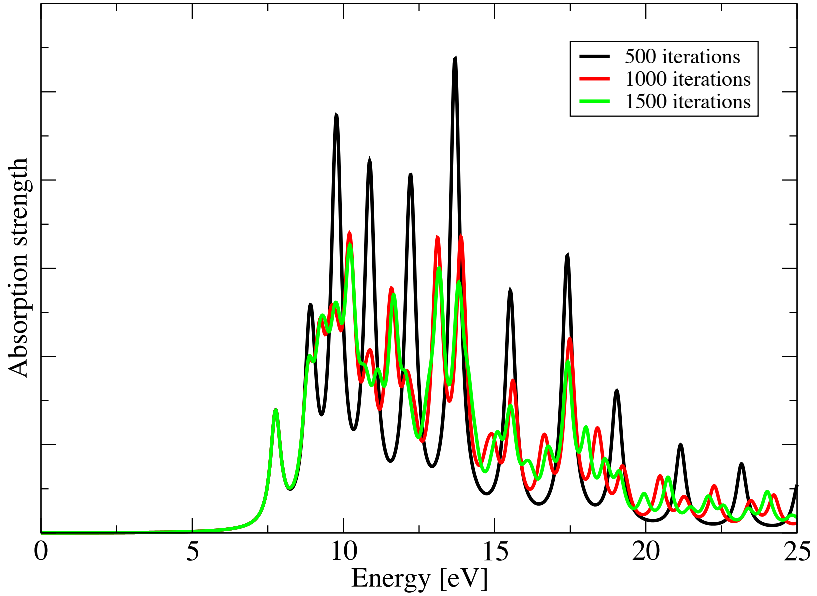
Effect of the terminator:

No terminator:



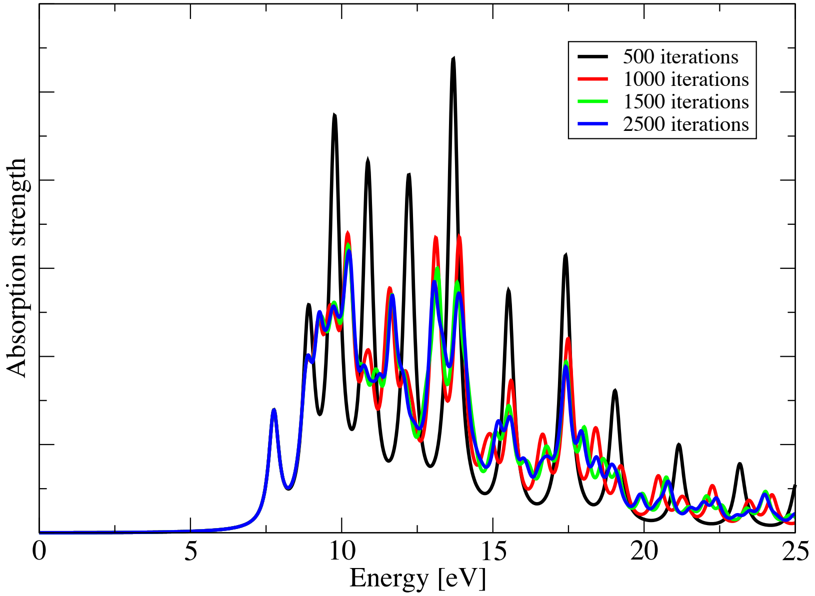
Effect of the terminator:

No terminator:



Effect of the terminator:

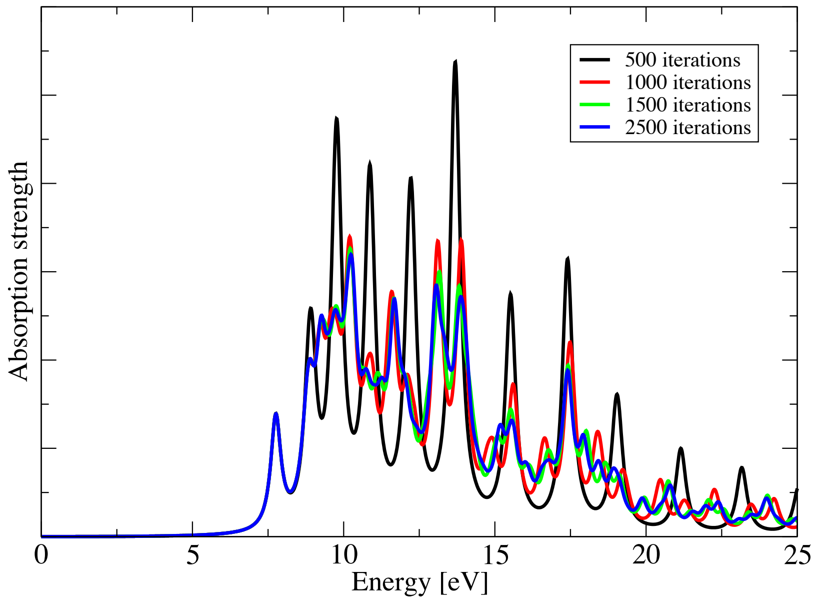
No terminator:

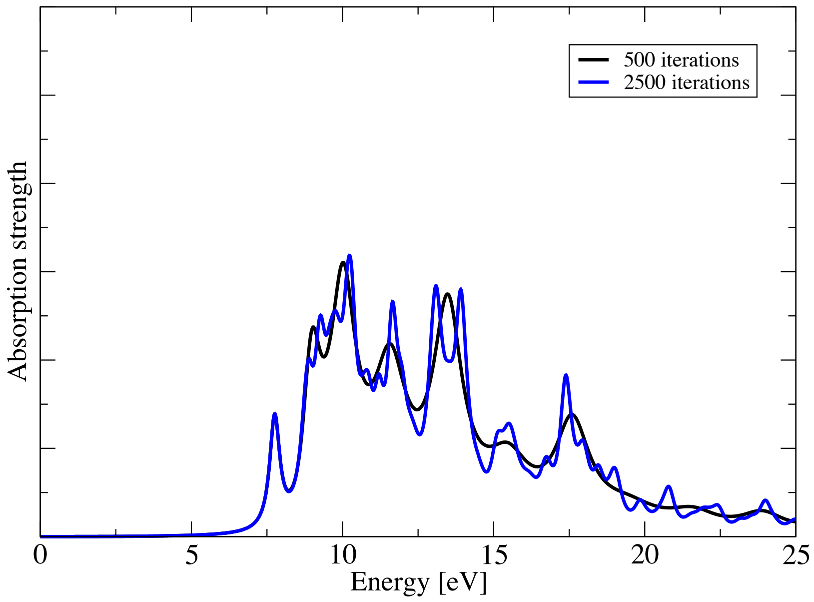


Effect of the terminator:

No terminator:

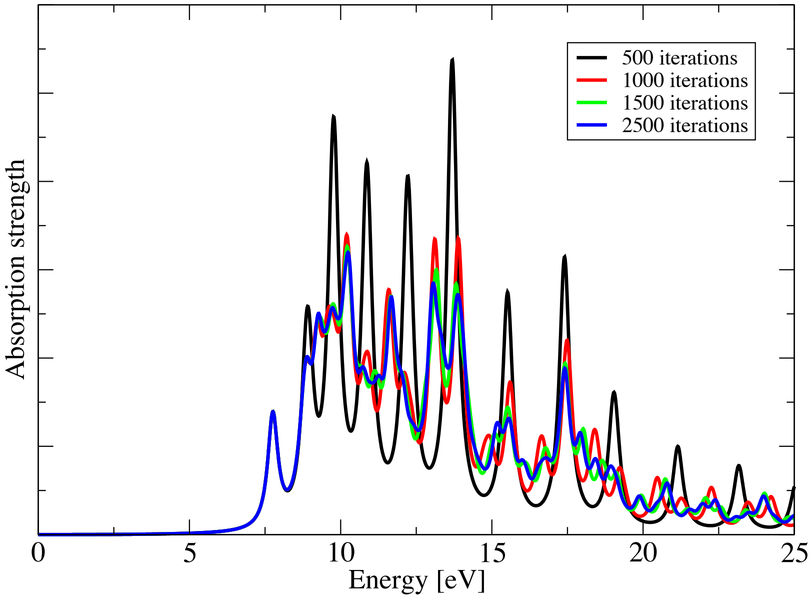
Terminator:



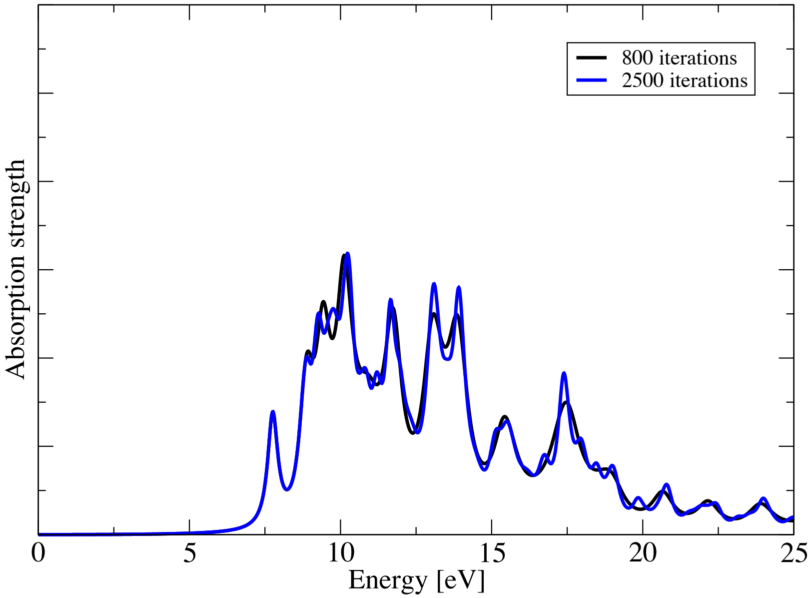


Effect of the terminator:

No terminator:

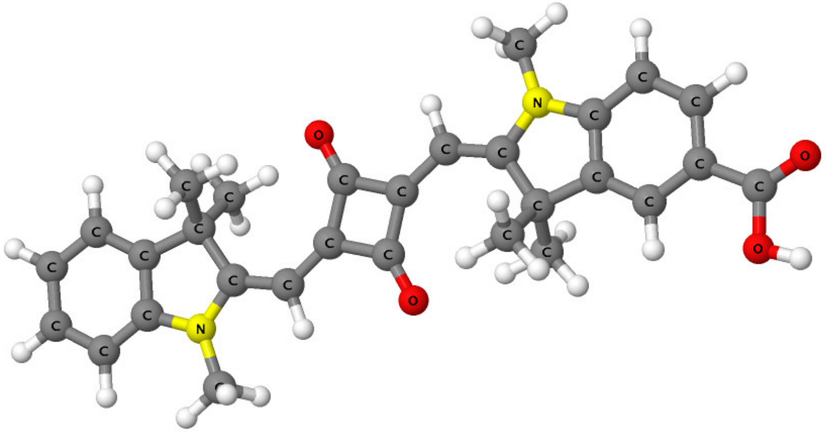


Terminator:



Analyzing the spectrum

Example of a squaraine dye:



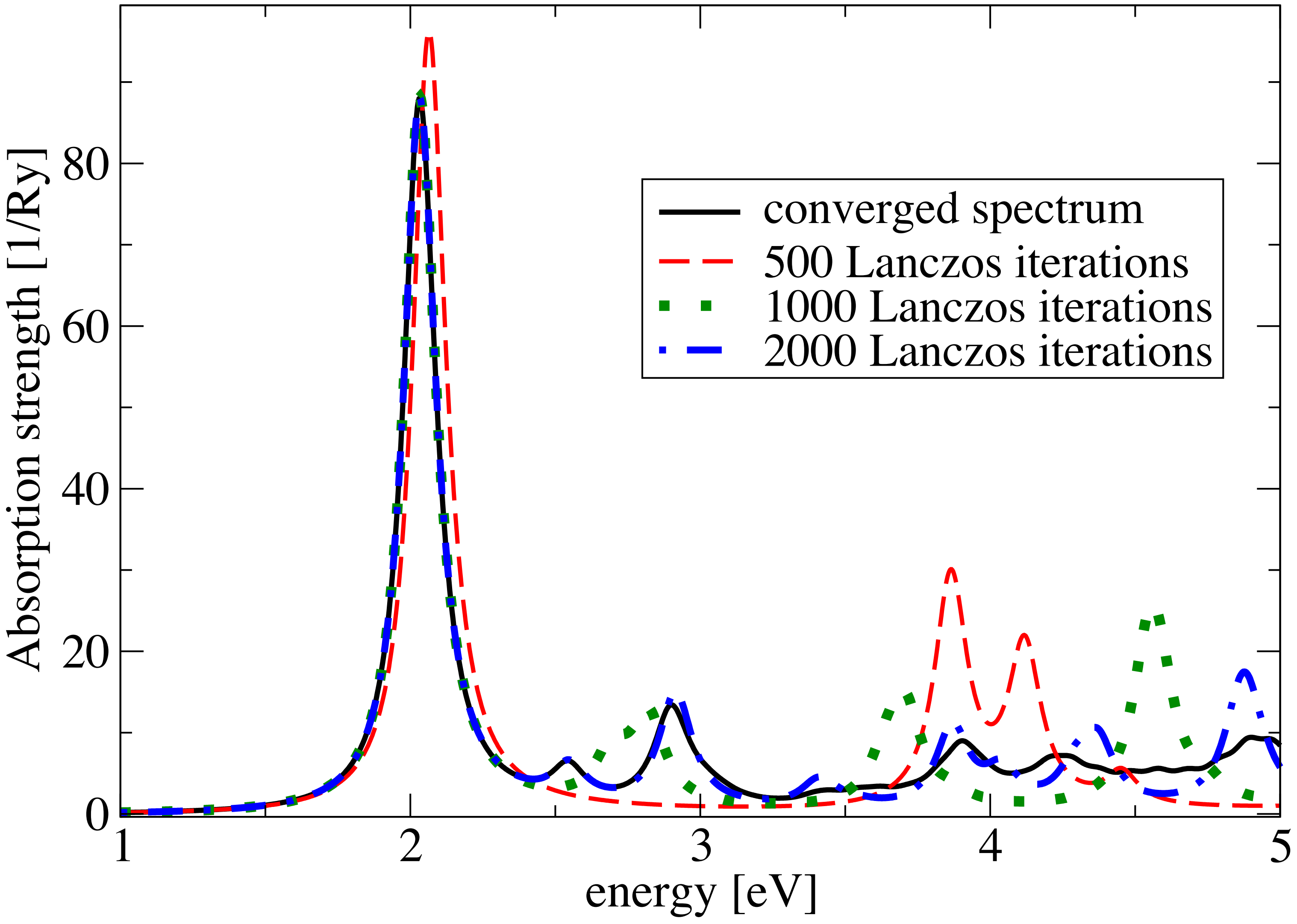
Can we analyze given features of the spectrum in terms of the electronic structure?

YES!

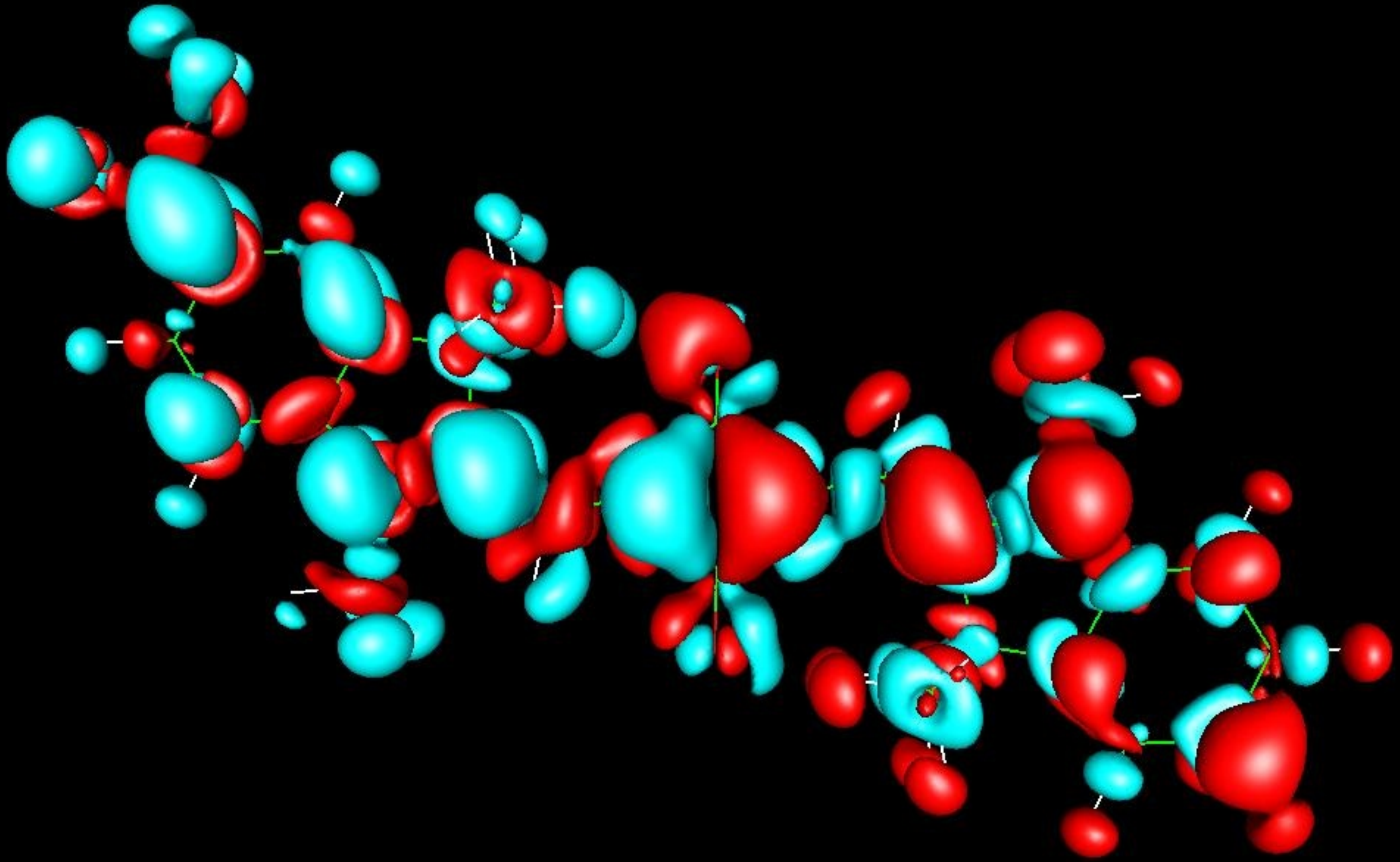
It is possible to compute the response charge density for any given frequency using a second recursion chain.

Convergence of the TDDFPT spectrum

Isolated squaraine molecule



Charge response at main absorption peak:



Conclusions

- TDDFT as a formally exact extension of ground-state DFT for electronic excitations

- Allows to follow the electronic dynamics in real time

- Using TDDFT in linear response allows one to calculate spectra

Thanks to:

- Filippo De Angelis (Perugia)
- Stefano Baroni (SISSA & DEMOCRITOS, Trieste)
- Brent Walker (University College, London)
- Dario Rocca (UC Davis)
- O. Baris Malcioglu (Univ. Liège)
- Arrigo Calzolari (Modena)
- Quantum ESPRESSO and its community