Topological strings and 5d N=1 gauge theories

Hirotaka Hayashi

(Tokai University)

Based on the collaboration with

- Kantaro Ohmori (IAS) [arXiv:1702.07263]

28th February 2017 at Physics and Geometry of F-theory in ICTP

1. Introduction

- The topological vertex is a powerful tool to compute the all genus topological string amplitudes for toric Calabi-Yau threefolds.

Iqbal 02, Aganagic, Klemm, Marino, Vafa 03
Awata, Kanno 05, Iqbal, Kozcaz Vafa 07

- The full topological string partition function has a physical meaning as the Nekrasov partition function through M-theory on toric Calabi-Yau threefolds.

- We can compute a large class of Nekrasov partition functions regardless of whether the theories have a Lagrangian description or not.

- It also became possible to apply the topological vertex to certain non-toric Calabi-Yau threefolds for example by making use of an RG flow induced by a Higgsing.

- Not only SU(N) gauge group but we can deal with USp(2N) gauge group.

HH, Zoccarato 16

- We can consider 5d theories which arise from circle compactifications of 6d SCFTs.
  Ex. M-strings, E-strings, etc

Haghighat, Iqbal, Kozcaz, Lockhart, Vafa 13, Kim, Taki, Yagi 15

- However there are still many interesting 5d theories to which we had not known how to apply the topological vertex.

Ex.

(1) 5d pure SO(2N) gauge theory

(2) 5d pure E_6, E_7, E_8 gauge theories

↑ ADHM construction is not known (Nevertheless, some results are known)

Benvenuti, Hanany, Mekareeya 10, Keller, Mekareeya, Song, Tachikawa 11, Gaiotto, Razamat 12, Keller, Song 12, Hananay, Mekareeya, Razamat 12, Cremonesi, Hanany, Mekareeya, Zaffaroni 14, Zafrir 15

- In this talk, we will present a powerful prescription of using the topological vertex to compute the partition functions of 5d pure SO(2N), E_6, E_7, E_8 gauge theories by utilizing their dual descriptions.

- In fact, the technique can be also applied to 5d theories which arise from a circle compactification of 6d “pure” SU(3), SO(8), E_6, E_7, E_8 gauge theories with one tensor multiplet.

1. Introduction
2. A dual description of 5d DE gauge theories
3. Trivalent gluing prescription
4. Applications to 5d theories from 6d
5. Conclusion

2. A dual description of 5d DE gauge theories

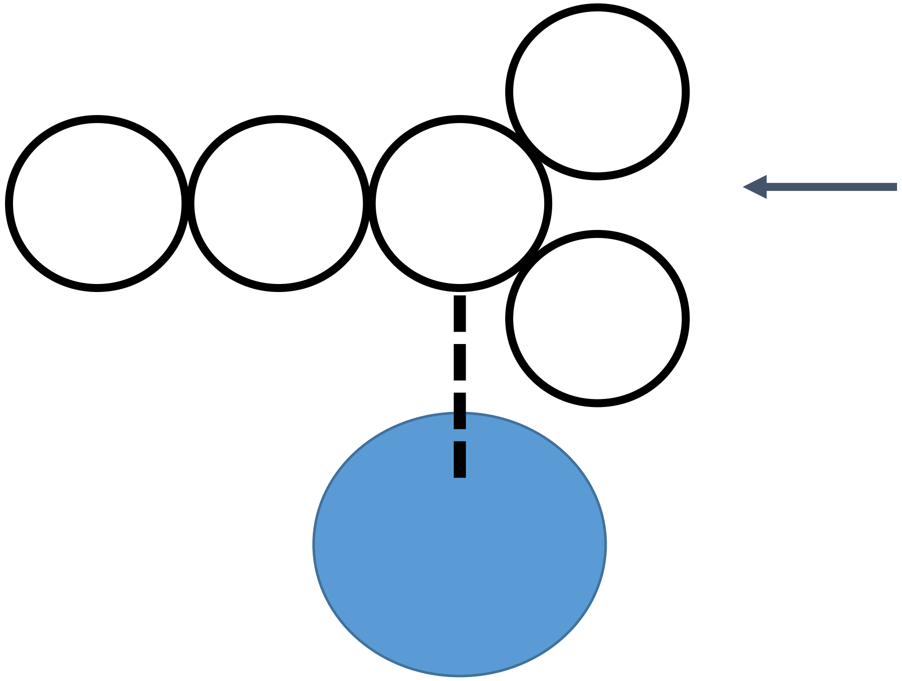
- Five-dimensional gauge theories can be realized by M-theory on Calabi-Yau threefolds or on 5-brane webs in type IIB string theory.

Witten 96, Morrison Seiberg 96,
Douglas, Katz, Vafa 96
Aharony, Hanany 97,
Aharony, Hanany, Kol 97

- Since we consider D, E gauge groups, we first start from M-theory configurations.

- Basically, ADE gauge groups are obtained from ADE singularities over a curve in a Calabi-Yau threefold

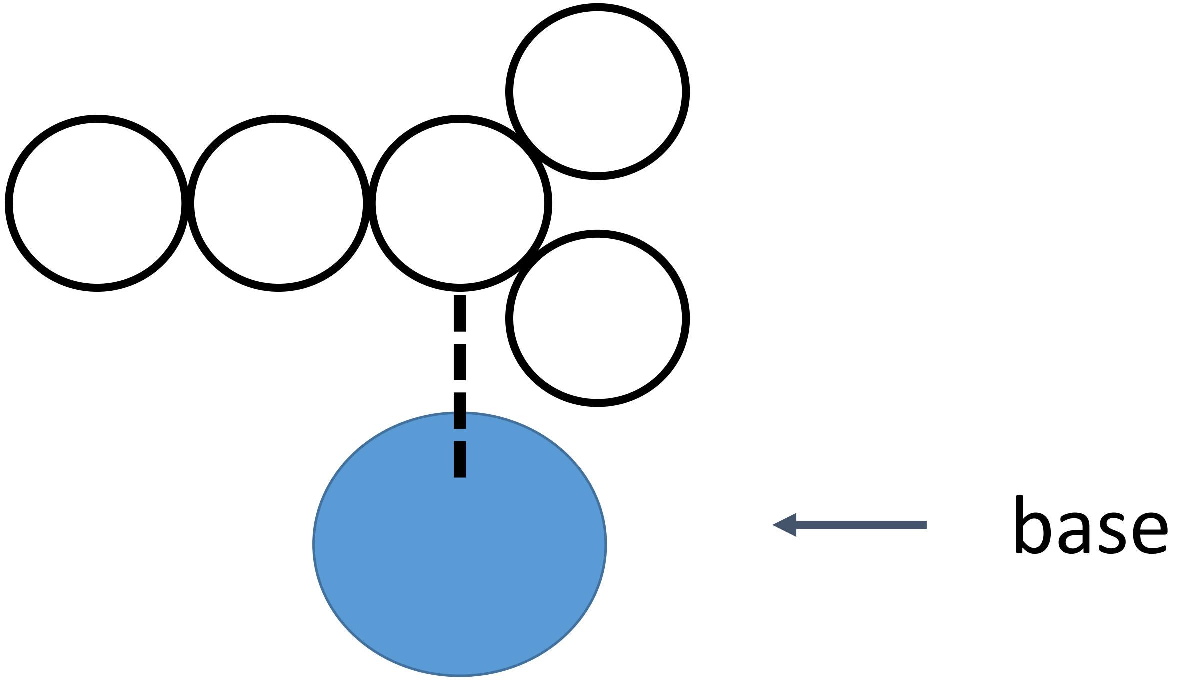
- Ex. 5d pure SO(2N+4) gauge theory
  → D_{N+2} singularities over a sphere



Dynkin diagram of SO(10)

- We can take a different way to see the same geometry for a dual description.
  "fiber-base duality"

Katz, Mayr, Vafa 97
Aharony, Hanany, Kol 97
Bao, Pomoni, Taki, Yagi 11



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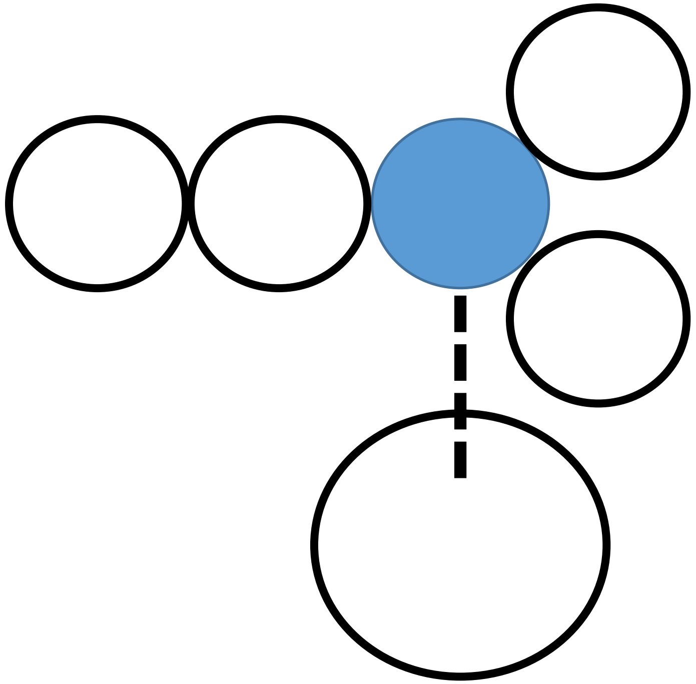
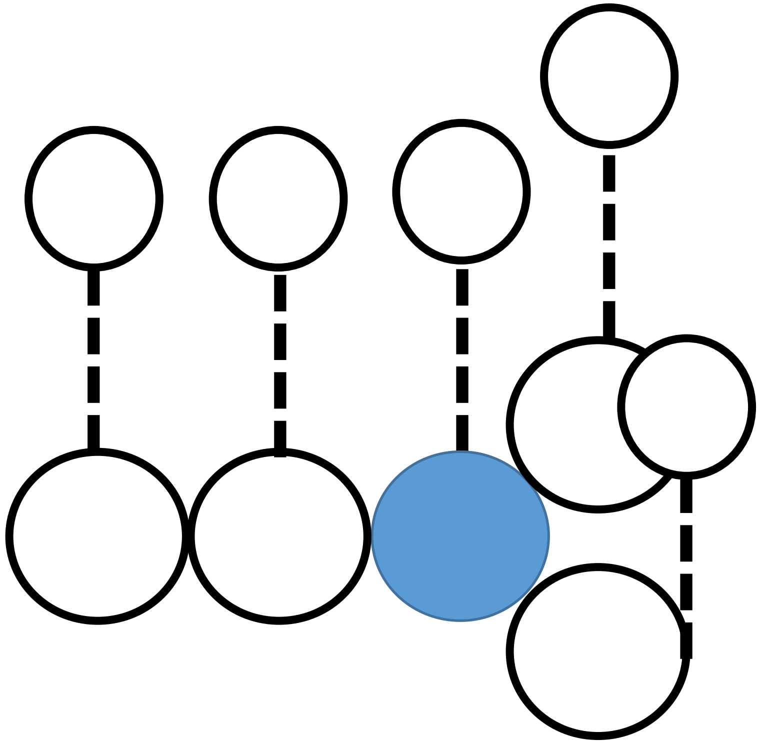


← base

⬇

shrink other spheres

- A schematic picture



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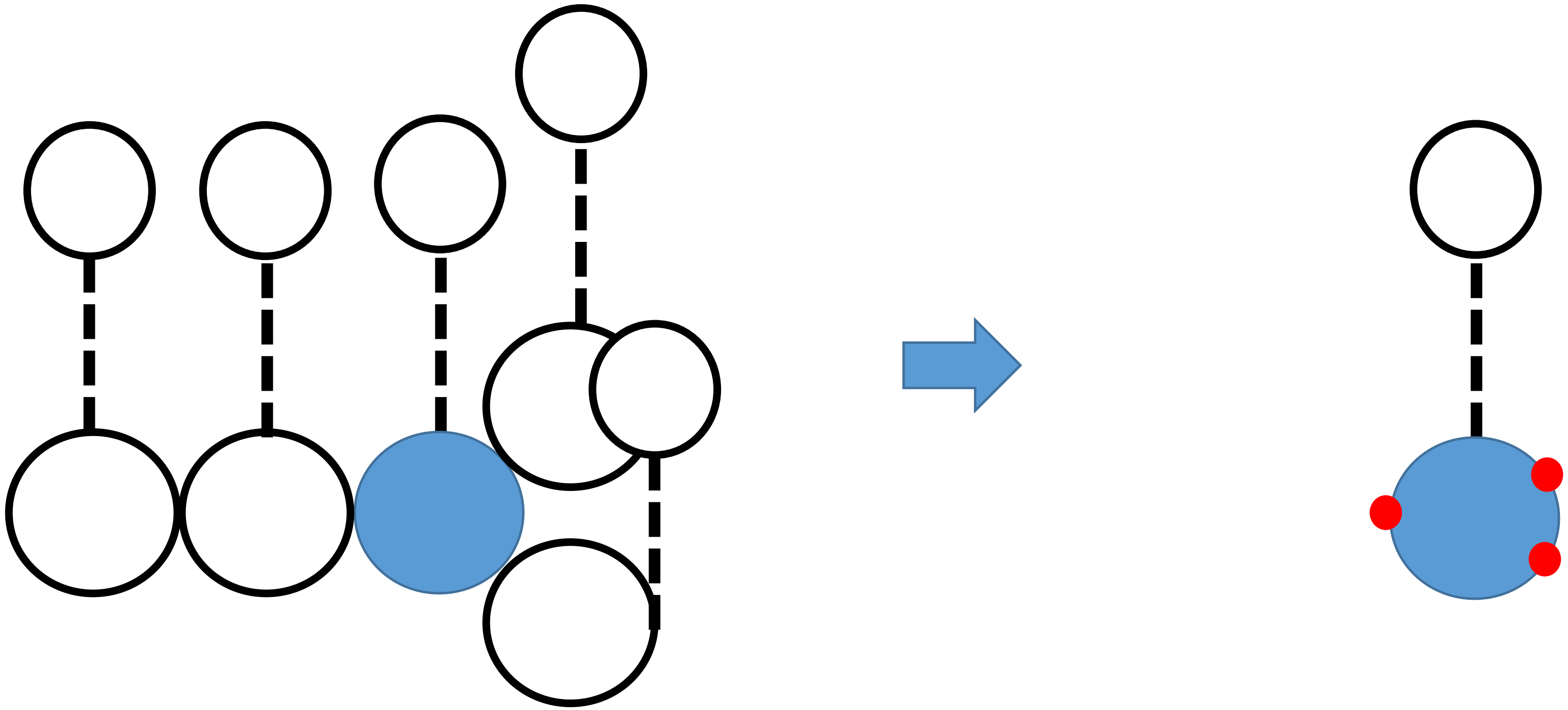

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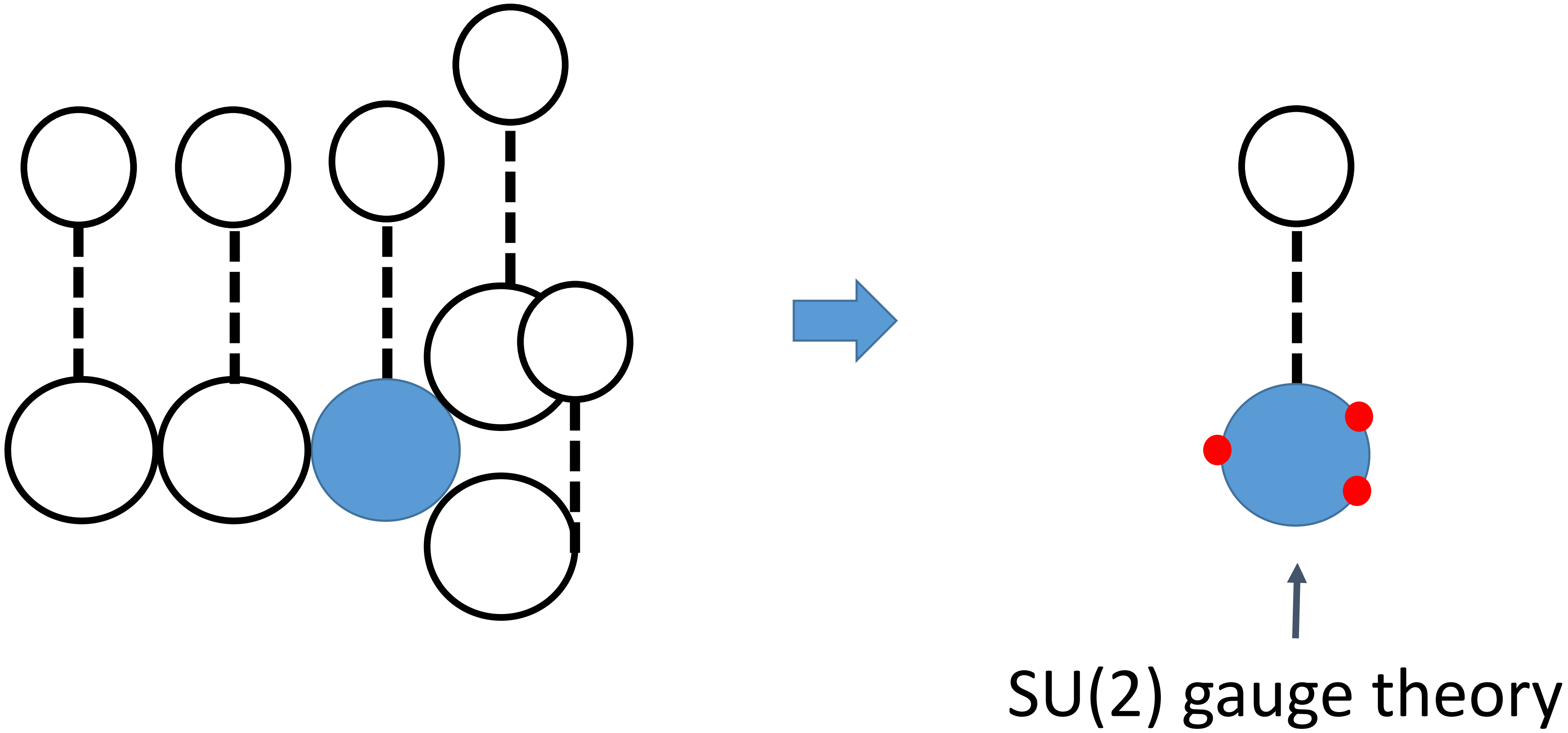

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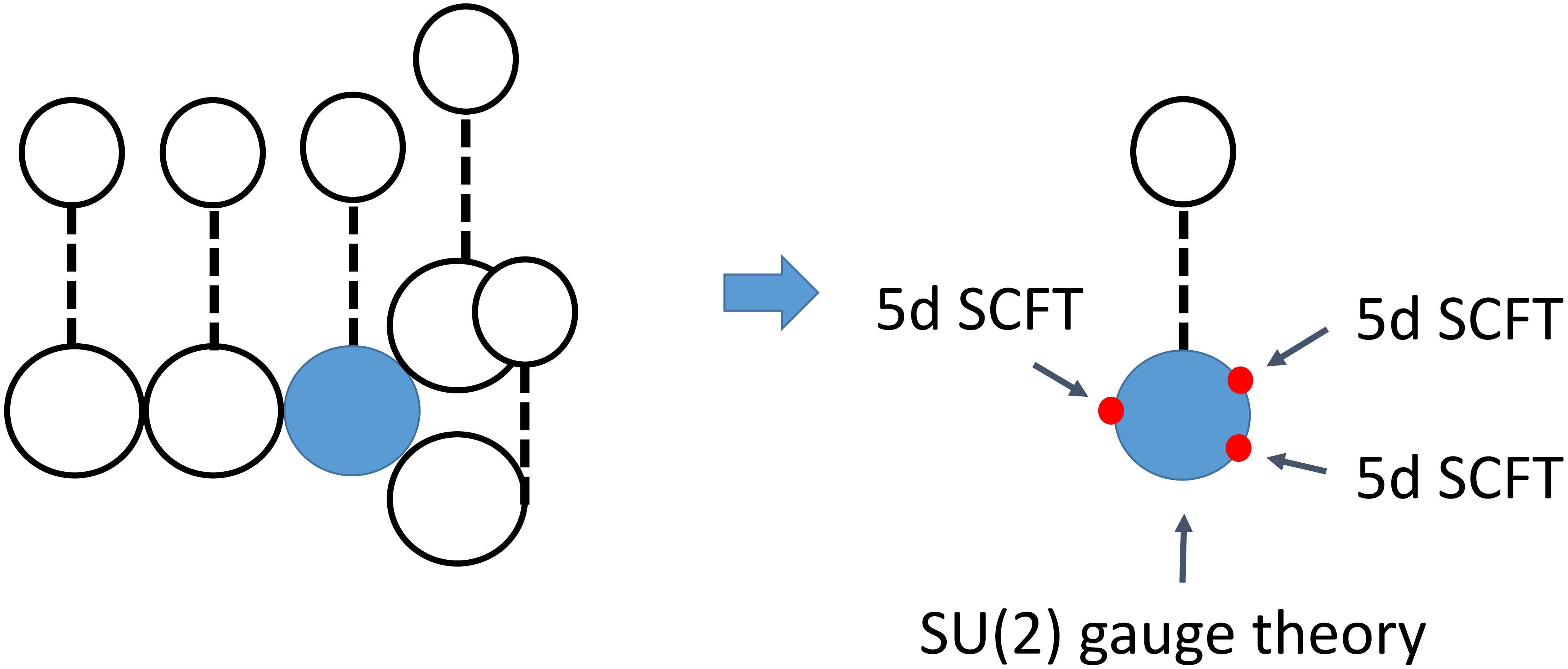
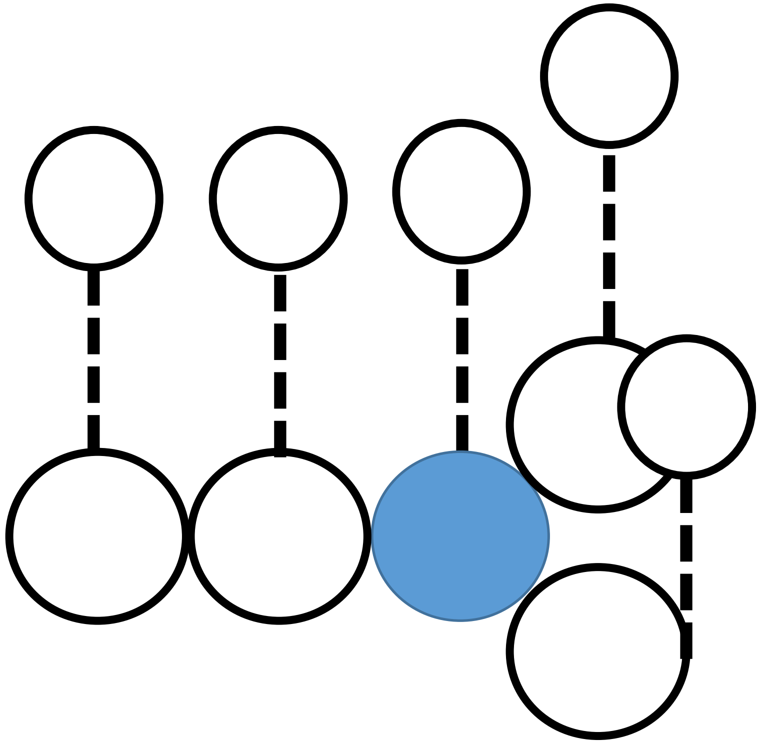


- The 5d SCFTs may be thought of as “matter” for the SU(2) gauge theory.
- Due to the SU(2) gauge symmetry, each of the 5d SCFTs should have an SU(2) flavor symmetry.
- What are the matter SCFTs?

- Going back to the schematic picture



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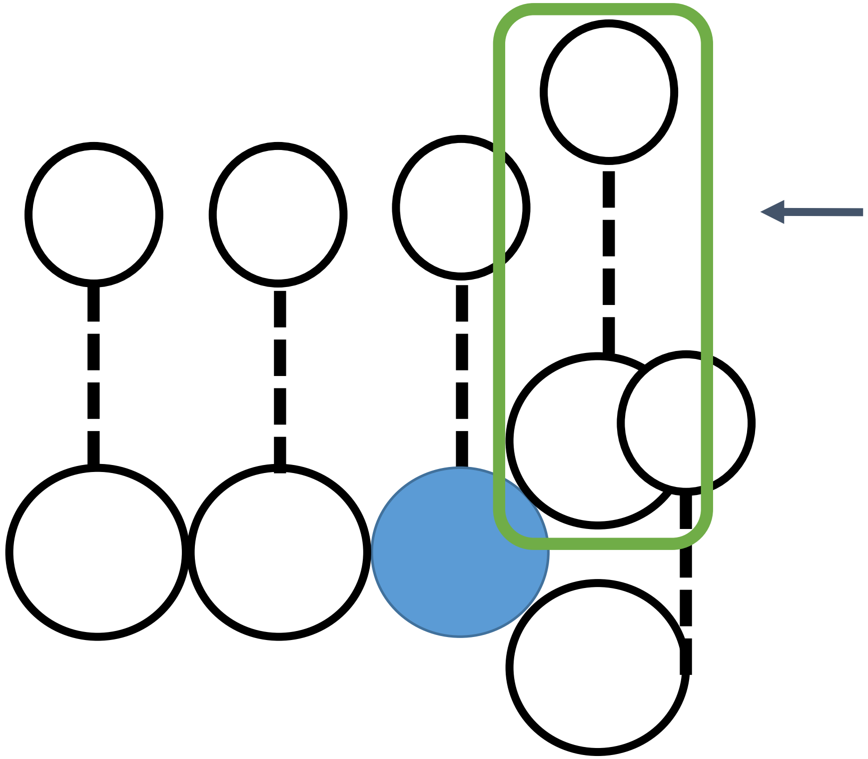

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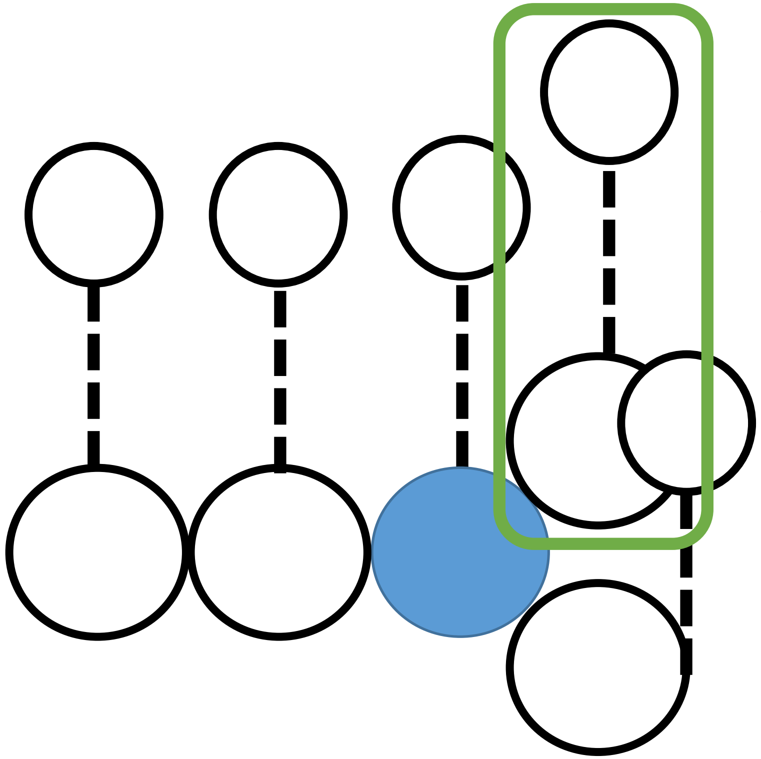

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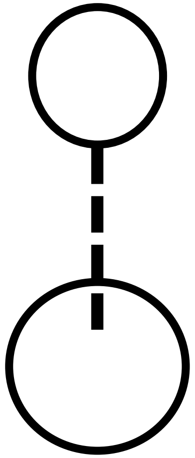

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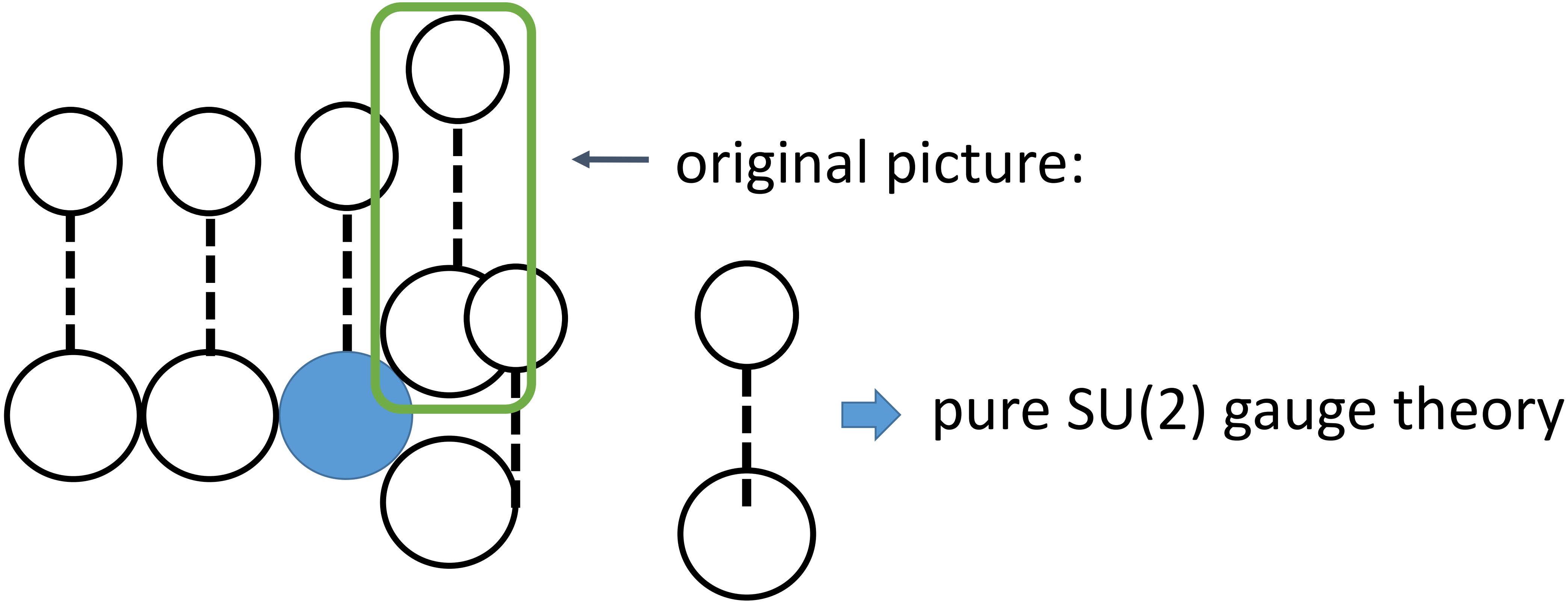


- In fact, there are two pure SU(2) gauge theories depending on the discrete theta angle θ.

Seiberg 96
Morrison Seiberg 96,
Douglas, Katz, Vafa 96

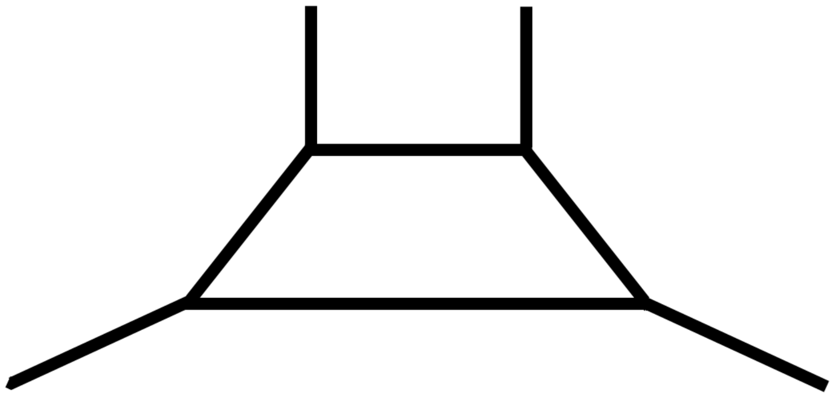
- The UV completion of the two theories are 5d SCFTs but their flavor symmetries are different.

(i). $\theta = 0 \rightarrow$ SU(2) flavor symmetry (E_1 theory)
(ii). $\theta = \pi \rightarrow$ U(1) flavor symmetry ($\tilde{E}_1$ theory)

- Therefore, the 5d SCFT should be the E_1 theory.

- It is illustrative to see it from 5-brane webs.

- A 5-brane web for E_1 theory.



pure SU(2)

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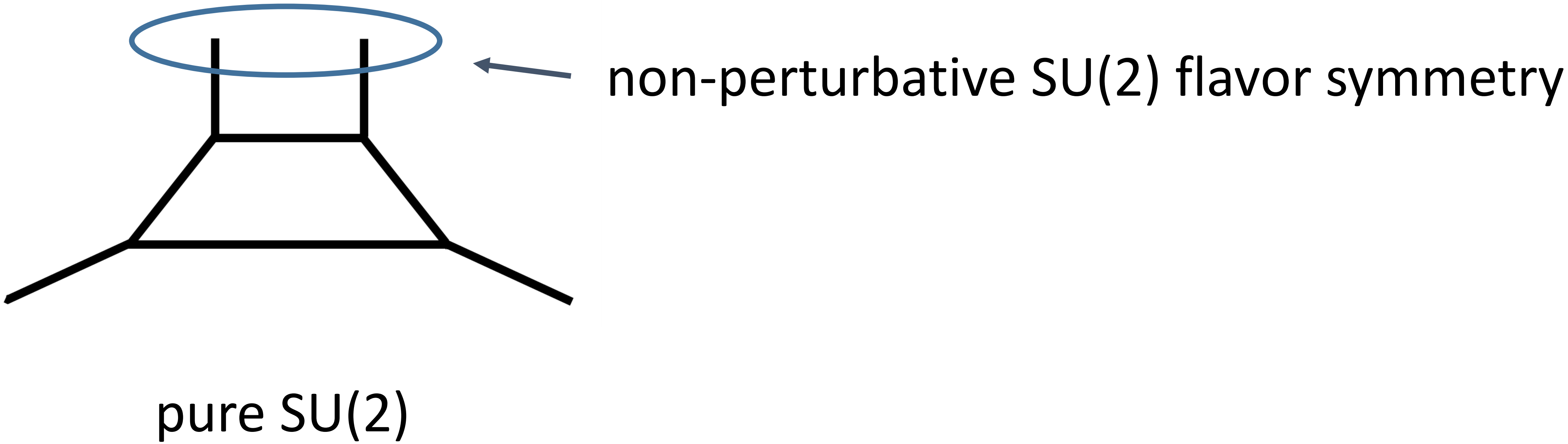

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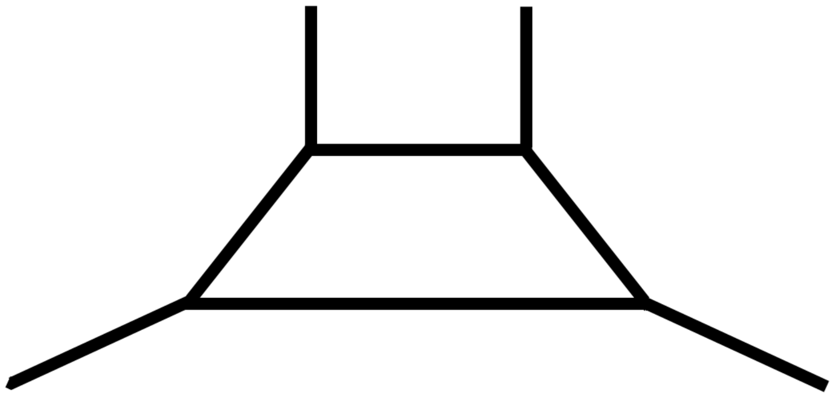
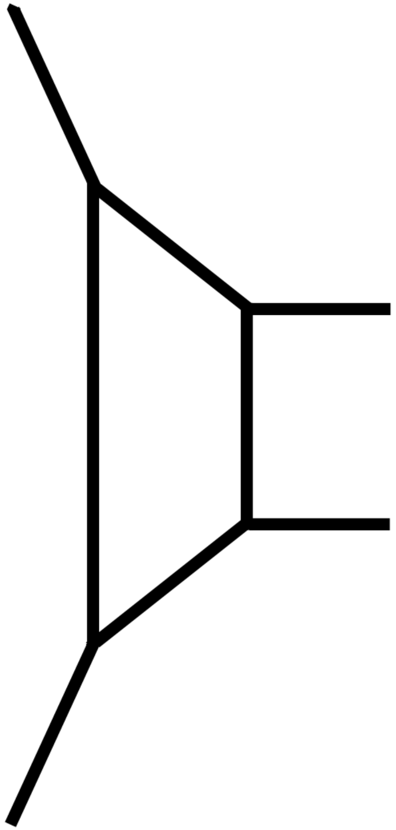


pure SU(2)

S-dual



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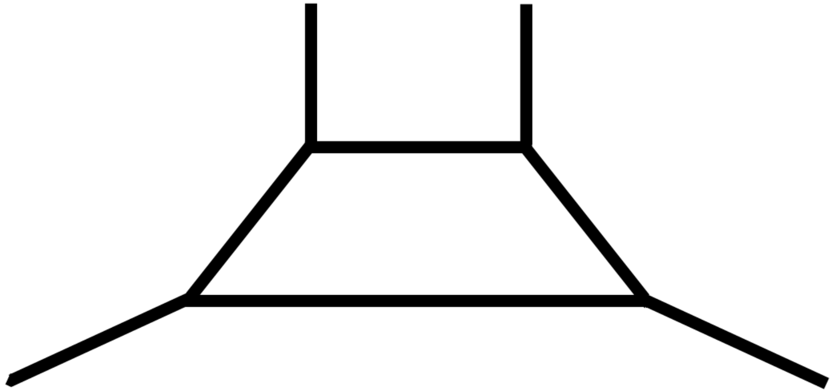

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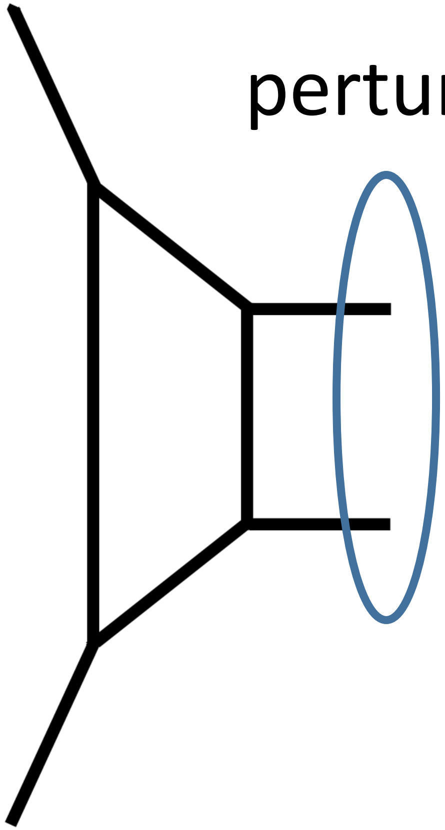

perturbative SU(2)

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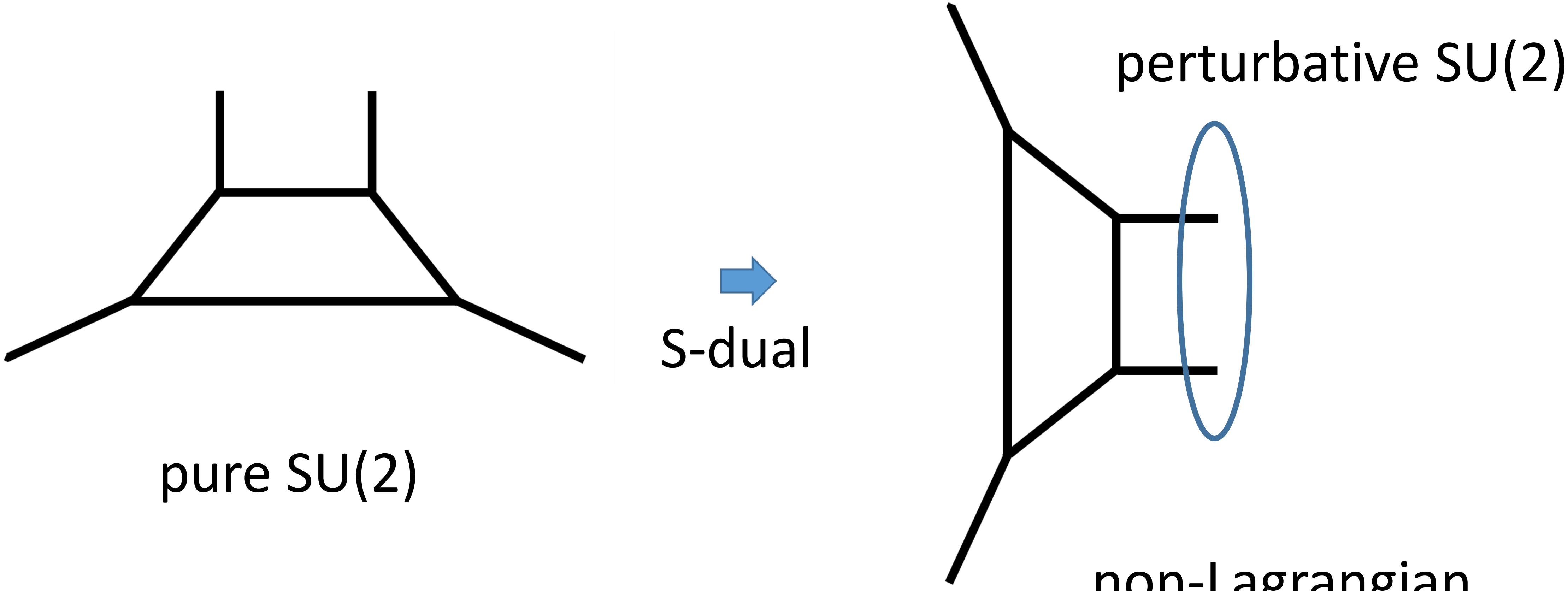

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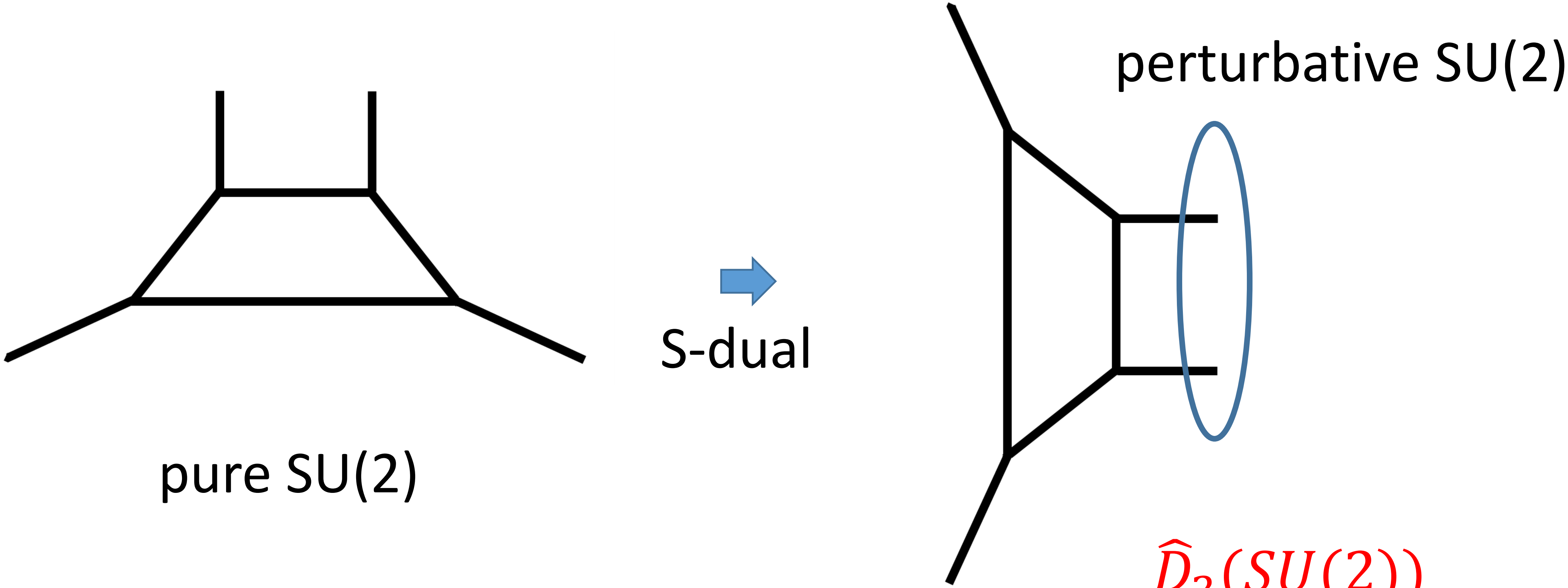
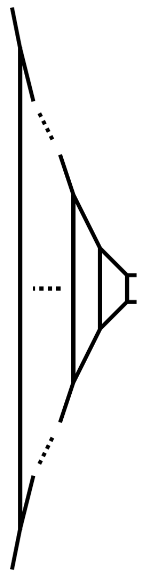
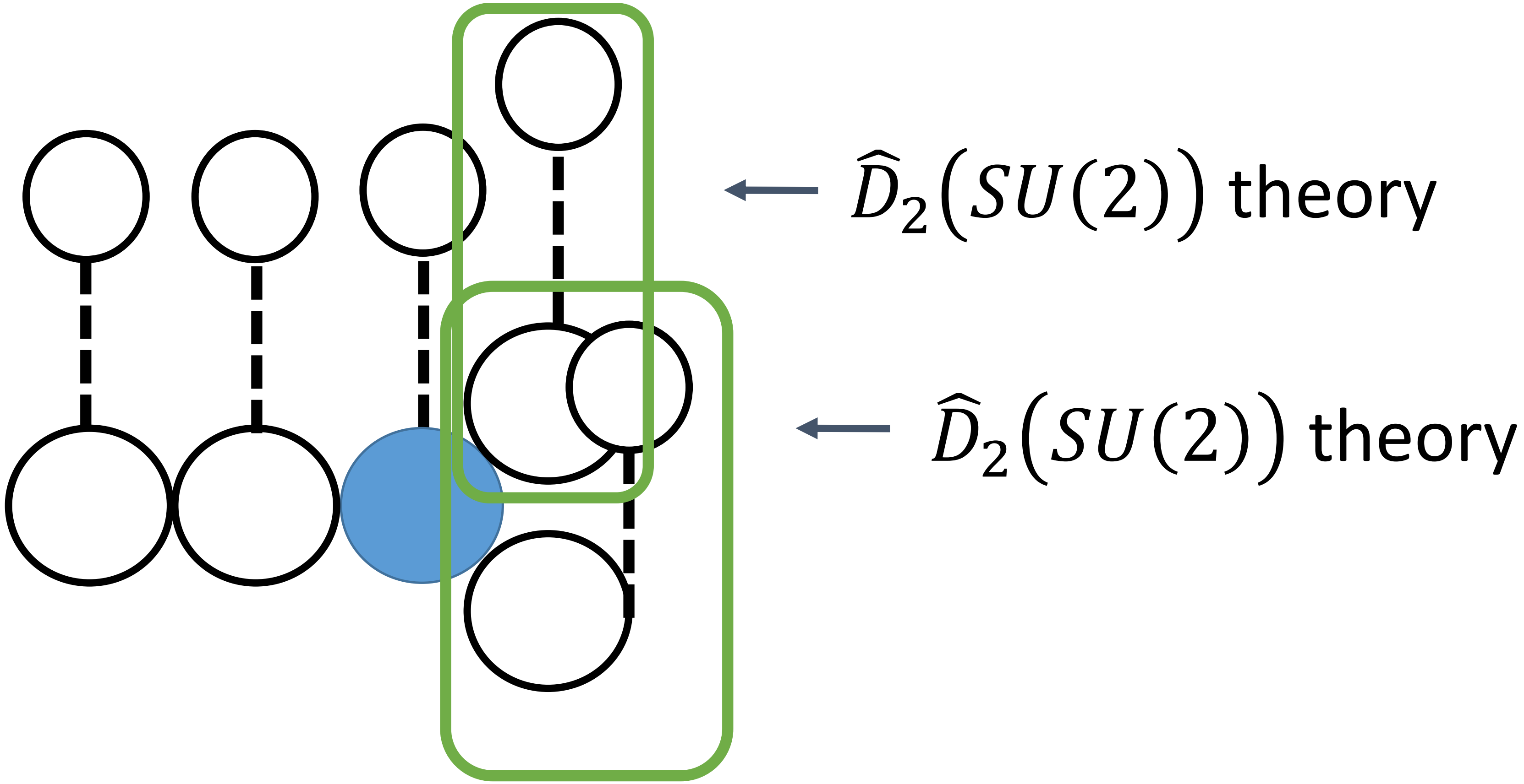


- $\widehat{D}_N\big(SU(2)\big)$ is a 5d SCFT with (N-1)-dimensional Coulomb branch moduli space and has an SU(2) flavor symmetry.

Del Zotto, Vafa, Xie 15

- When the SU(2) flavor symmetry is perturbative the theory is S-dual to a pure SU(N) gauge theory with the CS level N or –N.




$\widehat{D}_2(SU(2))$ theory
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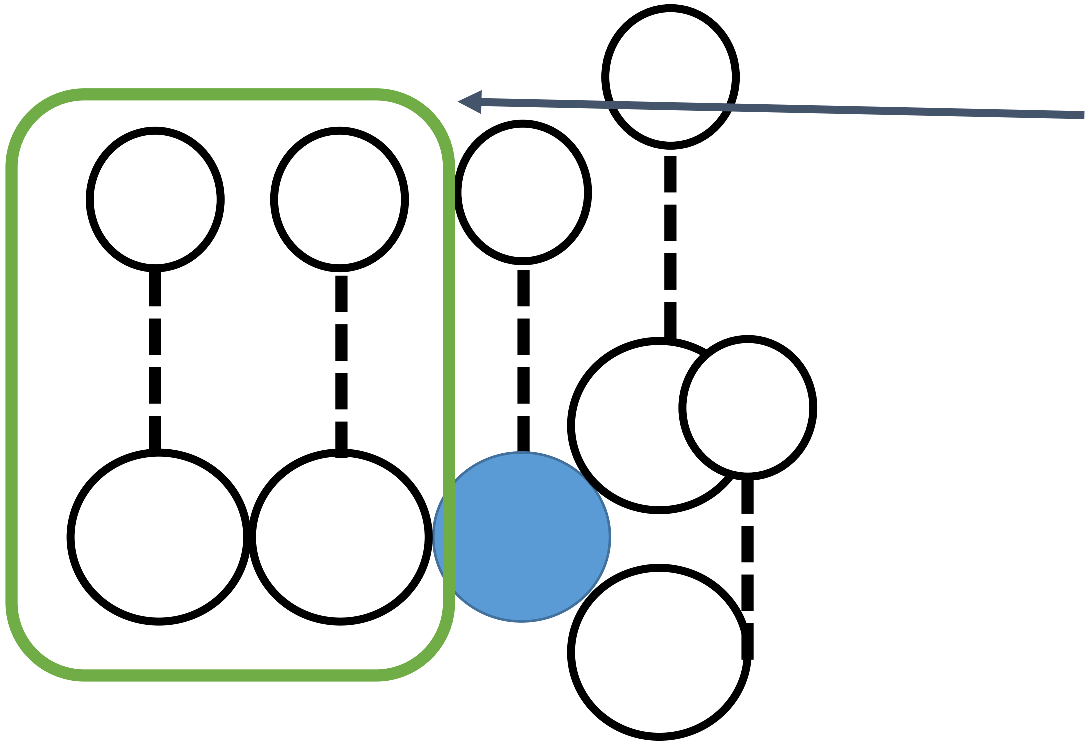

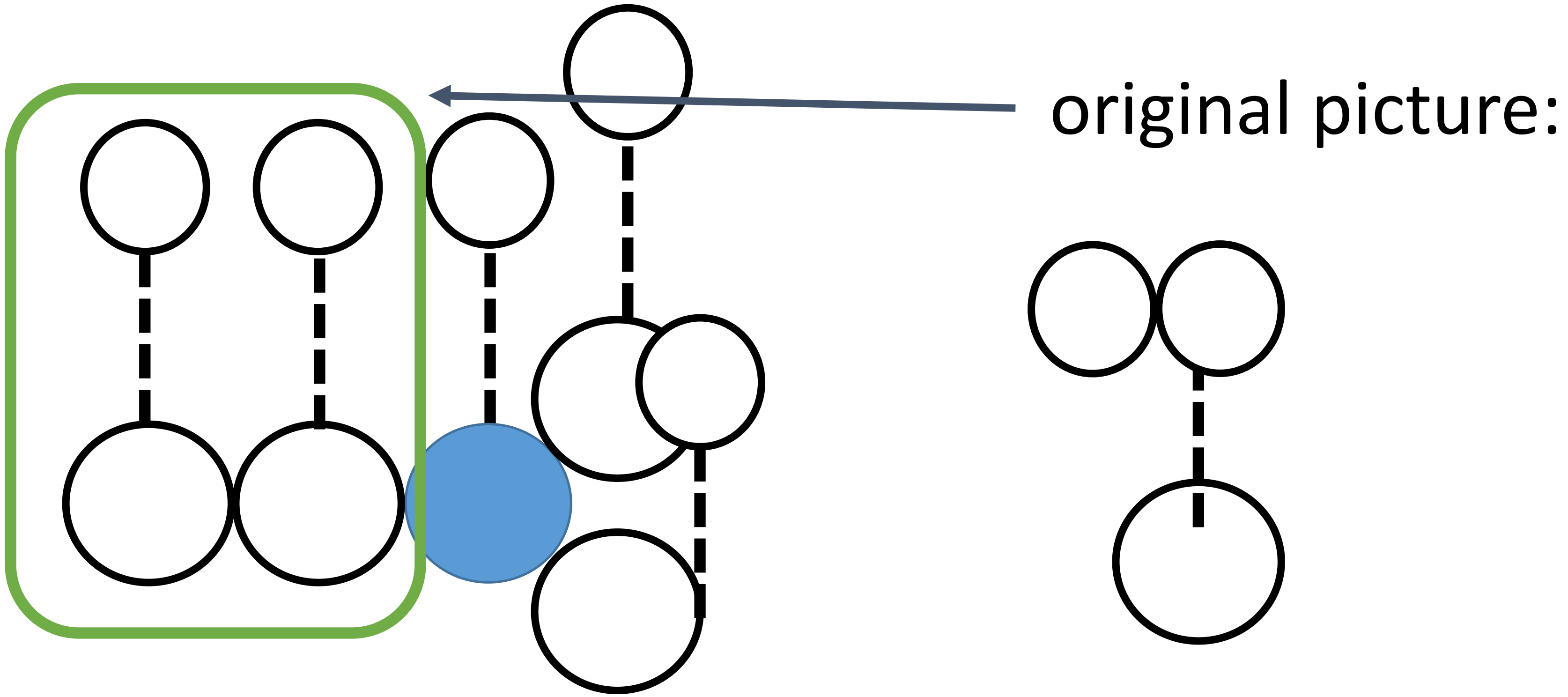
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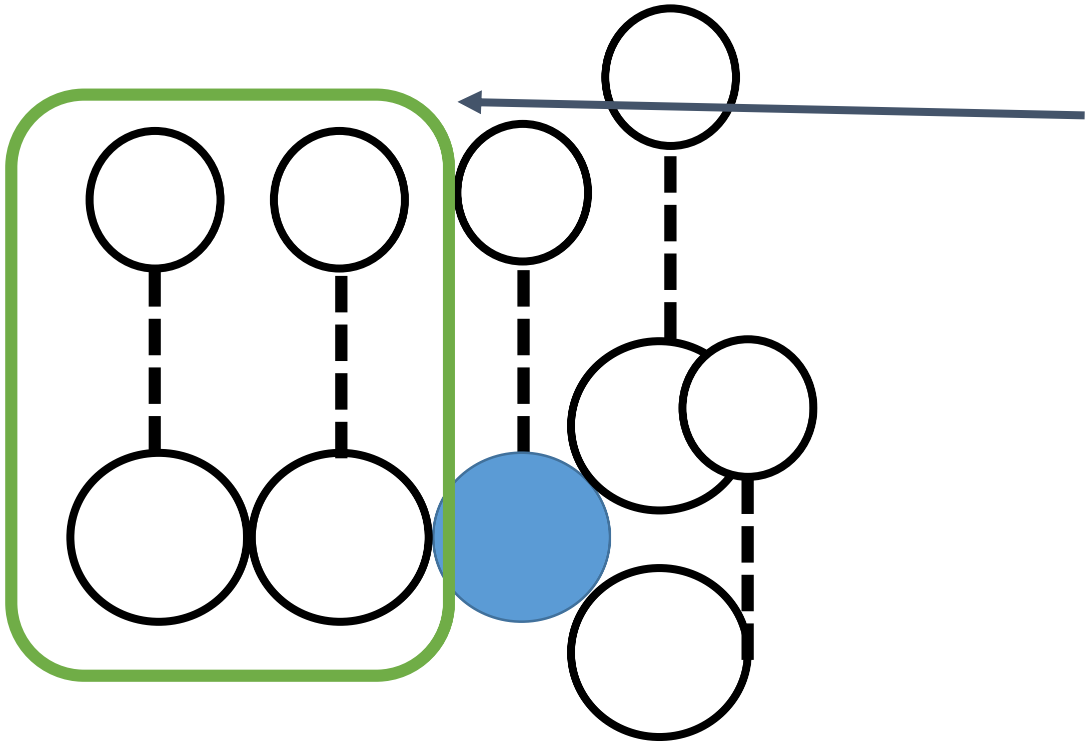

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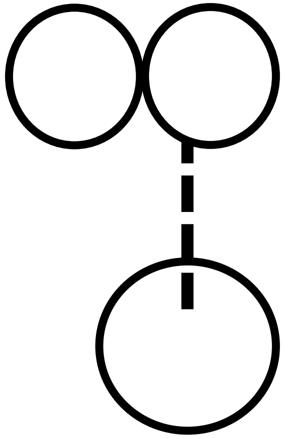
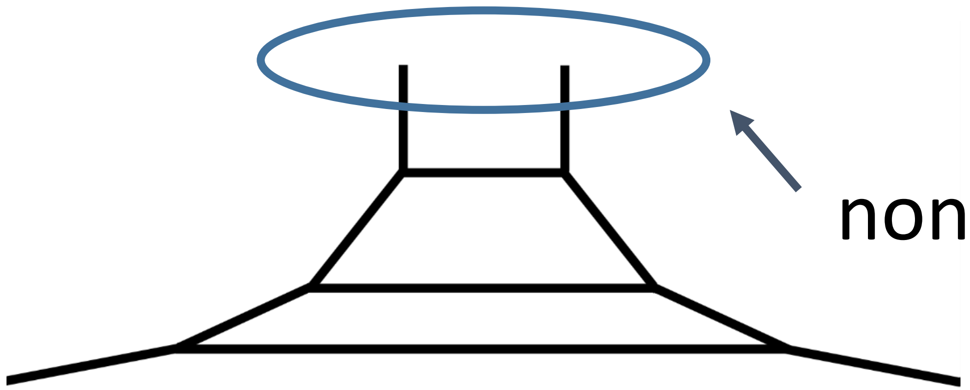


➡ pure SU(3) gauge theory

- The pure SU(3) gauge theory should have an SU(2) flavor symmetry hence the Chern-Simons level should be 3 or –3.

- A 5-brane web picture:



non-perturbative SU(2) flavor symmetry

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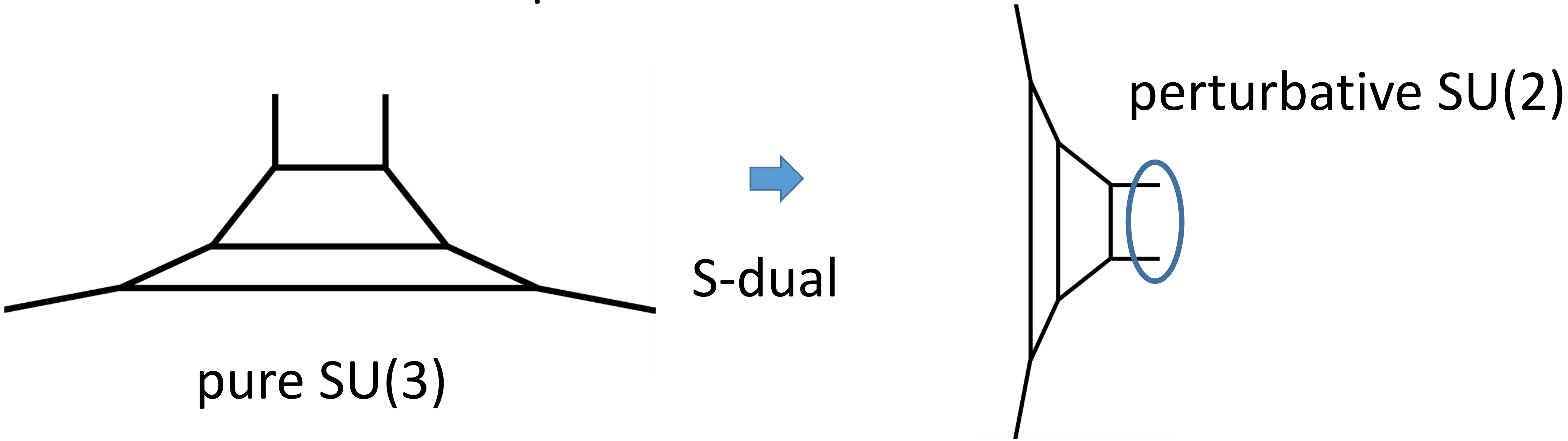

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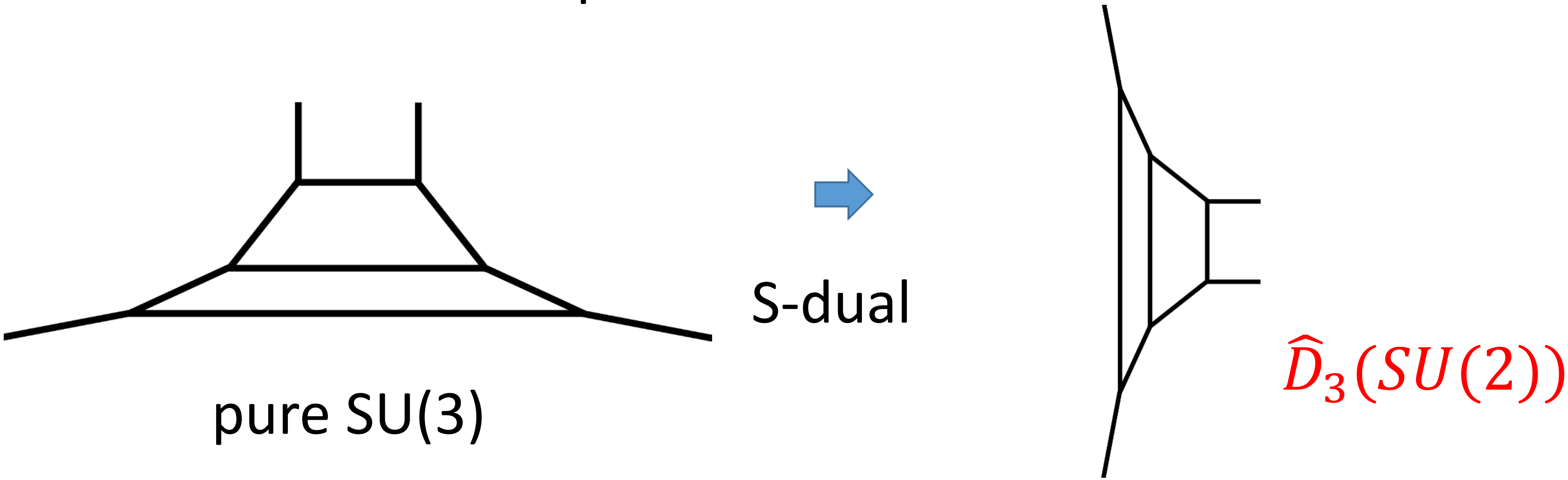
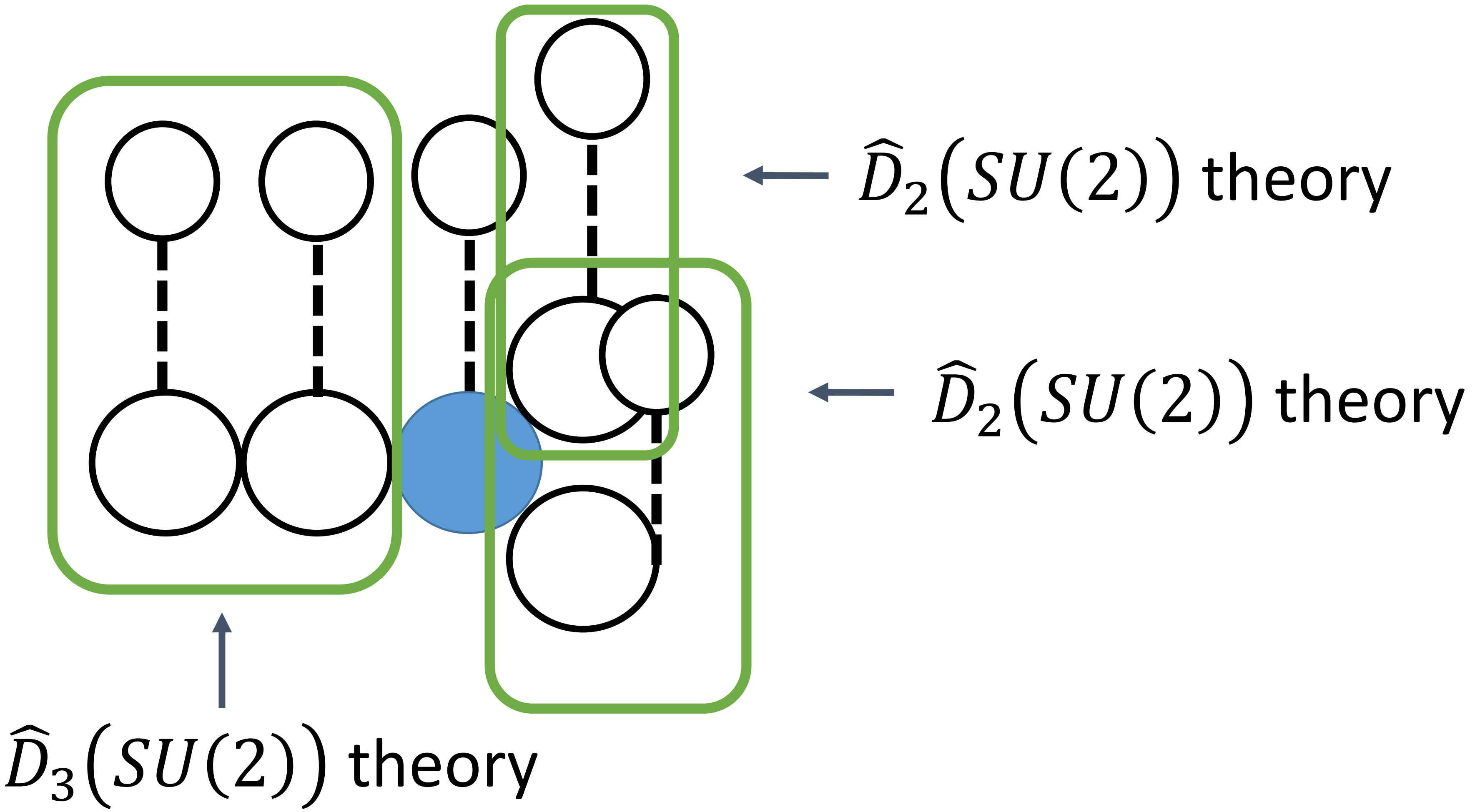
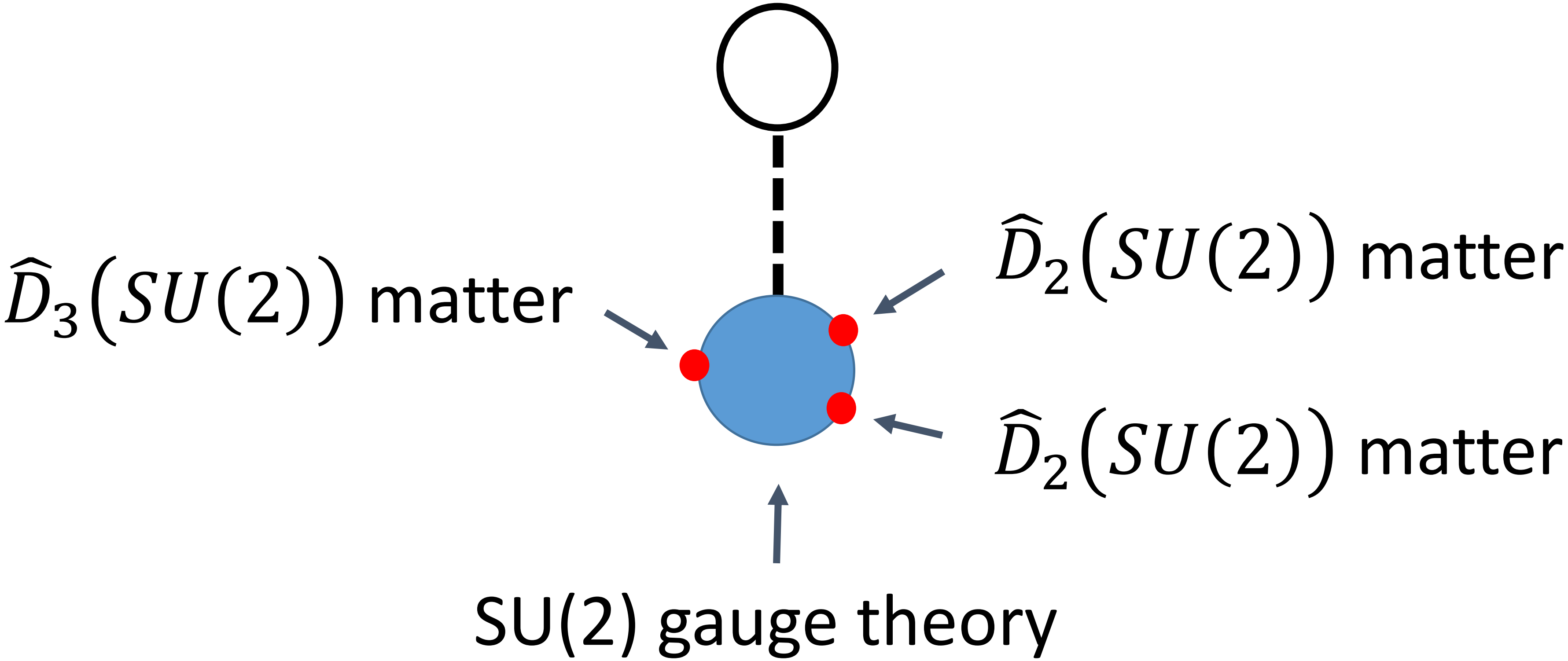


- The geometric picture

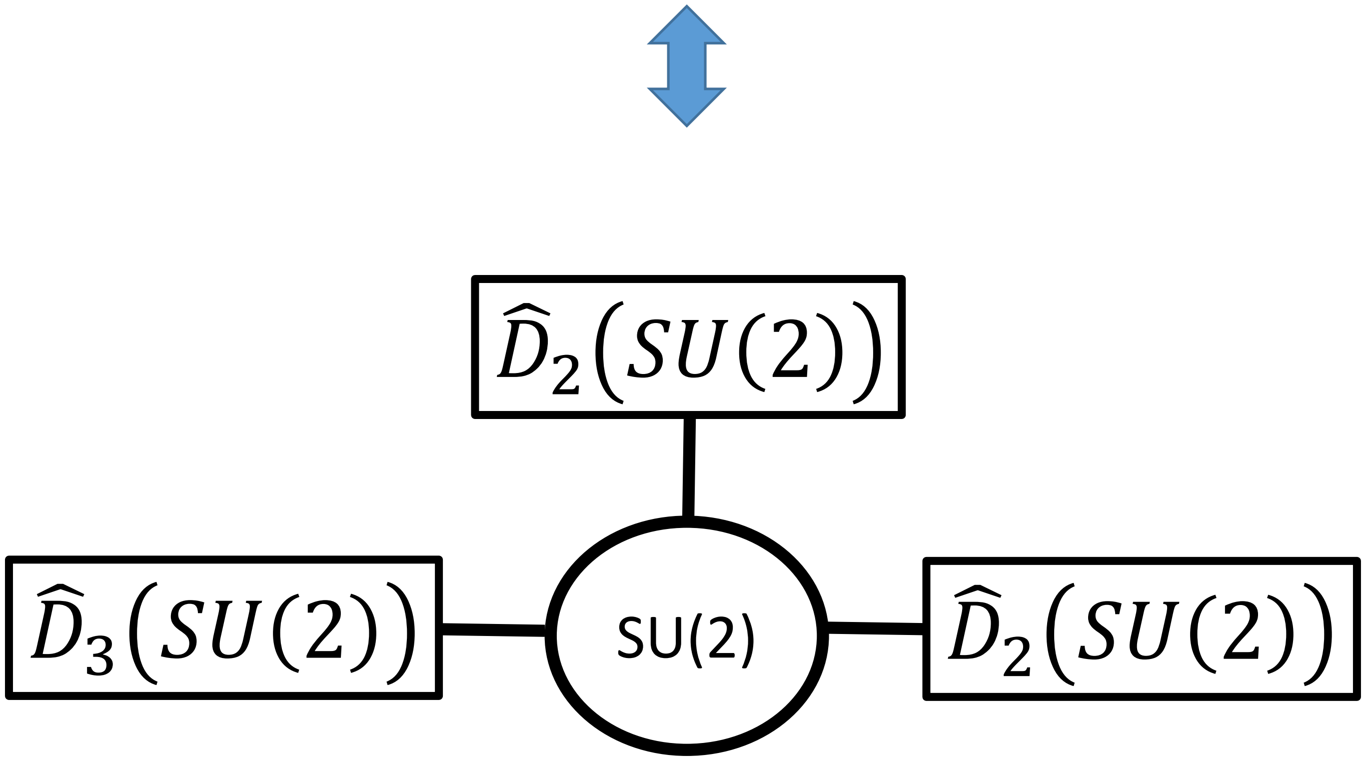


- The shrinking limit leads to:



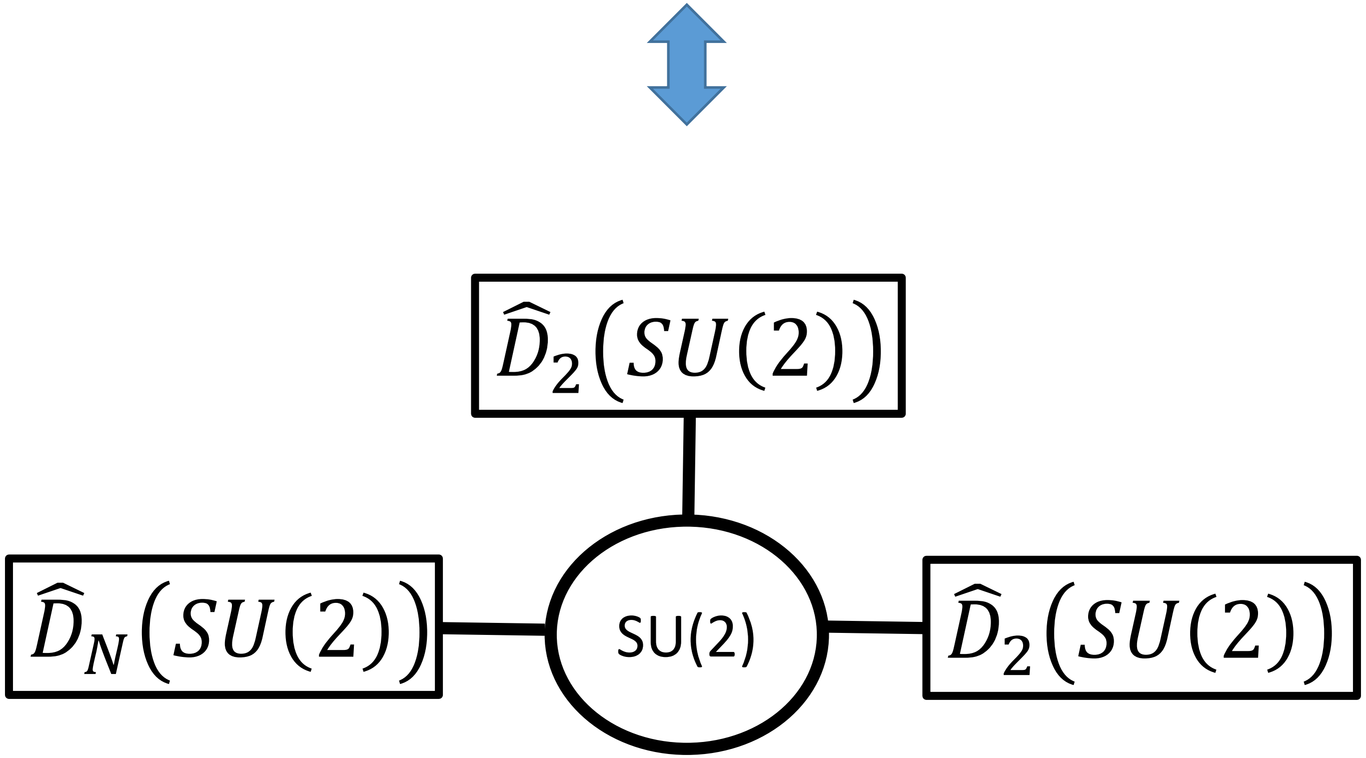
- A duality

pure SO(10) gauge theory



- In general

pure SO(2N+4) gauge theory



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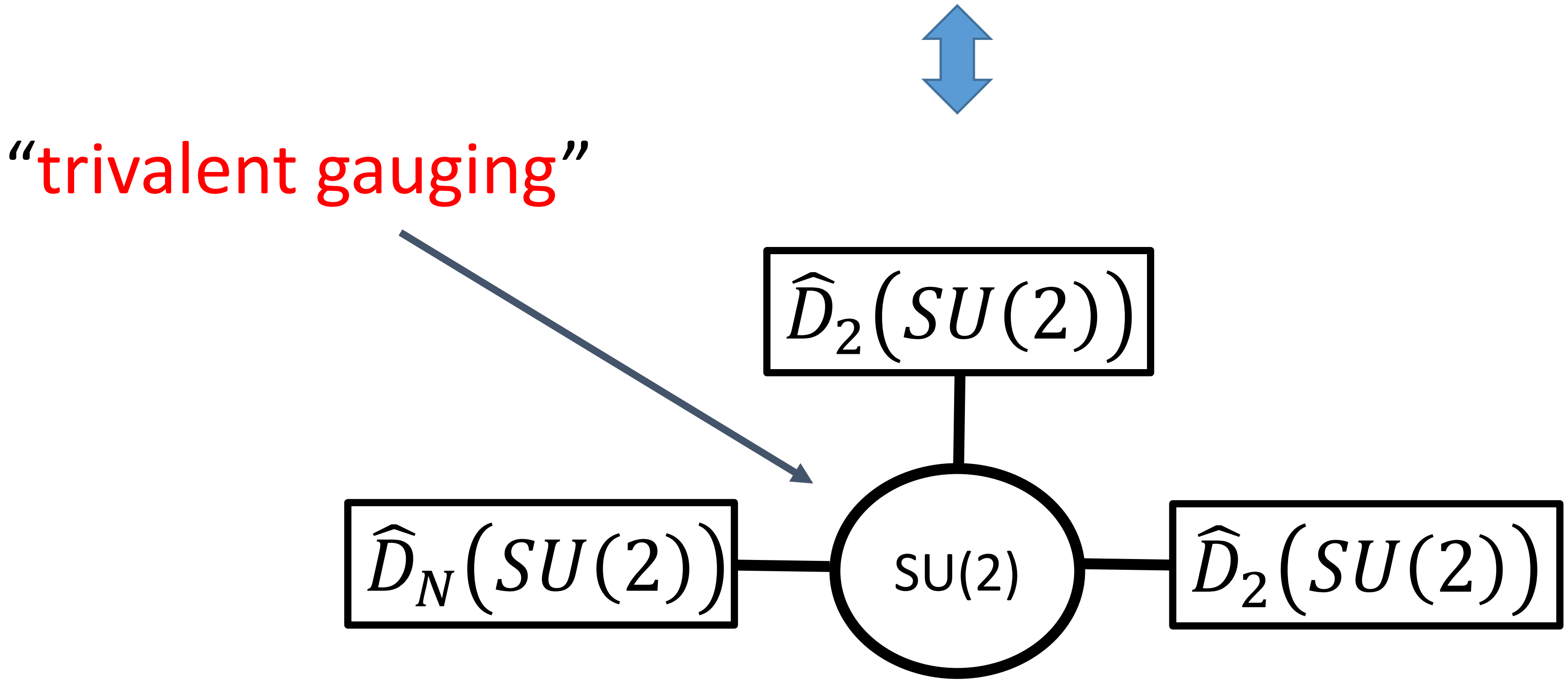
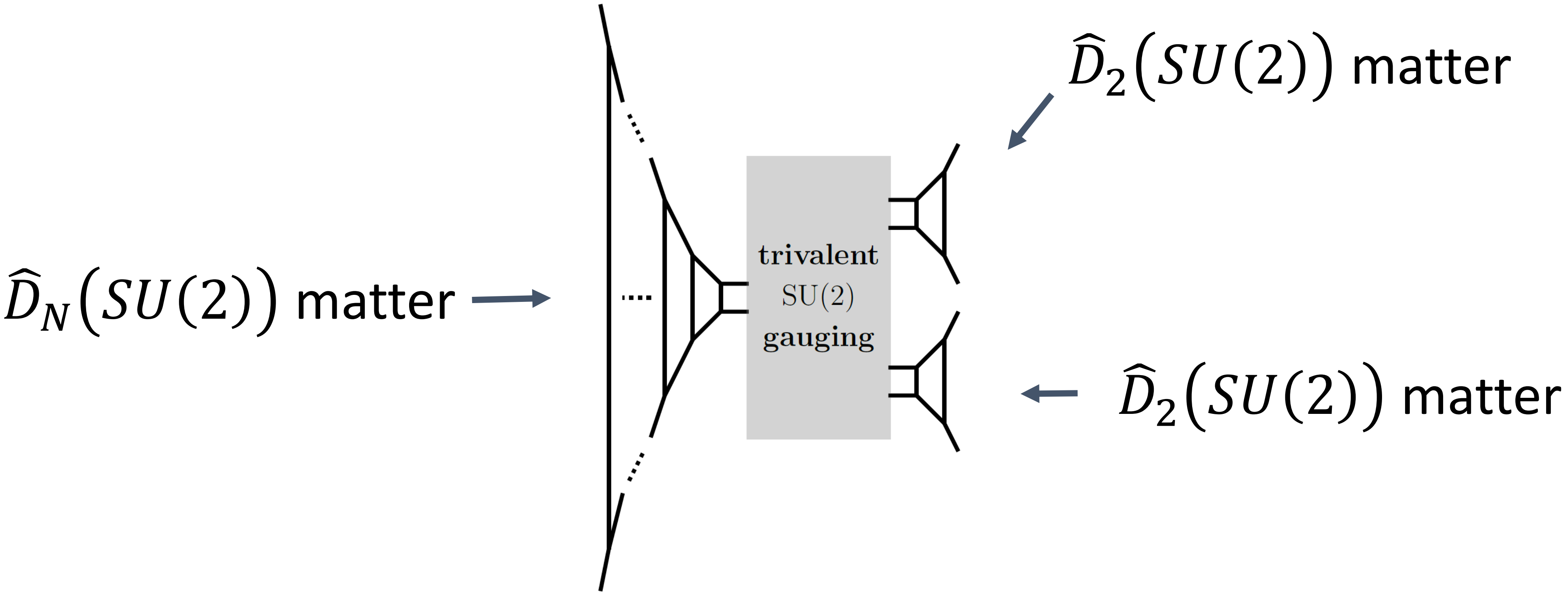


- A web-like description



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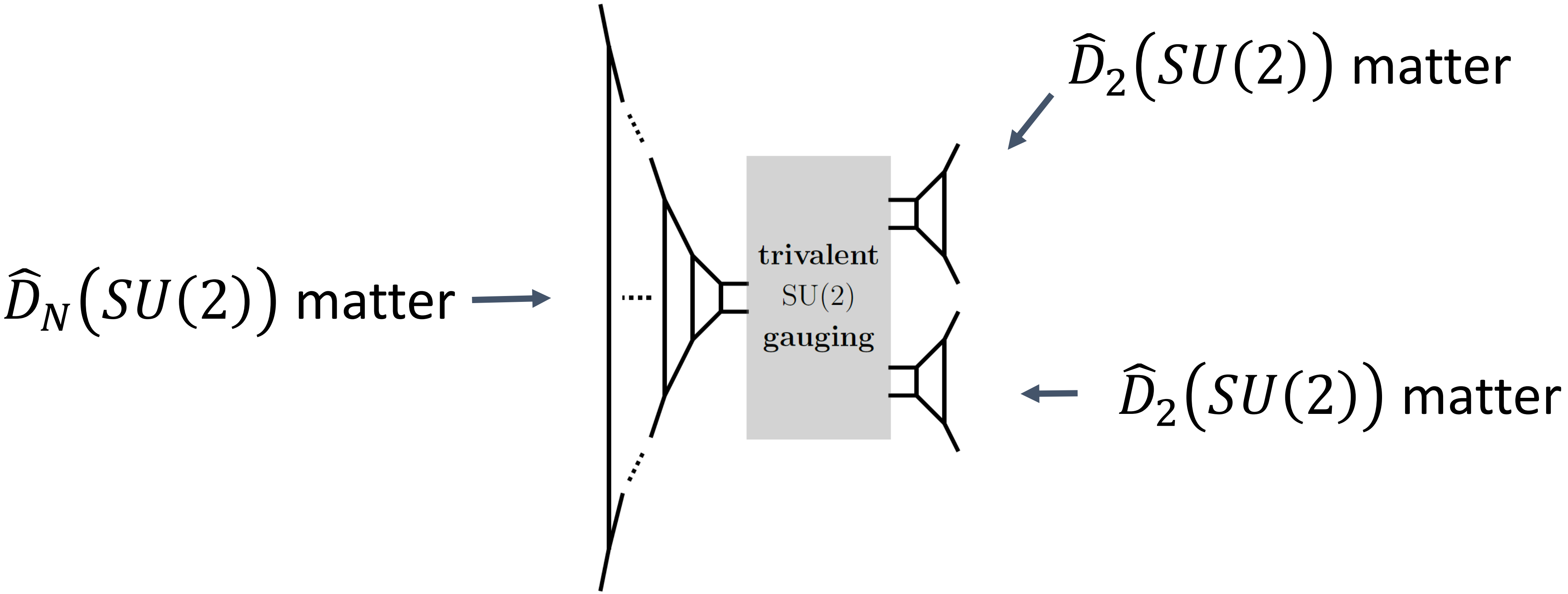

- We will make use of this picture for the later computations by topological strings.

- In fact, this realization of a duality can be easily extended to pure E_6, E_7, E_8 gauge theories.

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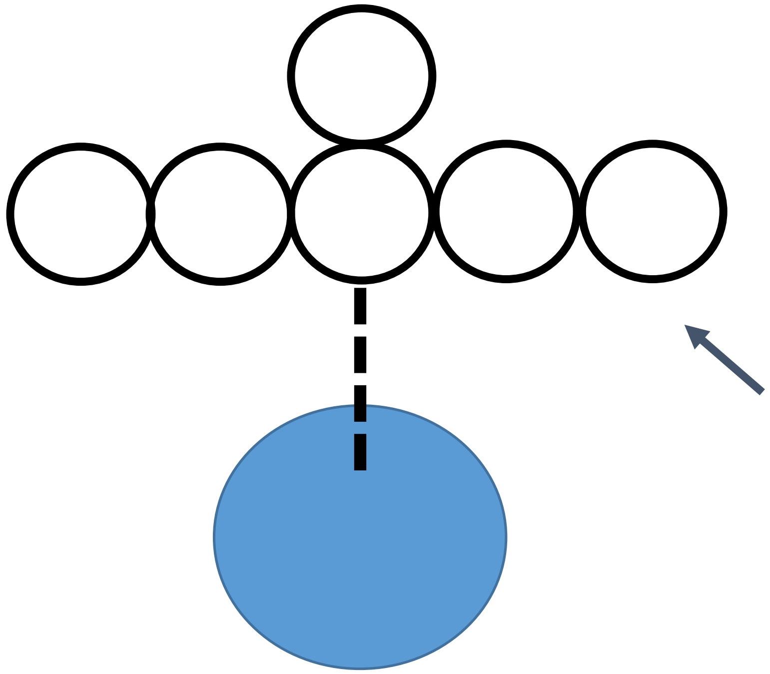
- Ex. pure E_6 gauge theory



Dynkin diagram of E_6

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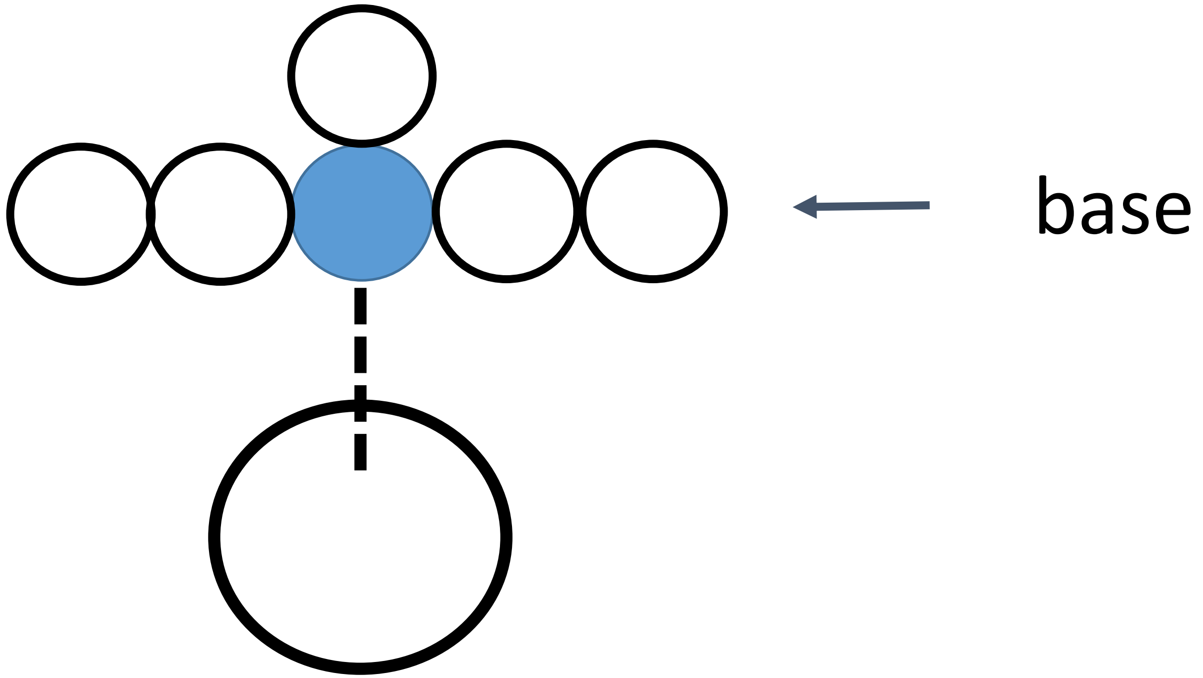
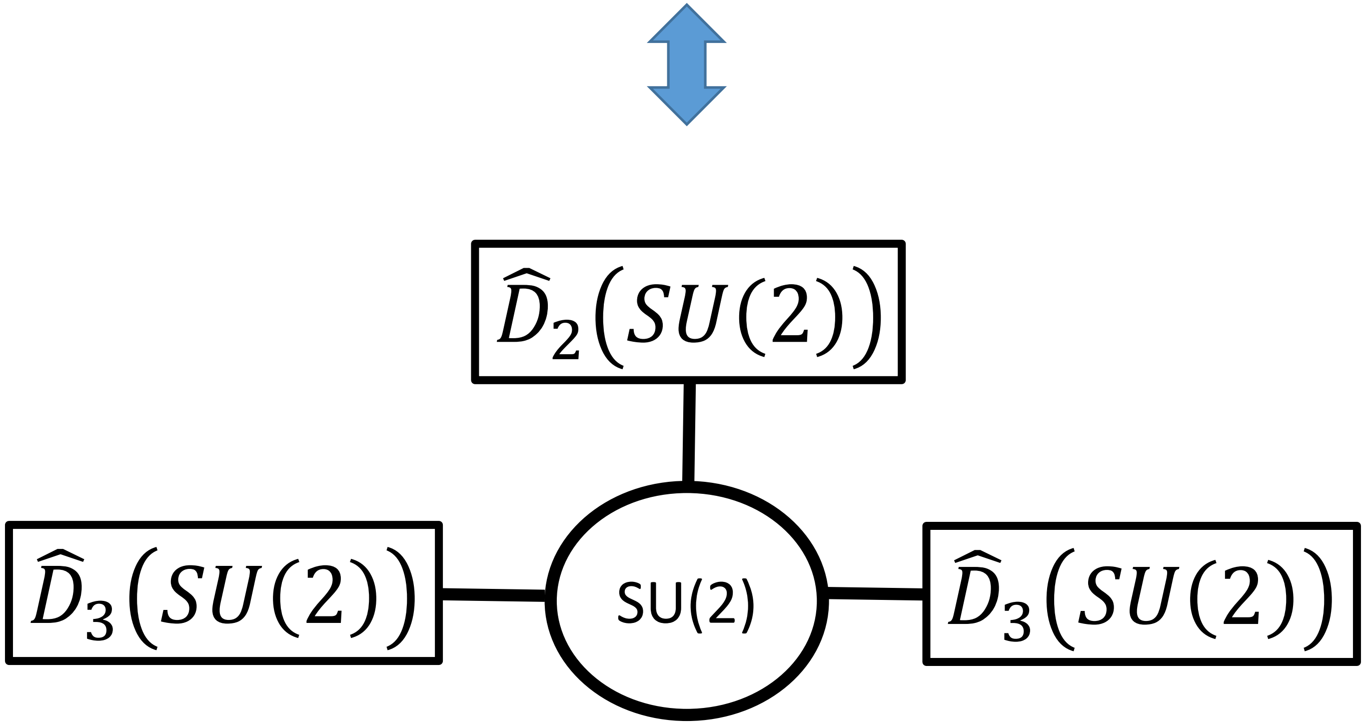
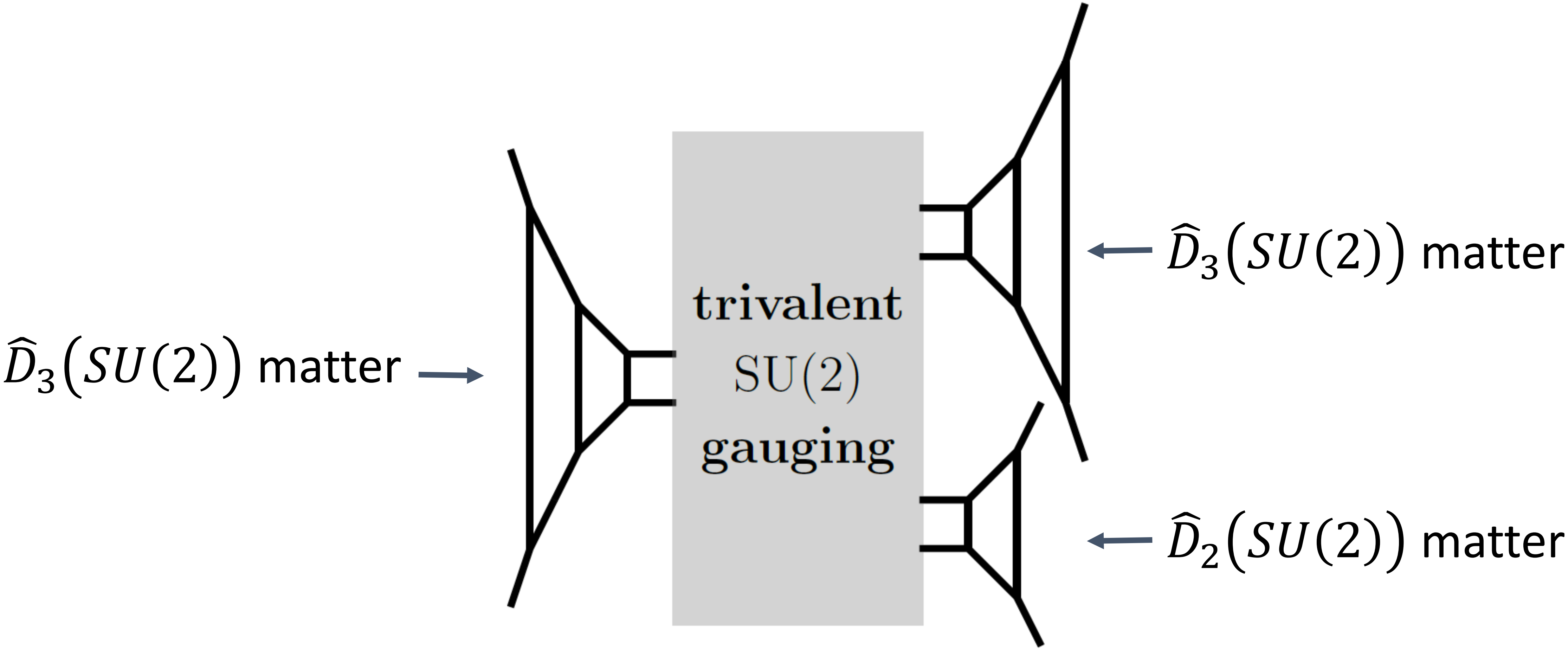


- A duality

Pure E_6 gauge theory

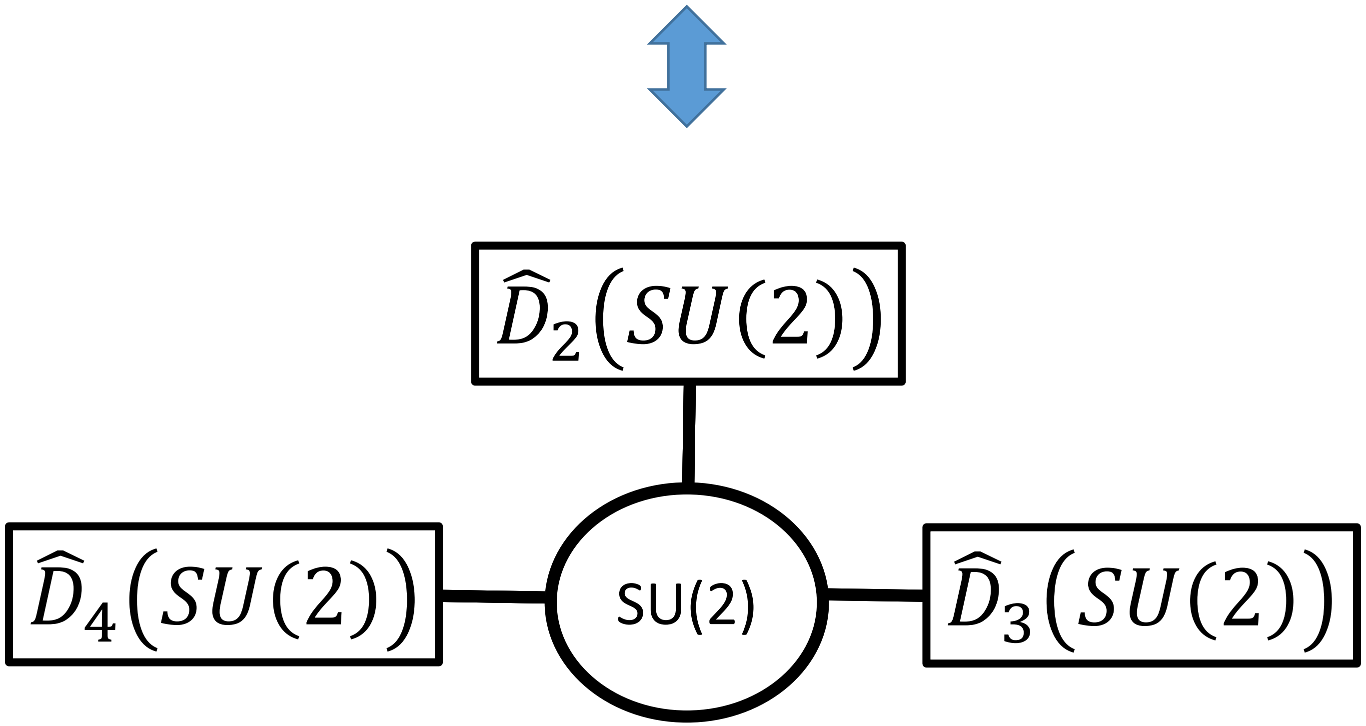


- A web-like picture



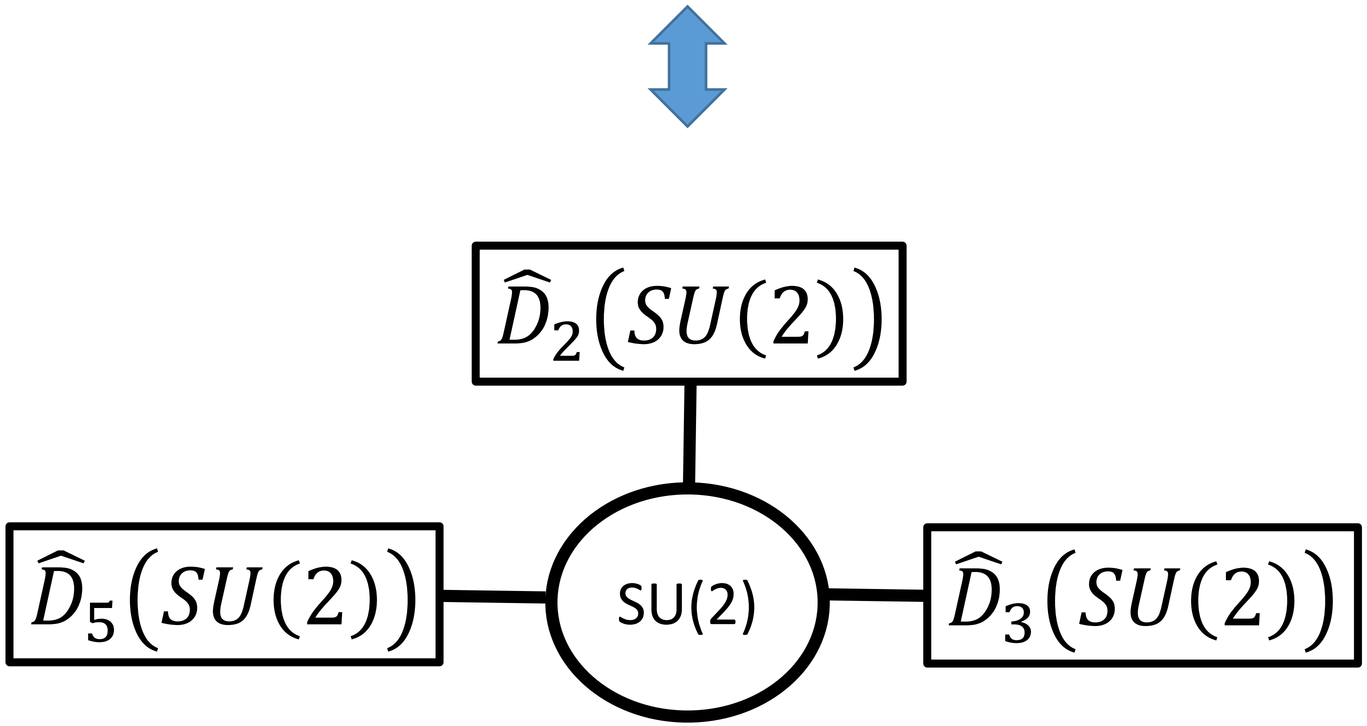
- A duality for pure E_7 gauge theory

Pure E_7 gauge theory



- A duality for pure E_8 gauge theory

Pure E_8 gauge theory



3. Trivalent gluing prescription

- We propose a prescription for computing the partition functions of the dual theories which are constructed by the trivalent gauging.

- For that let us consider a simpler case of an SU(2) gauge theory with one flavor.

- The Nekrasov partition function of an SU(2) gauge theory with one flavor is schematically written by

$$Z_{Nek} = \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} \, {Z^{SU(2)}}_{\lambda,\mu} \, {Z^{hyper}}_{\lambda,\mu}$$

Young diagrams describing the fixed points of U(1) in the U(2) instanton moduli space.

SU(2) vector multiplets

Nekrasov 02,
Nekrasov, Okounkov 03

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Young diagrams describing the fixed points of U(1) in the U(2) instanton moduli space.

SU(2) instanton background

Nekrasov 02,
Nekrasov, Okounkov 03

- Therefore, we would like to generalize this expression to

$$Z_{Nek} = \sum_{\lambda,\mu} \underbrace{Q^{|\lambda|+|\mu|} Z^{SU(2)}{}_{\lambda,\mu} Z^{T_1}{}_{\lambda,\mu} Z^{T_2}{}_{\lambda,\mu} Z^{T_3}{}_{\lambda,\mu}}$$

Trivalent SU(2) gauging of three 5d SCFTs

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Trivalent SU(2) gauging of three 5d SCFTs

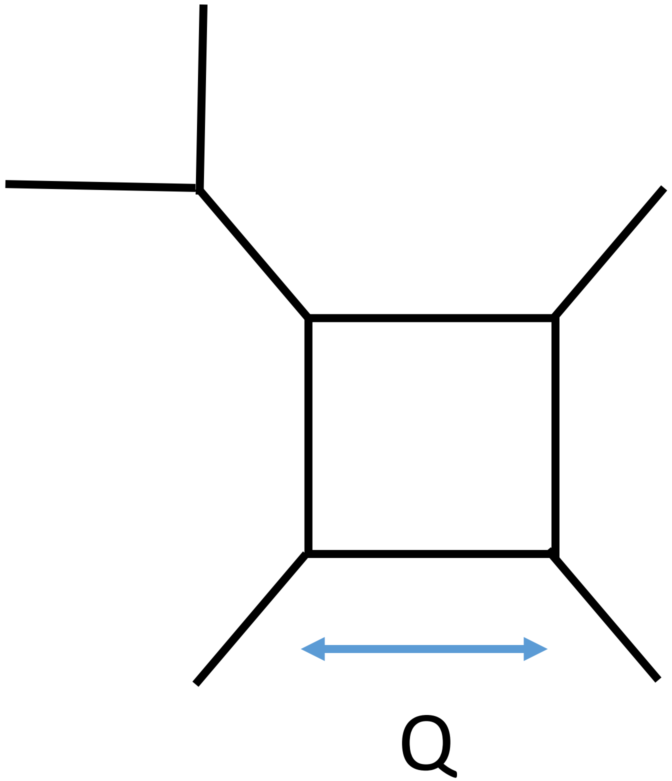
How can we compute these partition functions?

- However, obtaining the partition functions for the matter theories with an SU(2) instanton background will be difficult from a Lagrangian point of view since the SU(2) flavor symmetry appears non-perturbatively.

- We argue that the topological vertex methods helps us to compute the partition functions of the matter theories with an SU(2) instanton background.

- Let us see it first from a simpler example.

- Ex. SU(2) gauge theory with one flavor



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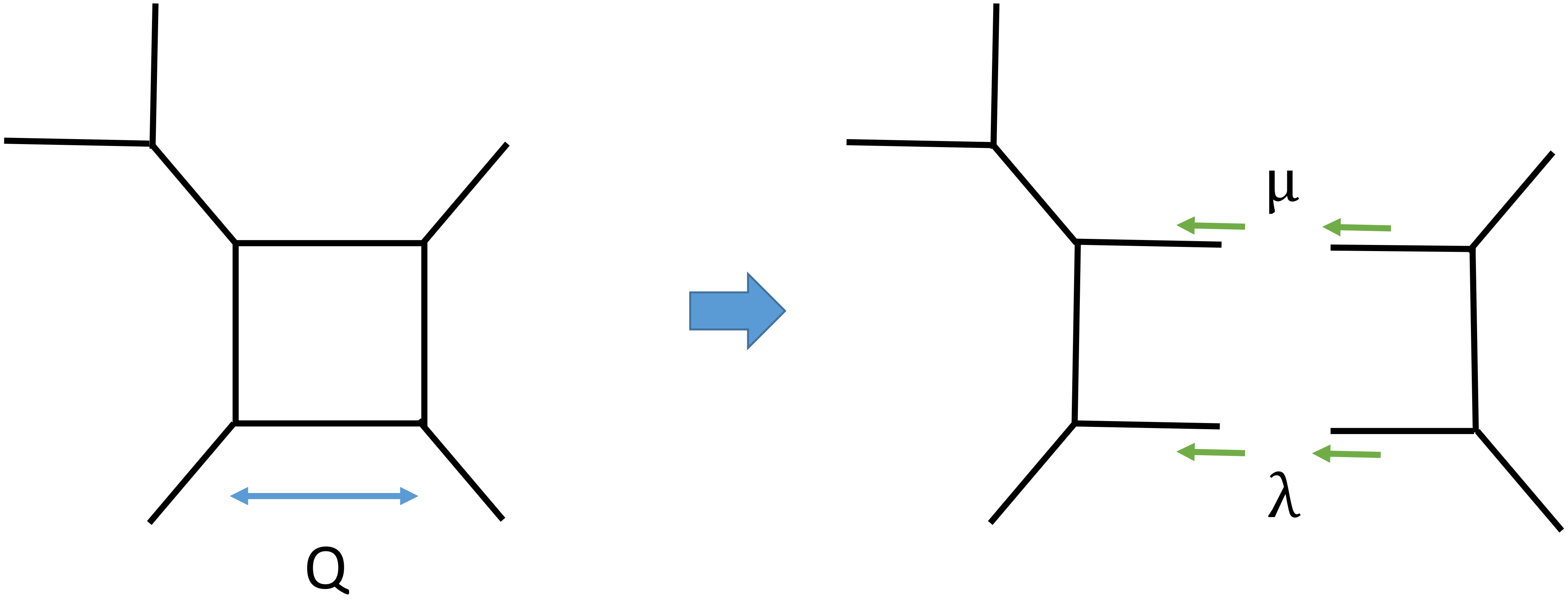
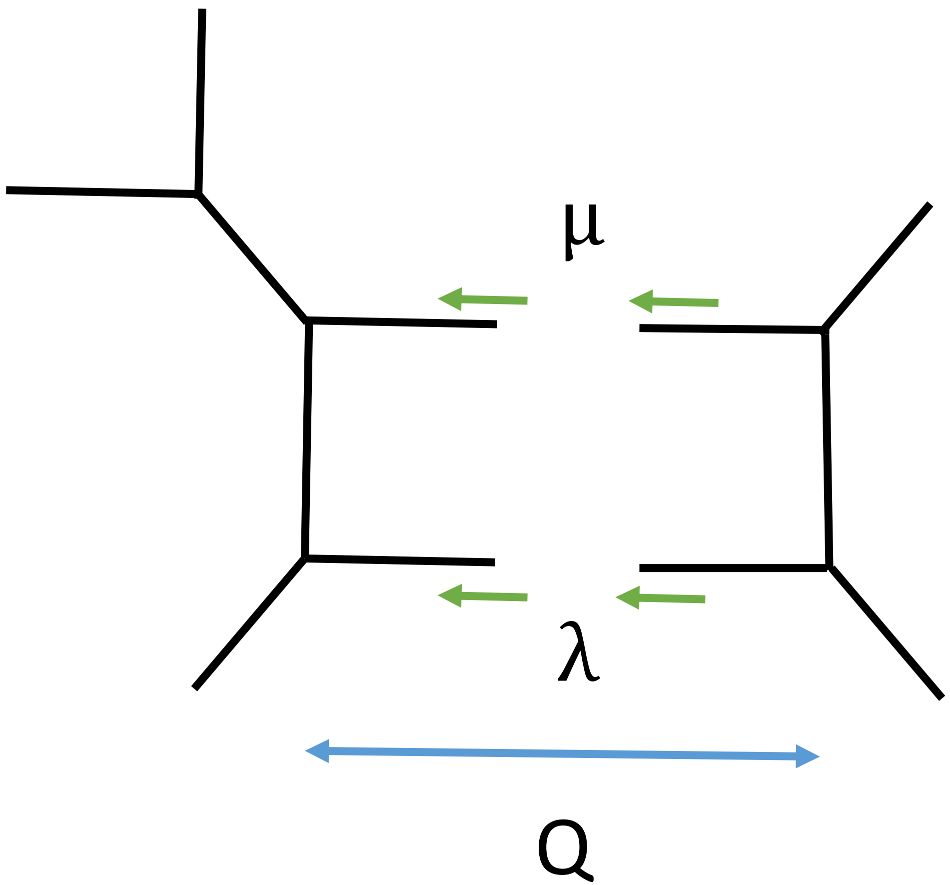
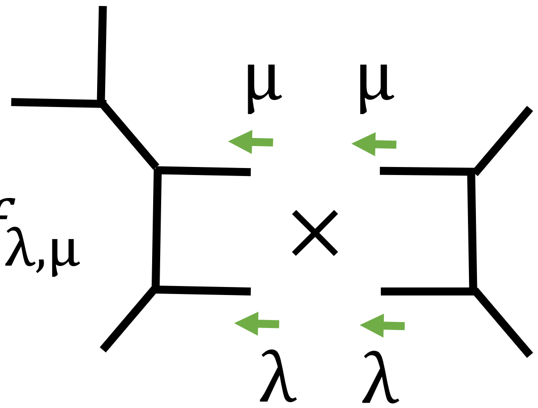


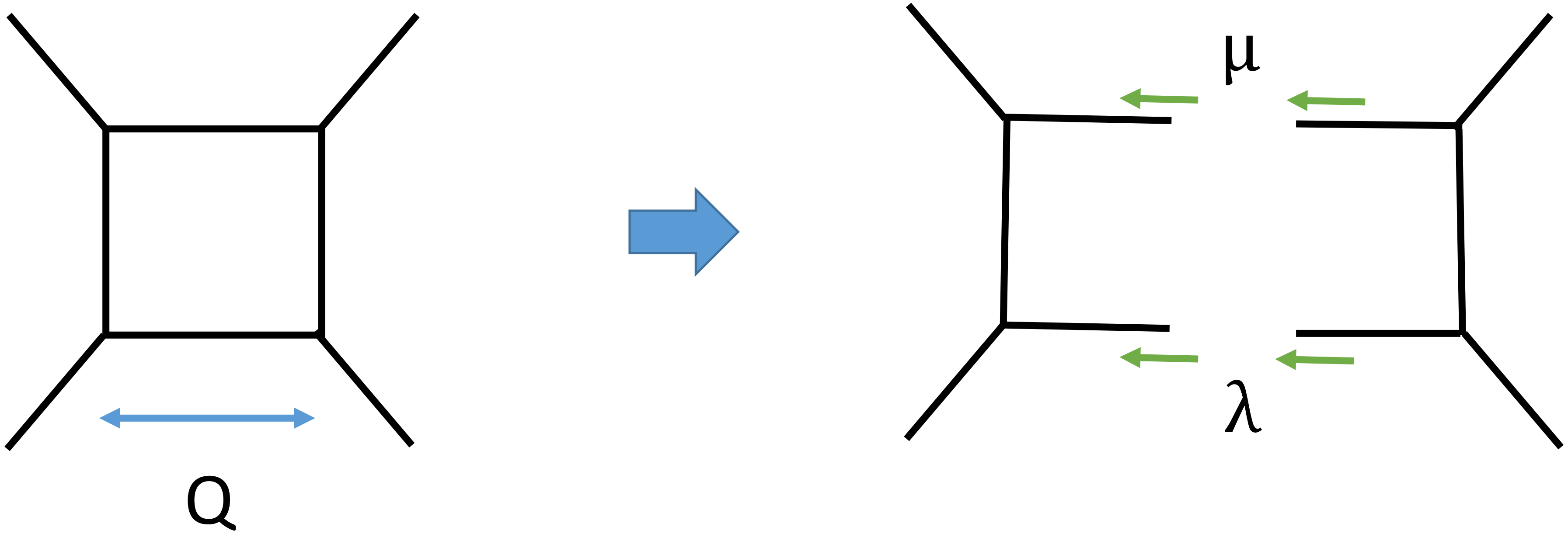


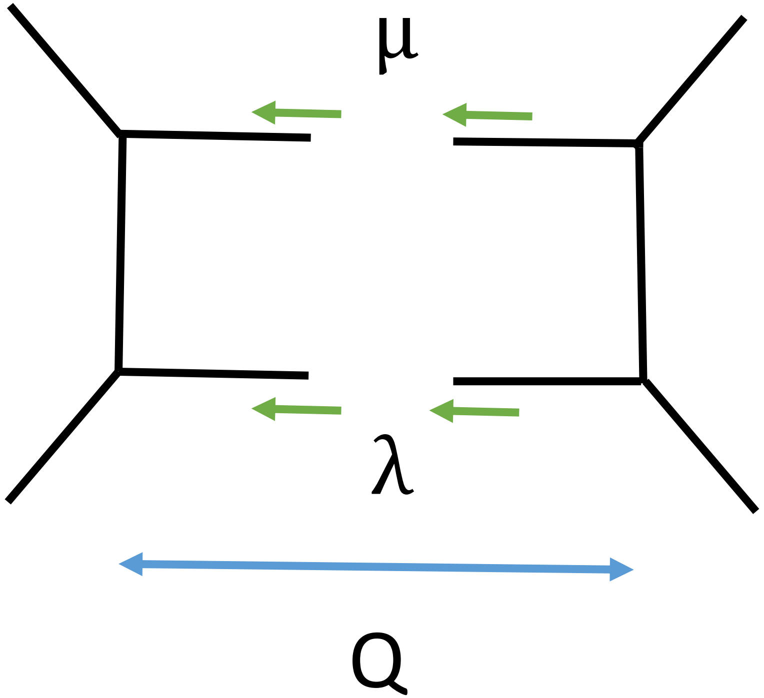
$$= \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|}\, Z^{SU(2)}{}_{\lambda,\mu}\, Z^{hyper}{}_{\lambda,\mu}$$

$$= \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} f_{\lambda,\mu}$$



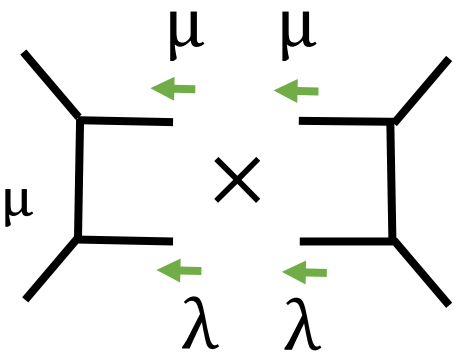
- On the other hand, the partition function of a pure SU(2) gauge theory is given by





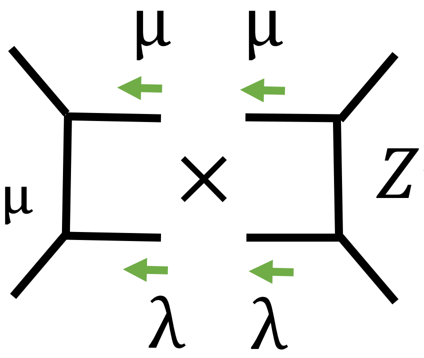
$$= \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} \, Z^{SU(2)}{}_{\lambda,\mu}$$

$$= \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} f_{\lambda,\mu}$$



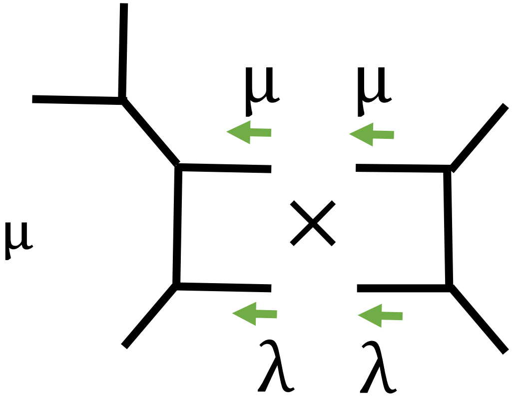
- Comparing the two equations, we can obtain the partition function of a hypermultiple on an SU(2) instanton background.

$$= \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|}\, {Z^{SU(2)}}_{\lambda,\mu}\, {Z^{hyper}}_{\lambda,\mu} = \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} f_{\lambda,\mu}$$



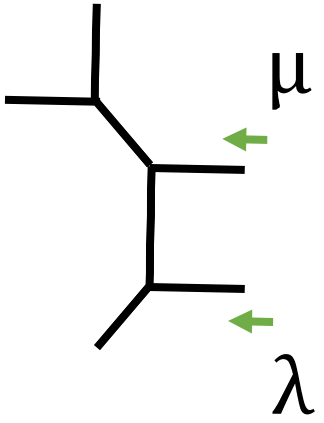
$${Z^{hyper}}_{\lambda,\mu}$$

$$= \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} f_{\lambda,\mu}$$



- Therefore, the partition function of a hypermultiple on an SU(2) instanton background is then given by

$$Z^{hyper}{}_{\lambda,\mu} =$$



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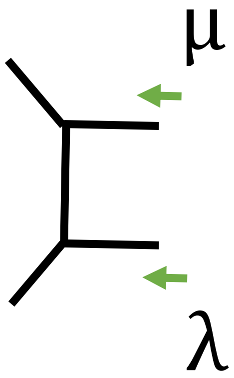
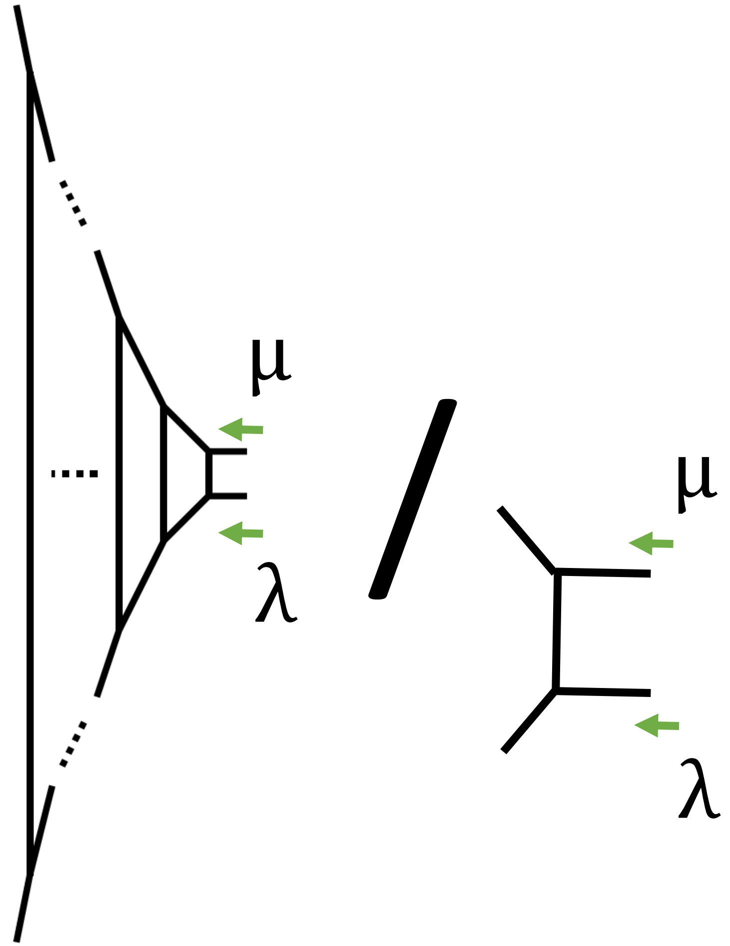


- We propose that the same prescription works for computing the partition function of the $\widehat{D}_N(SU(2))$ matter theory.

$$Z^{\widehat{D}_N(SU(2))}{}_{\lambda,\mu} =$$



- Hence when we consider the trivalent SU(2) gauging of three 5d SCFTs, $\widehat{D}_{N_1}(SU(2))$, $\widehat{D}_{N_2}(SU(2))$, $\widehat{D}_{N_3}(SU(2))$, we argue that the partition function is given by

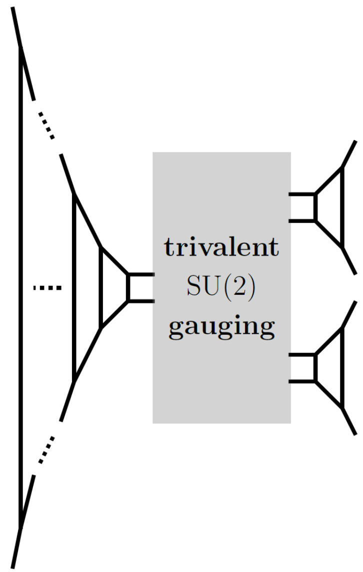
$$Z_{Nek} = \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} {Z^{SU(2)}}_{\lambda,\mu}$$

$$\times \underbrace{{Z^{\widehat{D}_{N_1}(SU(2))}}_{\lambda,\mu} {Z^{\widehat{D}_{N_2}(SU(2))}}_{\lambda,\mu} {Z^{\widehat{D}_{N_3}(SU(2))}}_{\lambda,\mu}}$$

partition functions of three 5d SCFT matter

- With this prescription, it is now straightforward to compute the partition functions of 5d pure SO(2N+4), E_6, E_7, E_8 gauge theories.

(1). Pure SO(2N+4) gauge theories



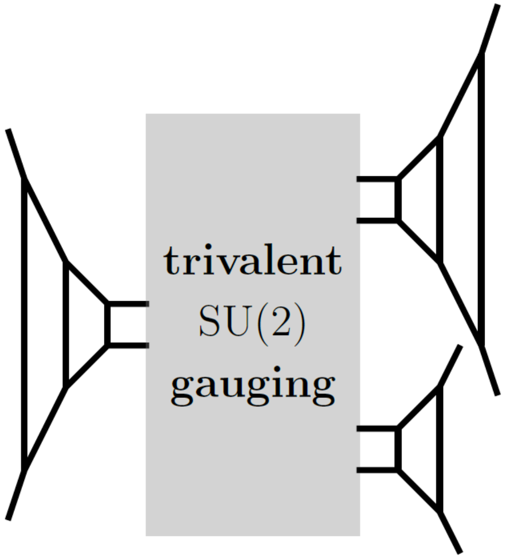
- The partition function:

$$Z_{Nek} = \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} Z^{SU(2)}{}_{\lambda,\mu}$$

$$\times \ Z^{\widehat{D}_N(SU(2))}{}_{\lambda,\mu} Z^{\widehat{D}_2(SU(2))}{}_{\lambda,\mu} Z^{\widehat{D}_2(SU(2))}{}_{\lambda,\mu}$$

- We checked that this indeed agrees with the localization result in the unrefined limit until the order Q^8 for the perturbative part and also until the order Q^5 for the one-instanton part and the two-instanton part for the case of SO(8).

(2). Pure E_6 theory



- The partition function

$$Z_{Nek} = \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} Z^{SU(2)}{}_{\lambda,\mu}$$

$$\times \; Z^{\widehat{D}_3(SU(2))}{}_{\lambda,\mu} Z^{\widehat{D}_3(SU(2))}{}_{\lambda,\mu} Z^{\widehat{D}_2(SU(2))}{}_{\lambda,\mu}$$

- We checked the result in the unrefined limit agrees with the localization result.
  Perturbative part : until Q^6
  One-instanton part : until Q^2

- The computation for the E_7 and E_8 partition functions is straightforward and we performed non-trivial checks.

Remarks:

1. It is possible to include matter in the vector representation for the SO(2N+4) gauge theory.

2. We can compute the partition function of SO(2N+3) gauge theory by a Higgsing from the partition function of SO(2N+4) gauge theory with vector matter.

3. We can extend the computation to the refined topological vertex. We checked the validity for SO(8).

4. Applications to 5d theories from 6d

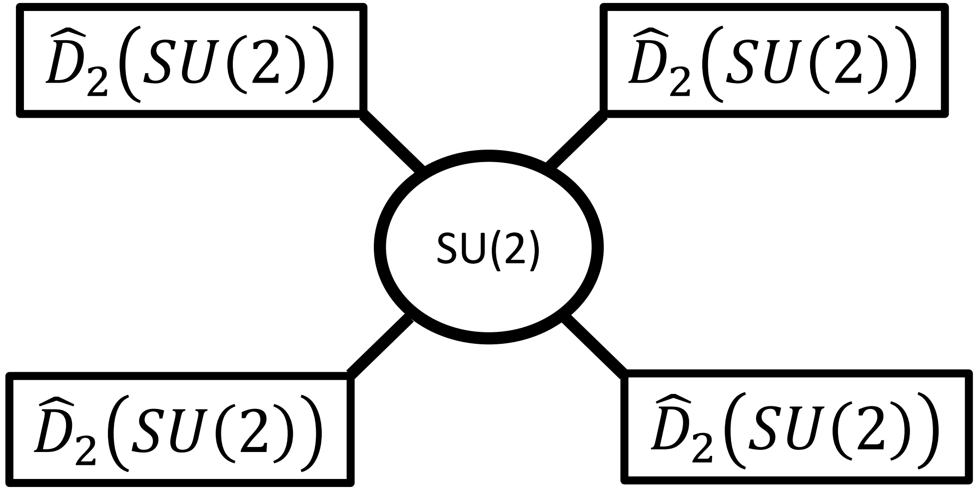
- The trivalent gauging method can be also applied to 5d theories which arise from 6d SCFTs on a circle.
- We consider 6d pure SU(3), SO(8), E_6, E_7, E_8 gauge theories with one tensor multiplet.
- They are examples of non-Higgsable clusters and important building blocks for constructing general 6d SCFTs.

Morrison, Taylor 12,
Heckman, Morrison Vafa 13
Del Zotto, Heckman, Tomasiello, Vafa 14
Heckman, Morrison Rudelius, Vafa 15

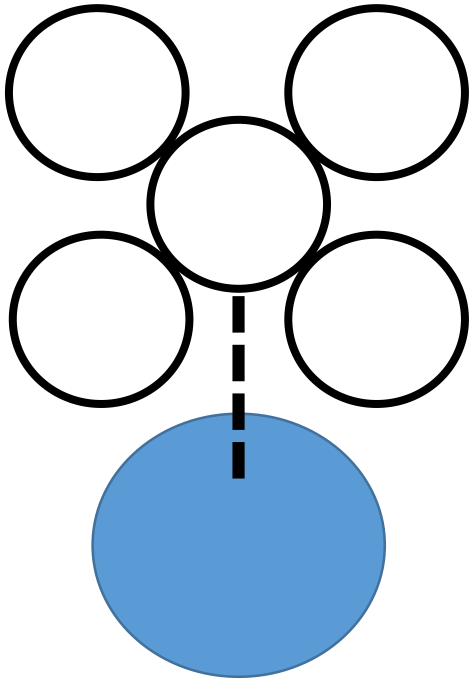
- 5d descriptions for the 6d pure SO(8), E_6, E_7, E_8 gauge theories have been already known.

Del Zotto, Vafa, Xie 15

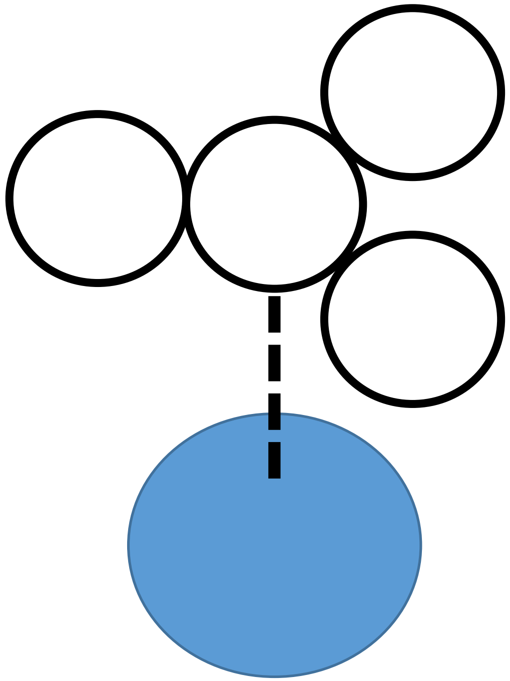
- A 5d description of 6d SO(8) gauge theory without matter:



- Affine Dynkin (6d) vs Dynkin (5d)

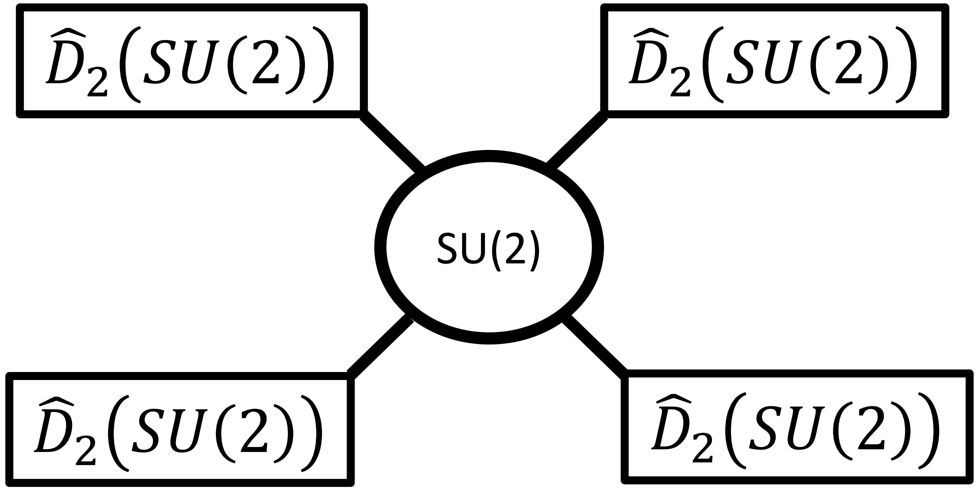


6d SCFT

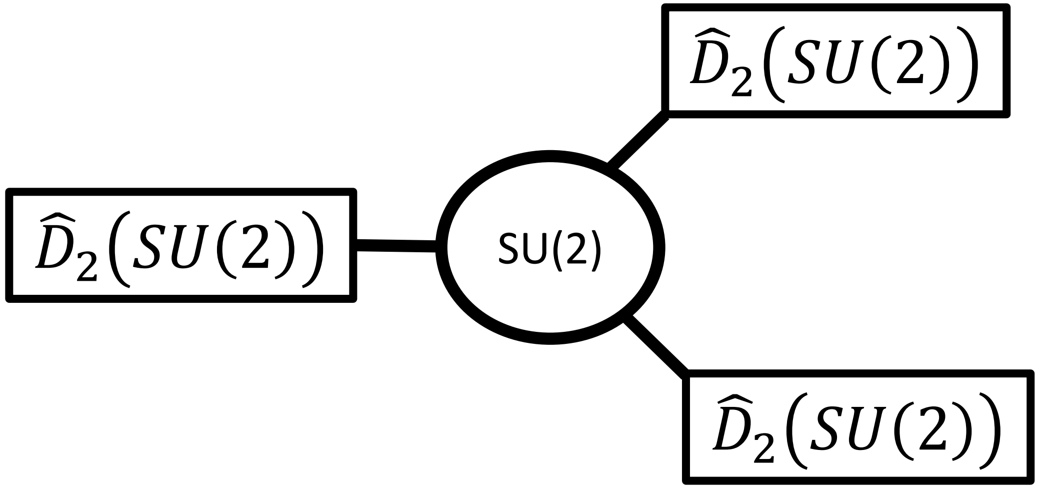


5d SCFT

- Affine Dynkin (6d) vs Dynkin (5d)



6d SCFT



5d SCFT

- The partition function of the 5d theory is given by

$$Z_{Nek} = \sum_{\lambda,\mu} Q^{|\lambda|+|\mu|} {Z^{SU(2)}}_{\lambda,\mu}$$

$$\times \; {Z^{\widehat{D}_2(SU(2))}}_{\lambda,\mu} {Z^{\widehat{D}_2(SU(2))}}_{\lambda,\mu} {Z^{\widehat{D}_2(SU(2))}}_{\lambda,\mu} {Z^{\widehat{D}_2(SU(2))}}_{\lambda,\mu}$$

- The elliptic genus of this 6d SCFT has been computed.

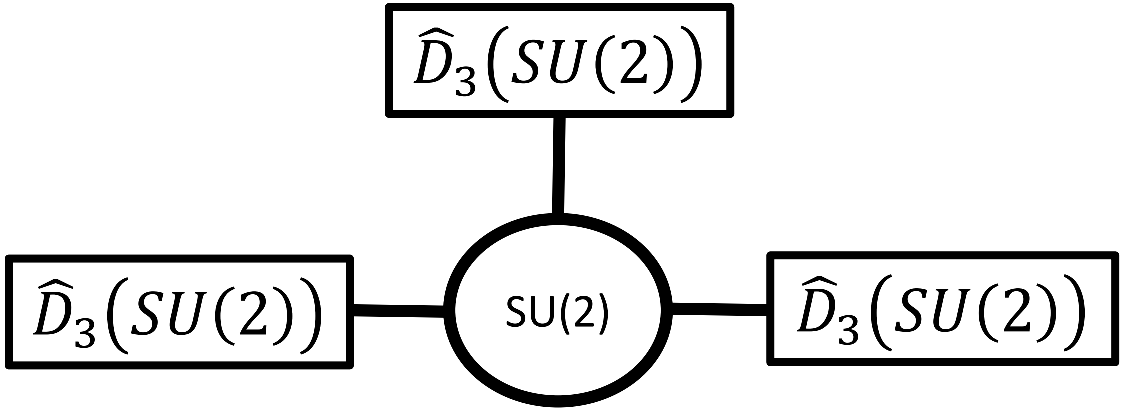
Haghighat, Klemm, Lockhart, Vafa 14

- We checked that the result agrees with the one-string elliptic genus in the unrefined limit until the order $Q^2 {Q_4}^2$.

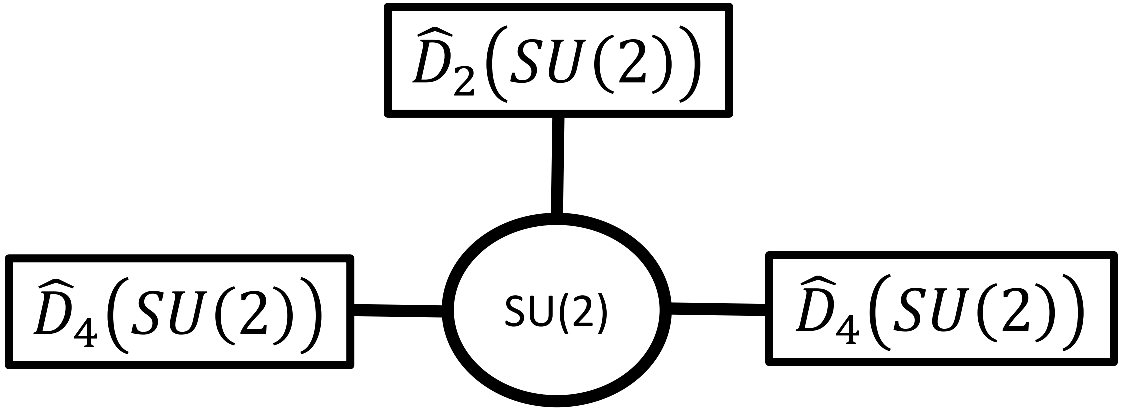
- It is straightforward to extend the analysis to the cases of 6d pure E_6, E_7, E_8 gauge theories with one tensor multiplet.

- Namely, we extend the Dynkin fibers of E_6, E_7, E_8 to the affine Dynkin fibers.

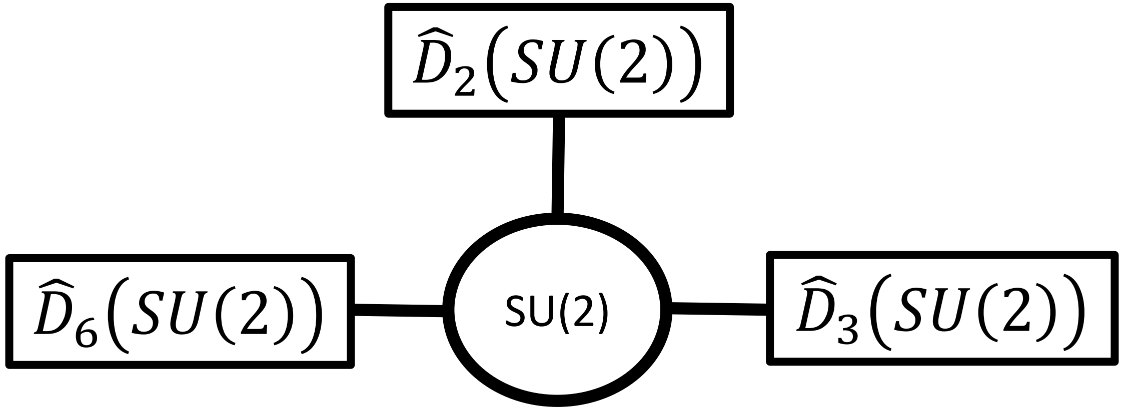
- Ex. E_6



- E_7



- E_8



- We computed the partition functions from the trivalent gauging prescription.

- Finally, we consider the 6d pure SU(3) gauge theory with one tensor multiplet.

- The structure of the geometry is different from the previous cases and we start from its geometry.

- The Calabi-Yau threefold for the 6d theory can be realized by an orbifold.

Witten 96

- The orbifold geometry is given by ${}^{T^2\times\mathbb{C}^2}\!/_{\Gamma}$ with an orbifold action

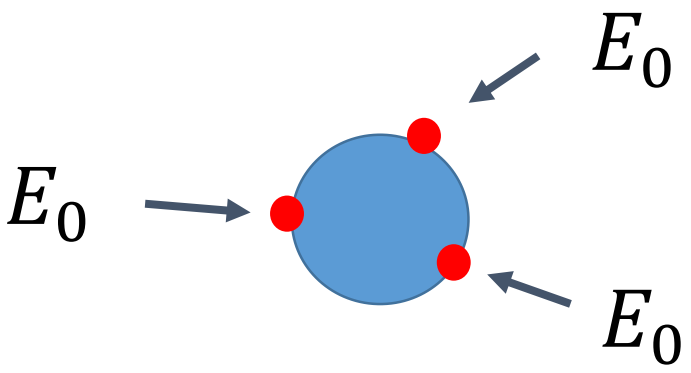
$$\underset{T^2}{(\omega^2};\underset{\mathbb{C}}{\omega},\underset{\mathbb{C}}{\omega}) \quad \text{with } \omega^3 = 1$$

- The torus becomes a sphere with three fixed points. But there is no singularity over the sphere.

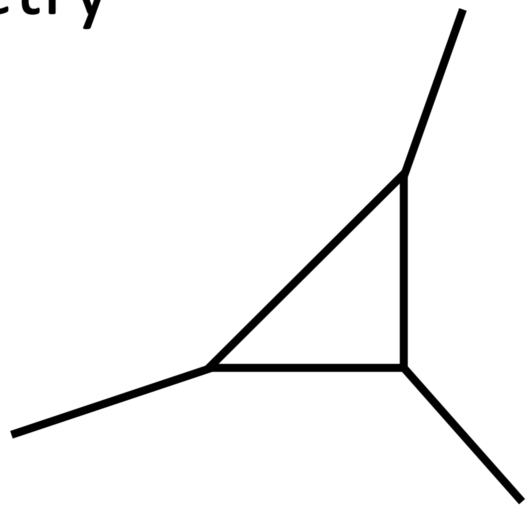
- The fixed point geometry is locally given by ${}^{\mathbb{C}^3}\!/_{\mathbb{Z}_3}$, which is local $\mathbb{P}^2$ geometry.

- A 5d description can be obtained by considering M-theory on the same Calabi-Yau threefold. Vafa 96

- Then each of the fixed points gives a 5d SCFT, E_0 theory, coming from the local $\mathbb{P}^2$. And three are coupled with each other.



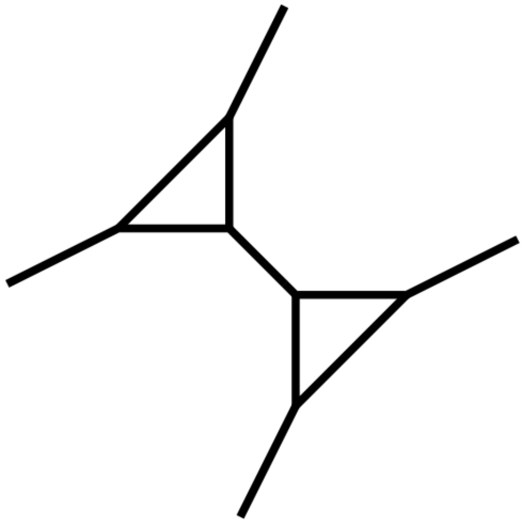
- Local $\mathbb{P}^2$ geometry



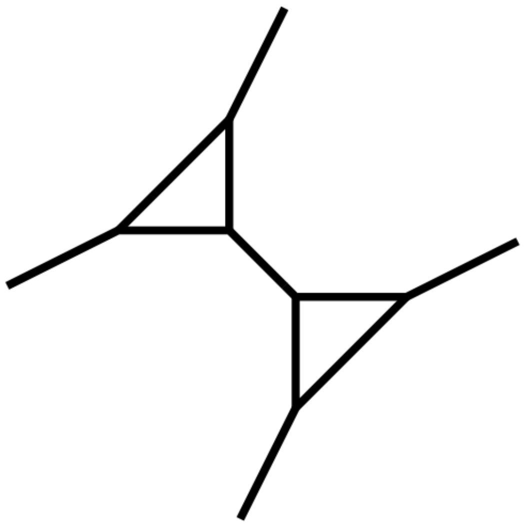
non-Lagrangian

- The Calabi-Yau geometry is given by gluing three local $\mathbb{P}^2$ geometries.

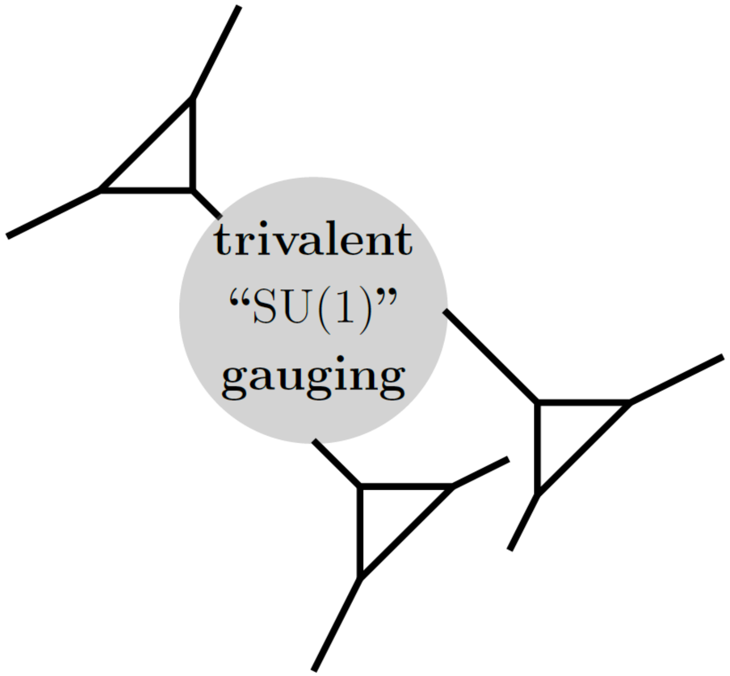
- Gluing two local $\mathbb{P}^2$ geometries.



- Gluing two local $\mathbb{P}^2$ geometries.

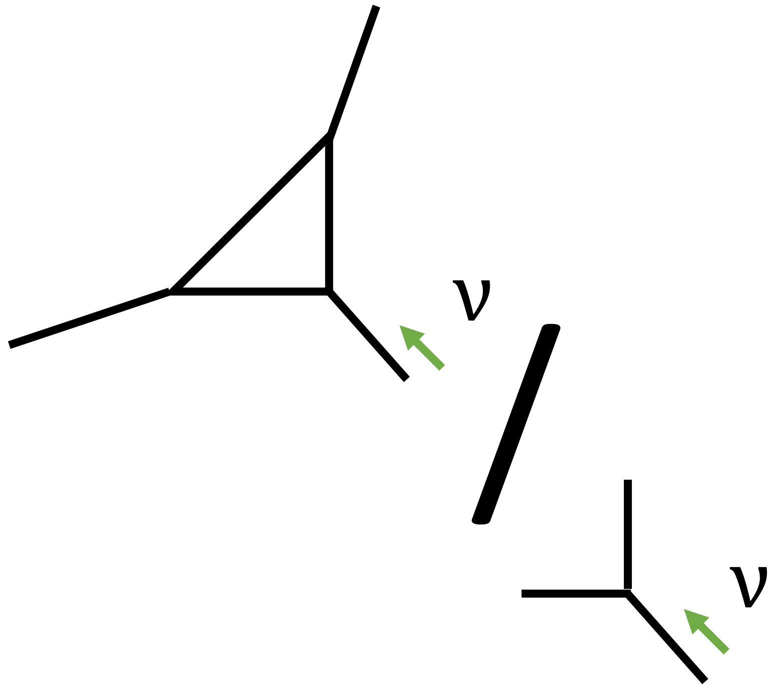


- Gluing three local $\mathbb{P}^2$ geometries



- We can use the same gluing technique to compute the partition function of the SCFT form a local $\mathbb{P}^2$ geometry.

$$Z^{E_0}{}_{\nu} =$$



- Then the partition function of the 5d theories from the 6d pure SU(3) gauge theory is given by

$$Z_{Nek} = \sum_{\nu} Q^{|\nu|} Z^{SU(1)}{}_{\nu}\, Z^{E_0}{}_{\nu}\, Z^{E_0}{}_{\nu}\, Z^{E_0}{}_{\nu}$$

↑

partition function of a resolved conifold

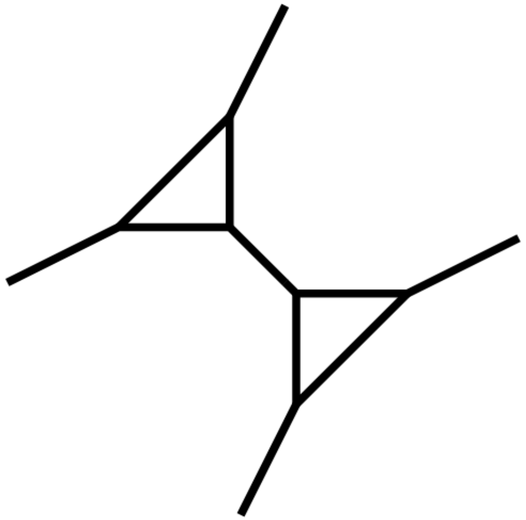
- The elliptic genus of this 6d SCFT has been recently calculated.

Kim, Kim, Park 16

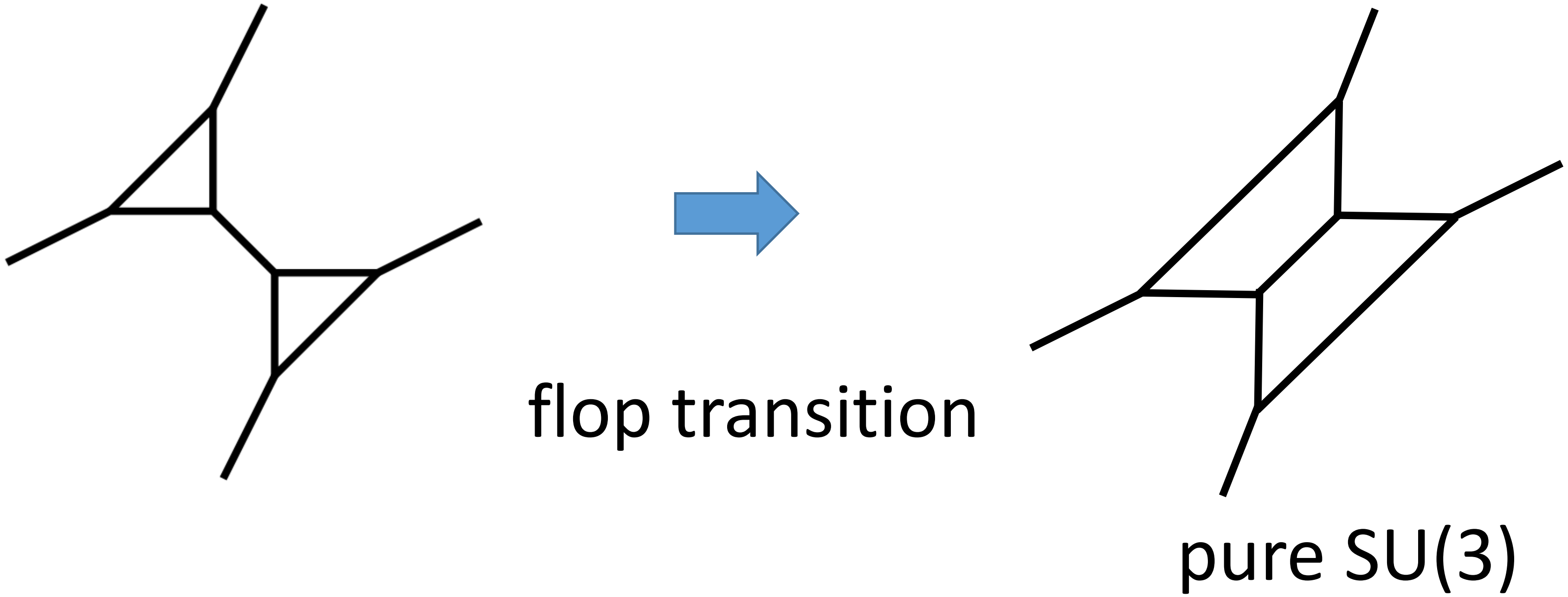
- For comparison we in fact need to perform flop transitions.

- When we take a 5d limit by taking the size of the compactification circle to infinity then the 6d theory reduces to a pure SU(3) gauge theory.

- In the current case, decoupling one local $\mathbb{P}^2$ reproduces the geometry glued by two local $\mathbb{P}^2$.



- In the current case, decoupling one local $\mathbb{P}^2$ reproduces the geometry glued by two local $\mathbb{P}^2$.



- Therefore, we need to perform the flop transition for the partition function obtained from the trivalent SU(1) gauging of three local $\mathbb{P}^2$ geometries.

- After the flop transition, indeed we found agreement with the elliptic genus of one-string until the order of ${Q_1}^2{Q_2}^2{Q_3}^2$.

Remarks:

1. Among the other non-Higgsable clusters, the one with gauge groups SU(2) x SO(7) x SU(2) has an orbifold construction. We determined the 5d description and it is again given by the trivalent SU(2) gauging.

2. We can extend the computation to the refined topological vertex. We checked the case of SO(8) until the order $Q\ {Q_1}^2 {Q_2}^2 {Q_3}^3$ for the one-string part.

5. Conclusion

- We proposed a new prescription to compute the partition functions of 5d theories constructed by trivalent gauging.

- This method gives the Nekrasov partition functions of (B)DE gauge theories in addition to AC.

- Furthermore, we computed the partition functions of 5d theories from circle compactifications of 6d pure SU(3), SO(8), E_6, E_7, E_8 gauge theories.