Average density of eigenvalues for GOE: quenched calculation

$$\rho_N(\lambda) = \frac{-2}{\pi N} \lim_{\varepsilon \to 0^+} \mathrm{Im} \frac{\partial}{\partial \lambda} \left\langle \mathrm{Log} \int_{\mathbb{R}^N} d\mathbf{y} \exp\left[-\frac{i}{2} \mathbf{y}^T (\lambda_\varepsilon \mathbb{1} - X) \mathbf{y}\right] \right\rangle$$

where $\langle (\cdots) \rangle = \int dx_{11} \ldots dx_{NN} \, \mathbb{P}(x_{11}, \ldots, x_{NN}) (\cdots)$ and $\lambda_\varepsilon = \lambda - i\varepsilon$.

We need to find a way to exchange the order of integrations in

$$\int dx_{11} \ldots dx_{NN} \, \mathbb{P}(x_{11}, \ldots, x_{NN}) \, \mathrm{Log} \int_{\mathbb{R}^N} d\mathbf{y} \exp\left[-\frac{i}{2} \mathbf{y}^T (\lambda_\varepsilon \mathbb{1} - X) \mathbf{y}\right].$$

where

$$\mathbb{P}(x_{11}, \ldots, x_{NN}) = \prod_{i=1}^{N} \frac{e^{-N x_{ii}^2/2}}{\sqrt{2\pi/N}} \prod_{i<j} \frac{e^{-N x_{ij}^2}}{\sqrt{\pi/N}}.$$

Idea Use the "replica identity"

$$\langle \mathrm{Log}\, Z(\lambda) \rangle = \lim_{n \to 0} \frac{1}{n} \mathrm{Log} \langle Z^n(\lambda) \rangle$$

Proof: $Z^n(\lambda) = e^{n \mathrm{Log} Z(\lambda)} = 1 + n \,\mathrm{Log}\, Z(\lambda) + \mathcal{O}(n^2)$

$$\Downarrow$$

$$\langle Z^n(\lambda) \rangle = 1 + n \langle \mathrm{Log}\, Z(\lambda) \rangle + o(n^2)$$

$$\mathrm{Log} \langle Z^n(\lambda) \rangle = \mathrm{Log}\left(1 + n \langle \mathrm{Log}\, Z(\lambda) \rangle + o(n^2)\right) \overset{n \to 0}{\sim} n \langle \mathrm{Log}\, Z(\lambda) \rangle.$$

The idea is to "replicate" the partition function 'n' (integer) times, exchange the integrals, and then analytically continue the result to the vicinity of $n=0$ (this step is mathematically quite delicate!)

$$\langle Z^n(\lambda)\rangle = \int \prod_{i\le j} dx_{ij} \prod_{i=1}^{N} \frac{e^{-N x_{ii}^2/2}}{\sqrt{2\pi/N}} \prod_{i<j} \frac{e^{-N x_{ij}^2}}{\sqrt{\pi/N}} \times$$

$$\times \underbrace{\int_{\mathbb{R}^{Nn}} \left(\prod_{a=1}^{n} d\mathbf{y}_a\right) \exp\left[-\frac{i}{2}\sum_{i,j}^{N}\sum_{a=1}^{n} y_{ia}(\lambda_\varepsilon \delta_{ij} - x_{ij}) y_{ja}\right]}_{\text{replicated partition function}}$$

$\boxed{\text{Remark:}}$ the logarithm - which was in the way - has completely disappeared!

Now, we can indeed exchange the order of integrations, and perform the average over disorder first.

$$\langle Z^n(\lambda)\rangle = \int_{\mathbb{R}^{Nn}} \left(\prod_{a=1}^{n} d\mathbf{y}_a\right) e^{-\frac{i}{2}\lambda_\varepsilon \sum_{i=1}^{N}\sum_{a=1}^{n} y_{ia}^2} \times \int \prod_{i=1}^{N} \frac{dx_{ii}}{\sqrt{2\pi/N}} e^{-\frac{N}{2}\sum_i x_{ii}^2 + \frac{i}{2}\sum_i x_{ii}\sum_a y_{ia}^2}$$

$$\times \int \prod_{i<j} \frac{dx_{ij}}{\sqrt{\pi/N}} e^{-N\sum_{i<j} x_{ij}^2 + i\sum_{i<j}\sum_{a=1}^{n} y_{ia} x_{ij} y_{ja}}$$

We can perform the Gaussian integrals using

$$\int_{-\infty}^{+\infty} dq \exp\left[-\alpha q^2 + i\gamma q\right] \propto \exp\left(-\frac{\gamma^2}{4\alpha}\right)$$

repeatedly, with $\alpha = \frac{N}{2}$ (or N) and $\gamma = \frac{1}{2}\sum_a y_{ia}^2$ (or $\gamma = \sum_a y_{ia} y_{ja}$)

to get (exercise!)

$$\langle Z^n(\lambda)\rangle = \int_{\mathbb{R}^{Nn}} \left(\prod_{a=1}^{n} d\mathbf{y}_a\right) \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{i=1}^{N}\sum_{a=1}^{n} y_{ia}^2 - \frac{1}{8N}\sum_{i=1}^{N}\left(\sum_a y_{ia}^2\right)^2 - \frac{1}{4N}\sum_{i<j}\left(\sum_a y_{ia} y_{ja}\right)^2\right]$$

these squares are annoying, because they couple variables belonging to different "sites" (i,j).

$$= \int_{\mathbb{R}^{Nn}} \left(\prod_a d\mathbf{y}_a\right) \exp\left[-\frac{i}{2}\lambda_\varepsilon \sum_{i=1}^{N}\sum_{a=1}^{n} y_{ia}^2 \underbrace{- \frac{1}{8N}\sum_{i,j}\left(\sum_a y_{ia} y_{ja}\right)^2}_{(*)}\right].$$

To decouple the site, we introduce the following normalized density

$$\mu(\vec{y}) = \frac{1}{N}\sum_{i=1}^{N}\prod_{a=1}^{n}\delta(y_a - y_{ia}) \qquad (**)$$

where the symbol $\vec{y} = (y_1, \dots, y_n)$.

You can now check by direct substitution that the term $(*)$ can be rewritten as (exercise!)

$$-\frac{1}{8N}\sum_{i,j}\left(\sum_a y_{ia}\, y_{ja}\right)^2 = -\frac{N}{8}\int d\vec{y}\, d\vec{w}\; \mu(\vec{y})\,\mu(\vec{w})\left(\sum_a y_a w_a\right)^2 .$$

where $d\vec{y} = \prod_{a=1}^{n} dy_a$.

We can enforce the definition $(**)$ using the following functional-integral representation of the identity

$$1 = \int \mathcal{D}\mu\, \mathcal{D}\hat{\mu}\; \exp\left[-i\int d\vec{y}\, \hat{\mu}(\vec{y})\left(N\mu(\vec{y}) - \sum_i \prod_a \delta(y_a - y_{ia})\right)\right]$$

which leads to

$$\left\langle Z^n(\lambda)\right\rangle = \int \mathcal{D}\mu\, \mathcal{D}\hat{\mu}\; \exp\left[-iN\int d\vec{y}\, \mu(\vec{y})\,\hat{\mu}(\vec{y}) - \frac{N}{8}\int d\vec{y}\, d\vec{w}\, \mu(\vec{y})\mu(\vec{w}) \cdot \left(\sum_a y_a w_a\right)^2\right] \times$$

$$\times \int_{\mathbb{R}^{Nn}} \left(\prod_a dy_a\right) \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{i=1}^{N}\sum_{a=1}^{n} y_{ia}^2 + i\sum_{i=1}^{N}\int d\vec{y}\, \hat{\mu}(\vec{y}) \prod_a \delta(y_a - y_{ia})\right]$$

The N-fold integral $\int_{\mathbb{R}^{Nn}} \left(\prod_a d\mathbf{y}_a\right)(\dots)$ is just a collection of N identical copies of a single integral, hence

$$\int_{\mathbb{R}^{Nn}} \left(\prod_a d\mathbf{y}_a\right)(\dots) = \left\{ \int_{\mathbb{R}^n} d\vec{y}_1 \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{n} y_{1a}^2 + i\int d\vec{y}\,\hat{\mu}(\vec{y}) \prod_a \delta(y_a - y_{1a}) \right. \right\}^N$$

$$= \left\{ \int_{\mathbb{R}^n} d\vec{y}_1 \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{n} y_{1a}^2 + i\,\hat{\mu}(\vec{y}_1) \right] \right\}^N .$$

$$= \left\{ \int_{\mathbb{R}^n} d\vec{y} \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{n} y_a^2 + i\,\hat{\mu}(\vec{y}) \right] \right\}^N = e^{N \operatorname{Log}\left(\int_{\mathbb{R}^n} d\vec{y} \dots\right)}$$

Therefore

$$\left\langle Z^n(\lambda) \right\rangle = \int \mathcal{D}\mu\, \mathcal{D}\hat{\mu}\, \exp\left[N\, \mathcal{S}_n[\mu, \hat{\mu}; \lambda] \right], \qquad (\square)$$

where

$$S_n[\mu, \hat{\mu}; \lambda] = -i \int d\vec{y}\, \mu(\vec{y})\, \hat{\mu}(\vec{y}) - \frac{1}{8}\int d\vec{y}\, d\vec{w}\, \mu(\vec{y})\, \mu(\vec{w}) \left(\sum_{a=1}^{n} y_a w_a \right)^2$$

$$+ \operatorname{Log}\left[\int_{\mathbb{R}^n} d\vec{y} \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_{a=1}^{n} y_a^2 + i\,\hat{\mu}(\vec{y}) \right] \right] .$$

The integral ($\square$) lends itself to a nice saddle-point approximation for $N \to \infty$. The only catch is that in doing so, we are effectively exchanging the order of limits: instead of taking $n \to 0$ first, and $N \to \infty$ afterwards, we are doing the opposite! This is not mathematically justified.

Finding the stationary points of the action

$$\frac{\delta S_n}{\delta \mu} = 0 \implies -i\hat{\mu}^*(\vec{y}) = \frac{1}{4}\int d\vec{w}\, \mu^*(\vec{w}) \Big(\underbrace{\sum_{a=1}^{n} y_a w_a}_{\vec{y}\cdot\vec{w}}\Big)^2$$

$$\frac{\delta S_n}{\delta \hat{\mu}} = 0 \implies \cancel{-i}\,\mu^*(\vec{y}) = \frac{\exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_a y_a^2 + i\hat{\mu}^*(\vec{y})\right]\cdot \cancel{(-i)}}{\int_{\mathbb{R}^n} d\vec{y}\,' \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_a y_a'^2 + i\hat{\mu}^*(\vec{y}\,')\right]}$$

Combining the two, we get

$$\boxed{-i\hat{\mu}^*(\vec{y}) = \frac{\frac{1}{4}\int d\vec{w}\, \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_a w_a^2 + i\hat{\mu}^*(\vec{w})\right](\vec{y}\cdot\vec{w})^2}{\int d\vec{w}\, \exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_a w_a^2 + i\hat{\mu}^*(\vec{w})\right]}} \quad (\square\square)$$

$\downarrow$

an integral equation for $\hat{\mu}^*(\vec{y})$.

How to solve it?

We need to make an assumption on the behaviour of μ, μ^* upon permutation of the replica indices. There is a good body of research - although not yet a formal proof - pointing to the exactness of the "replica-symmetric" high-temperature solution, i.e. the one preserving permutation-symmetry among replicas, and rotational symmetry in the space of replicas.

This means that we should look for a solution of (□□) in the form $\hat{\mu}^*(\vec{y}) = \hat{\mu}^*(y)$, with $y = |\vec{y}|$, and similarly for μ^*.

Therefore we need to introduce n-dimensional spherical coordinates

$$
\begin{aligned}
y_1 &= y \cos\phi_1 \\
y_2 &= y \sin\phi_1 \cos\phi_2 \\
y_3 &= y \sin\phi_1 \sin\phi_2 \cos\phi_3 \\
&\vdots \\
y_n &= y \sin\phi_1 \cdots \sin\phi_{n-2} \sin\phi_{n-1} ,
\end{aligned}
$$

and calling ϕ the angle between $\vec{y}$ and $\vec{\omega}$, we have that all angular integrals cancel out between numerator and denominator

$$
-i \underbrace{\hat{\mu}^*(y)}_{\text{scalar variable}} = \frac{y^2}{4} \frac{\displaystyle\int_0^\infty d\omega\, \omega^{n-1} \exp\left[-i\frac{\lambda_\varepsilon}{2}\omega^2 + i\hat{\mu}^*(\omega)\right] \omega^2 \int_0^\pi d\phi\, (\sin\phi)^{n-2} (\cos\phi)^2}{\displaystyle\int_0^\infty d\omega\, \omega^{n-1} \exp\left[-i\frac{\lambda_\varepsilon}{2}\omega^2 + i\hat{\mu}^*(\omega)\right] \int_0^\pi d\phi\, (\sin\phi)^{n-2}} ,
$$

where one uses the fact that the volume element in n-dim. spherical coordinates is $r^{n-1} \underbrace{\sin^{n-2}\phi_1 \sin^{n-3}\phi_2 \cdots \sin\phi_{n-2}}_{\text{these cancel out between numerator and denominator}}$;

Using
$$\int_0^{\pi} d\phi\, (\sin\phi)^{n-2} (\cos\phi)^2 = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}$$

$$\int_0^{\pi} d\phi\, (\sin\phi)^{n-2} = \frac{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma(n/2)} .$$

and calling $G(\omega) = e^{-\frac{i}{2}\lambda_\varepsilon \omega^2 + i\hat{\mu}^*(\omega)}$

one obtains

$$i\hat{\mu}^*(y) = \underbrace{\frac{\Gamma(n/2)\, n}{2\Gamma\left(1+\frac{n}{2}\right)}}_{\xrightarrow{n\to 0}\ 1} \frac{y^2}{4} \frac{\int_0^\infty d\omega\, \omega^{n+1} G(\omega)}{\int_0^\infty \omega^n G'(\omega)\, d\omega} ,$$

using an integration by parts in the denominator.

In the "replica limit" $n \to 0$, this yields

$$\boxed{i\hat{\mu}^*(y) = C(\lambda)\, y^2} ,$$

where $C(\lambda)$ can be determined self-consistently using

$$C(\lambda) = \frac{1}{4}\,\frac{\int_0^\infty d\omega\, \omega\, G(\omega)}{\int_0^\infty d\omega\, G'(\omega)} = \frac{1}{4}\,\frac{\int_0^\infty d\omega\, \omega\, \exp\left[-\frac{i}{2}\lambda_\varepsilon \omega^2 + C(\lambda)\omega^2\right]}{\int_0^\infty d\omega\, \exp\left[-\frac{i}{2}\lambda_\varepsilon\omega^2 + C(\lambda)\omega^2\right]\cdot 2\omega\left[-\frac{i}{2}\lambda_\varepsilon + C(\lambda)\right]}$$

$$\Big\Downarrow \quad \underline{\text{exercise}}\,!$$

$$\boxed{C(\lambda) = \frac{1}{4}\left(i\lambda_\varepsilon \pm \underbrace{\sqrt{2-\lambda_\varepsilon^2}}\right)}$$

the birth of a semicircle....

Let us recap:

$$\rho_N(\lambda) = \frac{-2}{\pi N} \lim_{\varepsilon\to 0^+} \mathrm{Im}\, \frac{\partial}{\partial\lambda} \underbrace{\left\langle \mathrm{Log}\, Z(\lambda)\right\rangle}_{\lim_{n\to 0}\frac{1}{n}\mathrm{Log}\left\langle Z^n(\lambda)\right\rangle}$$

$$\left\langle Z^n(\lambda)\right\rangle = \int \mathcal{D}\mu\, \mathcal{D}\hat{\mu}\, \exp\left[N\, S_n[\mu,\hat{\mu};\lambda]\right] \overset{N\to\infty}{\sim} \exp\left[N\, S_n[\mu^*,\hat{\mu}^*;\lambda]\right]$$

$$\Downarrow$$

$$\boxed{\rho_{N\to\infty}(\lambda) = -\frac{2}{\pi} \lim_{\varepsilon\to 0^+} \mathrm{Im} \lim_{n\to 0} \frac{1}{n}\frac{\partial}{\partial\lambda} S_n[\mu^*,\hat{\mu}^*;\lambda]\,.}$$

Recall that the action is the sum of three terms

$$S_n[\mu,\hat{\mu};\lambda] = -i\int d\vec{y}\,\mu(\vec{y})\hat{\mu}(\vec{y}) - \frac{1}{8}\int d\vec{y}\,d\vec{w}\,\mu(\vec{y})\mu(\vec{w})\left(\sum_a y_a w_a\right)^2$$

$$+\operatorname{Log}\left[\int_{\mathbb{R}^n} d\vec{y}\,\exp\left[-i\frac{\lambda_\varepsilon}{2}\sum_a y_a^2 + i\hat{\mu}(\vec{y})\right]\right],$$

so the derivative w.r.t. λ only acts over the last term because i) λ appears explicitly (not throug μ^* or $\hat{\mu}^*$) only there, and ii) the action is stationary at the saddle point

$$\frac{\partial S_n}{\partial \lambda} = \underbrace{\left.\frac{\delta S_n}{\delta \mu}\right|_{\mu=\mu^*}}_{=0}\frac{\partial \mu}{\partial \lambda} + \cdots$$

Therefore, taking the derivative of the third term and writing the integral in n-dim spherical coordinates again

cancel out

$$\rho_{N\to\infty}(\lambda) = \frac{-2}{\pi}\lim_{\varepsilon\to 0^+}\operatorname{Im}\lim_{n\to 0}\frac{1}{n}\,\frac{-\frac{i}{2}\int_0^\infty dy\,y^{n+1}\exp\left[-i\frac{\lambda_\varepsilon}{2}y^2 + C(\lambda)y^2\right]}{\int_0^\infty dy\,y^{n-1}\exp\left[-i\frac{\lambda_\varepsilon}{2}y^2 + C(\lambda)y^2\right]}$$

$$= \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\operatorname{Re}\frac{1}{-2C(\lambda)+i\lambda_\varepsilon}.$$

Recalling that $C(\lambda) = \frac{1}{4}\left[i\lambda_\varepsilon \pm \sqrt{2-\lambda_\varepsilon^2}\right]$ and $\lambda_\varepsilon = \lambda - i\varepsilon$

We can extract the real and imaginary parts of $C(\lambda)$

$$C(\lambda) = P_\varepsilon(\lambda) + i\, Q_\varepsilon(\lambda)$$

where

$$P_\varepsilon(\lambda) = \frac{1}{\sqrt{2}} \sqrt{2 - \lambda^2 + \varepsilon^2 + \sqrt{(2-\lambda^2+\varepsilon^2)^2 + (2\varepsilon\lambda)^2}}$$

$$Q_\varepsilon(\lambda) = \frac{\mathrm{sgn}(2\varepsilon\lambda)}{\sqrt{2}} \sqrt{\sqrt{(2-\lambda^2+\varepsilon^2)^2 + (2\varepsilon\lambda)^2} - (2-\lambda^2+\varepsilon^2)}$$

Hence

$$\mathrm{Re}\, \frac{1}{-2C(\lambda) + i\lambda_\varepsilon} = \frac{-2P_\varepsilon(\lambda)}{4P_\varepsilon^2(\lambda) + (\lambda - 2Q_\varepsilon(\lambda))^2}$$

In the limit $\varepsilon \to 0^+$ and for $-\sqrt{2} < \lambda < \sqrt{2}$, P_ε and Q_ε converge to

$$P_0(\lambda) = \pm \frac{\sqrt{2-\lambda^2}}{4}$$

$$Q_0(\lambda) = \frac{\lambda}{4} \, ,$$

from which $\rho_{N\to\infty}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$ as expected.