Second order conformally invariant elliptic equations

Yanyan Li

Rutgers University

May 25, 2017. ICTP, Trieste, Italy

• Theorem 3-1 (Luc Nguyen, L.) (Blow up analysis)
Assume $\{u_k\} \in C^2(B_2)$,

$$f(\lambda(A^{u_k})) = 1,\ u_k > 0,\ \text{in } B_2, \quad \sup_{B_1} u_k \to \infty.$$

Then $\forall\ \epsilon > 0$, after passing to a subsequence,
$\exists\ \{x_k^1, \cdots, x_k^m\} \subset B_2(0)$, $1 \le m \le \bar{m}$,

$$|x_k^i - x_k^j| \ge K^{-1} > 0, \qquad \forall k,\ i \ne j,$$

$$u_k(x_k^i) = \sup_{B_\delta(x_k^i)} u_k.$$

$$K^{-1} \le \frac{u_k(x_k^i)}{u_k(x_k^j)} \le K,\ \forall\ i, j, k,$$

$$\left|u_k(x) - U^{x_k^i, u_k(x_k^i)}(x)\right| \le \epsilon U^{x_k^i, u_k(x_k^i)}(x),\ \ \forall\ x \in B_\delta(x_k^i).$$

$$\frac{1}{K\delta^{n-2} u_k(x_k^1)} \le u_k(x) \le \frac{K}{\delta^{n-2} u_k(x_k^1)},\ \text{ in } B_{\frac{3}{2}}(0) \setminus \cup_{i=1}^m B_\delta(x_k^i),\ \ \forall\ k.$$

— $U^{\bar{x},\mu}(x) = \mu U(\mu^{\frac{2}{n-2}}(x - \bar{x}))$,
— $U(x) = (1 + |x|^2)^{\frac{2-n}{2}}$ satisfies $f(\lambda(A^U)) = 1$,
— $\bar{m}, K$ **depend only on** (f, Γ), δ depends on (f, Γ) and ϵ.

Proposition 3-1. (Strengthened Liouville type theorem) Assume $0 < v \in C^0(\mathbb{R}^n)$, $0 < v_k \in C^2(B_{R_k})$, $R_k \to \infty$,

$$f(\lambda(A^{v_k})) = 1 \text{ in } B_{R_k}, \qquad v_k \to v \text{ in } C^0_{loc}(\mathbb{R}^n).$$

$$\textit{Then} \qquad v(x) = \Big(\frac{a}{1 + a^2|x - \bar{x}|^2}\Big)^{\frac{n-2}{2}}, \quad a > 0, \bar{x} \in \mathbb{R}^n.$$

- v satisfies $f(\lambda(A^v)) = 1$, in $\mathbb{R}^n$ in viscosity sense.

Open Problem. Let $0 < v \in C^0_{loc}(\mathbb{R}^n)$ satisfy

$$f(\lambda(A^v)) = 1, \text{ in } \mathbb{R}^n \text{ in viscosity sense.}$$

Is it true that

$$v(x) = \Big(\frac{a}{1 + a^2|x - \bar{x}|^2}\Big)^{\frac{n-2}{2}}, \qquad a > 0, \bar{x} \in \mathbb{R}^n?$$

Proof of Proposition 3-1. $0 < v$ superharmonic, so

$$|y|^{n-2} v(y) \geq 2c_0 > 0, \ \ \forall \ |y| \geq 1.$$

Passing to subsequence, shrinking R_k, shrinking c_0, may assume

$$|v_k(y) - v(y)| \leq (R_k)^{-n}, \quad v_k(y) \geq c_0 (R_k)^{2-n}, \ \ \forall \ |y| \leq R_k.$$

- Define for $x \in \mathbb{R}^n$, $|x| + 1 \le R_k/4$,

$$\bar\lambda_k(x) = \sup\{0 < \mu \le \frac{R_k}{4} \mid (v_k)_{x,\lambda} \le v_k \text{ in } B_{R_k}(0)\backslash B_\lambda(x), \forall 0 < \lambda < \mu\},$$

where $(v_k)_{x,\lambda}(y) := (\frac{\lambda}{|y-x|})^{n-2} v_k(x + \frac{\lambda^2(y-x)}{|y-x|^2})$, the Kelvin transformation.

- $\bar\lambda_k(x)$ well defined and $\exists C(x) > 0$ such that

$$0 < \frac{1}{C(x)} \le \bar\lambda_k(x) \le \frac{R_k}{4}, \quad \forall\ k.$$

—— **Proof based on** :

- Local gradient estimates:

$$u \in C^2(B_2),\ f(\lambda(A^u)) = 1,\ 0 < u \le b,\ \text{in } B_2$$

implies

$$|\nabla \log u| \le C \quad \text{in } B_1$$

—— C depends only on (f, Γ) and b.

- Set $\bar{\lambda}(x) = \liminf_{k\to\infty} \bar{\lambda}_k(x) \in (0, \infty]$.

- Can prove (maximum principle, Hopf Lemma): either $\bar{\lambda}(x) \equiv \infty \ \forall\ x$ or $\bar{\lambda}(x) < \infty \ \forall\ x$.

$\bar{\lambda}(x) \equiv \infty$ leads to: $v \equiv$ Constant, which can be ruled out.

$\bar{\lambda}(x) < \infty \ \forall\ x$ leads to:

$$\lim_{|y|\to\infty} |y|^{n-2} v_{x,\bar{\lambda}(x)}(y) = \alpha := \liminf_{|y|\to\infty} |y|^{n-2} v(y) < \infty, \ \ \forall\ x.$$

• **We have arrived at**: $0 < v \in C^{0,1}_{loc}(\mathbb{R}^n)$, $\Delta v \le 0$ in $\mathbb{R}^n$, for every $x \in \mathbb{R}^n$, there exists $0 < \bar{\lambda}(x) < \infty$ such that

$$v_{x,\bar{\lambda}(x)}(y) \le v(y), \ \forall \ |y - x| \ge \bar{\lambda}(x),$$

$$\lim_{|y|\to\infty} |y|^{n-2} v_{x,\bar{\lambda}(x)}(y) = \alpha := \liminf_{|y|\to\infty} |y|^{n-2} v(y), \quad \forall \ x.$$

• **Claim.** We can deduce from the above that

$$v(x) = b\Big(\frac{a}{1 + a^2|x - \bar{x}|^2}\Big)^{\frac{n-2}{2}}, \qquad a, b > 0, \bar{x} \in \mathbb{R}^n.$$

Since $v_k \to v$ in $C^0_{loc}(\mathbb{R}^n)$ and $f(\lambda(A^{v_k})) = 1$, can prove that $b = 1$.

• **A Lemma.** For $n \geq 2$, $B_1 \subset \mathbb{R}^n$, $w_1, w_2 \in C^0(B_1)$, w_1, w_2 differentiable at 0, $u \in L^1_{loc}(B_1 \setminus \{0\})$, $\Delta u \leq 0$ in $B_1 \setminus \{0\}$,

$$u(y) \geq \max\{w_1(y), w_2(y)\}, \ y \in B_1 \setminus \{0\},$$

$$w_1(0) = w_2(0) = \liminf_{y \to 0} u(y).$$

Then

$$\nabla w_1(0) = \nabla w_2(0).$$

• Apply the lemma with $w^{(x)} := \left[v_{x,\bar{\lambda}(x)} \right]_{0,1}$, $u = v_{0,1}$.

• For some $V \in \mathbb{R}^n$,

$$\nabla w^{(x)}(0) = V, \ \forall \ x \in \mathbb{R}^n.$$

- A calculation yields

$$\nabla w^{(x)}(0) = (n-2)\alpha x + \alpha^{\frac{n}{n-2}} v(x)^{\frac{n}{n-2}} \nabla v(x).$$

- Thus

$$\nabla_x\Big(\frac{n-2}{2}\alpha^{\frac{n}{n-2}} v(x)^{-\frac{2}{n-2}} - \frac{n-2}{2}\alpha|x|^2 + V\cdot x\Big) = 0, \ \forall\ x \in D.$$

- Consequently, for some $\bar{x} \in \mathbb{R}^n$ and $d \in \mathbb{R}$,

$$v(x)^{-\frac{2}{n-2}} \equiv \alpha^{-\frac{2}{n-2}} |x-\bar{x}|^2 + d\alpha^{-\frac{2}{n-2}}.$$

- Since $v > 0$, we must have $d > 0$, so

$$v(x) \equiv \Big(\frac{\alpha^{\frac{2}{n-2}}}{d + |x-\bar{x}|^2}\Big)^{\frac{n-2}{2}}.$$

Proposition 3-2. Assume $\{v_k\} \in C^2(B_{R_k})$, $R_k \to \infty$,

$$f(\lambda(A^{v_k}))(y) = 1, \quad 0 < v_k(y) \le v_k(0) = 1, \quad |y| \le R_k. \qquad (1)$$

Then $\forall\ \epsilon > 0$, $\exists\ k_0' = k_0'(\epsilon)$ and $\delta' = \delta'(\epsilon)$ such that $\forall\ k > k_0'$,

$$|v_k(y) - U(y)| \le 2\epsilon U(y), \qquad \forall\ |y| \le \delta' R_k. \qquad (2)$$

Recall:

— $U(x) := \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}$

— $A^U \equiv 2I$, $f(\lambda(A^U)) \equiv 1$

By the local gradient estimates and by Proposition 3-1, after passing to subsequence,

$$v_k \to U, \quad \text{in } C^0_{loc}(\mathbb{R}^n).$$

Lemma 1. $\forall\ \epsilon > 0$, $\exists\ k_0$, such that $\forall\ k \geq k_0$,

$$\min_{|y|=r} v_k(y) \leq (1+\epsilon)U(y), \qquad \forall\ 0 < r < R_k.$$

Proof.

- Facts:

$$U_{0,\lambda}(y) < U(y), \quad \forall\ 0 < \lambda < 1, |y| > \lambda,$$

$$U_{0,1} \equiv U.$$

$$U_{0,\lambda}(y) > U(y), \quad \forall\ \lambda > 1, |y| > \lambda.$$

- Contradiction argument: If for some $\epsilon > 0$, $\exists\ r_k$

$$\min_{|y|=r_k} v_k(y) > (1+\epsilon)U(y).$$

- Then, using the above facts of U, $r_k \to \infty$, and

$$(v_k)_\lambda(y) \le v_k(y),\ \forall\ 0 < \lambda < 1+\epsilon^2, |y| = r_k.$$

- Sending $k \to \infty$.

$$U_\lambda(y) \le U(y),\ \forall\ 0 < \lambda < 1+\epsilon^2, \lambda < |y| < \infty.$$

Violating the above facts of U. Lemma 1 proved.

Lemma 2. $\forall\ \epsilon > 0$, $\exists$ small $\delta_1 > 0$, large $r_1 > 0$, such that for large k,

$$v_k(y) \geq (1-\epsilon)U(y), \qquad \forall\ |y| \leq \delta_1 R_k,$$

$$\int_{r_1 \leq |y| \leq \delta_1 R_k} v_k^{\frac{n+2}{n-2}} \leq \epsilon.$$

Proof. Since $v_k \to U$ in $C^0_{loc}(\mathbb{R}^n)$, $\exists\ r_1$ such that for large k

$$v_k(y) \geq (1-\epsilon^2)U(y), \qquad \forall\ |y| \leq r_1,$$

$$v_k(y) \geq (1-\epsilon^2)(r_1)^{2-n}, \qquad \forall\ |y| = r_1,$$

Superharmonicity of v_k, maximum principle, we have

$$v_k(y) \geq (1-\epsilon^2)\left(|y|^{2-n} - (R_k)^{2-n}\right), \qquad r_1 \leq |y| \leq R_k.$$

Thus, for any $\delta_1 \in (0, \epsilon^{\frac{2}{n-2}})$,

$$v_k(y) \geq (1-2\epsilon^2)|y|^{2-n}, \qquad r_1 \leq |y| \leq \delta_1 R_k.$$

The equation of v_k implies that $\exists\ \delta > 0$,

$$-\Delta v_k(y) \geq \frac{n-2}{2}\delta v_k(y)^{\frac{n+2}{n-2}} \qquad \text{in } r_1 \leq |y| \leq \delta_1 R_k.$$

This implies

$$\begin{aligned} v_k(y) \;&\geq\; (1-2\epsilon^2)|y|^{2-n} \\ &\quad + \frac{1}{C}|y|^{2-n}\int_{2r_1 \leq |x| \leq \delta_1 R_k/8} \delta v_k(x)^{\frac{n+2}{n-2}} dx, \ \forall\ |y| = \frac{\delta_1 R_k}{2}. \end{aligned}$$

By Lemma 1,

$$(1+2\epsilon^2)|y|^{2-n} \geq v_k(y), \qquad \forall\ |y| = \frac{\delta_1 R_k}{2}.$$

Lemma 2 follows from the above.

Since $v_k \leq 1$, by Lemma 2, for any $\epsilon > 0$, we have, for large k,

$$\int_{r_1 \leq |y| \leq \delta_1 R_k} v_k^{\frac{2n}{n-2}} \leq \epsilon.$$

• Small energy implies L^∞ bound — consequence of Liouville, as showed before.

Lemma 3. $\exists\ \delta_0 > 0$ and $C_0 > 1$ such that if $0 < u \in C^2(B_2)$,

$$f(\lambda(A^u)) = 1, \text{ in } B_2, \qquad \int_{B_2} u^{\frac{2n}{n-2}} \leq \delta_0,$$

then

$$u \leq C_0 \quad \text{in } B_1.$$

Lemma 4. $\exists\ C, \delta_4 > 0$, independent of k, such that

$$v_k(y) \le CU(y), \qquad \forall\ |y| \le \delta_4 R_k.$$

Proof. $\forall\ 4r_1 < r < \delta_1 R_k/4$, consider

$$\tilde{v}_k(z) = r^{\frac{n-2}{2}} v_k(rz), \quad \frac{1}{4} < |z| < 4.$$

For large k,

$$\int_{\frac{1}{4}<|z|<4} \tilde{v}_k(z)^{\frac{2n}{n-2}} = \int_{\frac{r}{4}<|\eta|<4r} v_k(\eta)^{\frac{2n}{n-2}} \le \epsilon := \delta_0,$$

where $\delta_0 > 0$ is the number in Lemma 3.

- By Lemma 3,

$$\tilde{v}_k(z) \leq C, \quad \frac{1}{3} < |z| < 3,$$

for some universal constant C.

- By local gradient estimates,

$$|\nabla \log \tilde{v}_k(z)| \leq C, \quad \frac{1}{2} < |z| < 2.$$

- Thus

$$\max_{|z|=1} \tilde{v}_k(z) \leq \min_{|z|=1} \tilde{v}_k(z).$$

i.e.

$$\max_{|x|=r} v_k(x) \leq C \min_{|x|=r} v_k(x) \leq CU(r).$$

—- used Lemma 1 for last inequality. Lemma 4 follows immediately.

Proof of Proposition 3-2. Only need to prove that there exists δ' and k_0' such that for any $k \geq k_0'$,

$$v_k(y) \leq (1+2\epsilon)U(y), \qquad \forall\ |y| \leq \delta' R_k.$$

Suppose the contrary, passing to subsequence, $\exists\ |y_k| = \delta_k R_k$,

$\delta_k \to 0^+$, but

$$v_k(y_k) = \max_{|y|=\delta_k R_k} v_k(y) \geq (1+2\epsilon)U(y_k).$$

Since $v_k \to v$ in $C^0_{loc}(\mathbb{R}^n)$, $|y_k| \to \infty$.
Consider rescaling of v_k:

$$\hat{v}_k(z) := |y_k|^{n-2} v_k(|y_k| z), \qquad |z| < \frac{\delta_4 R_k}{|y_k|} \to \infty.$$

We have

$$f_k(\lambda(A^{\hat{v}_k}))(z) := |y_k|^{-2} f(\lambda(A^{v_k}))(z) = |y_k|^{-2}, \qquad |z| < \frac{\delta_4 R_k}{|y_k|}.$$

Since $\hat{v}_k \leq C$, we can apply gradient estimates to f_k to obtain:

$\forall\ 0 < \alpha < \beta < \infty$, $\exists\ C(\alpha, \beta)$ such that for large k,

$$|\nabla \log \hat{v}_k(z)| \le C(\alpha, \beta), \qquad \forall\ \alpha < |z| < \beta.$$

We know from Lemma 1 and the above

$$\min_{|z|=1} \hat{v}_k(z) \le 1 + \frac{5\epsilon}{4},$$

and

$$\max_{|z|=1} \hat{v}_k(z) \ge 1 + \frac{3\epsilon}{2}.$$

Passing to subsequence, for some $0 < v^* \in C^{0,1}_{loc}(\mathbb{R}^n \setminus \{0\})$,

$$\hat{v}_k \to \hat{v}^* \qquad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^n \setminus \{0\}),\ \forall\ 0 < \alpha < 1,$$

and v^* satisfies in viscosity sense

$$\lambda(A^{\hat{v}^*}) \in \partial\Gamma, \qquad \mathbb{R}^n \setminus \{0\}.$$

Theorem

$$u \in C^{0,1}_{loc}(\mathbb{R}^n \setminus \{0\}),\ \lambda(A^u) \in \partial\Gamma \text{ in } \mathbb{R}^n \setminus \{0\},\ \text{viscosity sense}$$

implies

u radially symmetric about the origin 0.

So $\hat{v}^*$ radially symmetric.

Remark. If f is not assumed to be homogeneous, $\hat{v}^*$ does not necessarily satisfy $\lambda(A^{\hat{v}^*}) \in \partial\Gamma, \qquad \mathbb{R}^n \setminus \{0\}$.
Passing to subsequence,

$$\min_{|z|=1} \hat{v}^*(z) \le 1 + \frac{5\epsilon}{4},$$

$$\max_{|z|=1} \hat{v}^*(z) \ge 1 + \frac{3\epsilon}{2}.$$

Contradiction. Proposition 3-2 proved.

Proof of Theorem 1.

- By a previously known energy estimate of,

$$\int_{B_{1.9}} u_k^{\frac{2n}{n-2}} \le C.$$

- $\exists\ 1.8 < r_1 < r_2 < 1.9$,

$$\int_{B_{r_2}\setminus B_{r_1}} u_k^{\frac{2n}{n-2}} \le \delta_0.$$

- $\exists\ r_1 < r_3 < r_4 < r_2$ such that

$$u_k \le C, \quad \text{in } B_{r_4} \setminus B_{r_3},$$

- Go to a maximum point of u_k in B_{r_4}, and apply Proposition 3-2, ..., then apply Proposition 3-2 again in the region ... Since each time, it takes away a fixed amount of energy, it stops in finite times (the total energy is bounded by C).

Theorem 1 is proved.