Nonlinear tools in the fractional setting (and vice-versa)

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Part 1: Local Nonlinear Potential Theory and other nonlinear tools

- **In bounded domains one uses**

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho} \qquad\qquad \beta \in (0,n]$$

since

$$\begin{aligned}\mathbf{I}^{\mu}_{\beta}(x,R) &\lesssim \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-\beta}}\\ &= I_\beta(|\mu| \llcorner B_R(x))(x)\\ &\leq I_\beta(|\mu|)(x)\end{aligned}$$

for non-negative measures

Nonlinear potentials

- **The nonlinear Wolff potential is defined by**

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p]$$

which for $p = 2$ reduces to the usual Riesz potential

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B_\varrho(x))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n]$$

- **The nonlinear Wolff potential** plays in nonlinear potential theory the same role the Riesz potential plays in the linear one

The first nonlinear potential estimate

Theorem (Kilpeläinen & Malý, Acta Math. 1994)

If u solves

$$-\text{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B_R(x)} |u|^{p-1}\,dy\right)^{1/(p-1)}$$

holds

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For $p = 2$ we are back to the Riesz potential $\mathbf{W}^{\mu}_{1,p} = \mathbf{I}^{\mu}_{2}$ - the above estimate is non-trivial already in this situation

Controlling the Wolff potential

$$\int_0^\infty \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \lesssim I_\beta \left\{ [I_\beta(|\mu|)]^{1/(p-1)} \right\} (x)$$

The quantity in the right-hand side is usually called Havin-Mazya potential

A first gradient potential estimate

Theorem (Min., JEMS 2011)

When $p = 2$*, if* u *solves*

$$-\operatorname{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \fint_{B_R(x)} |Du|\, dy$$

holds

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holds

For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

The $p \neq 2$ case: a long path towards optimality

Theorem (Duzaar & Min., AJM 2011)

When $p \geq 2$*, if* u *solves*

$$-\text{div}\, a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{W}^{\mu}_{1/p,p}(x,R) + \fint_{B_R(x)} |Du|\, dy$$

holds

The $p \neq 2$ case: a long path towards optimality

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Indeed

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2013)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B_R(x)} |Du|\,dy \right)^{p-1}$$

holds

Indeed

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2013)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B_R(x)} |Du|\,dy\right)^{p-1}$$

holds

- The theorem still holds for general equations of the type $-\mathrm{div}\, a(Du) = \mu$
- The phenomenon is general: Baroni (Calc. Var. 2015) has given a far-reaching extension of this result to a family of very general operator with non-necessarily polynomial behaviour

A global estimate

Theorem (Kuusi & Min., CRAS 2011 + ARMA 2013)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

and decays naturally, then

$$|Du(x)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

Part 2: Nonlocal Nonlinear Potential Theory

The classical fractional Laplacean

$$(-\triangle)^{\alpha} u = f \qquad \text{for} \qquad 0 < s < 1$$

means that

$$\langle (-\triangle)^{\alpha} u, \varphi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u(x) - u(y)][\varphi(x) - \varphi(y)]}{|x-y|^{n+2\alpha}} \, dx \, dy = \int_{\mathbb{R}^n} f \varphi \, dx$$

for every $\varphi \in C_0^{\infty}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[u(x)-u(y)][\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy=\int_{\mathbb{R}^n}f\varphi\,dx$$

where

$$\frac{1}{\Lambda|x-y|^{n+2\alpha}}\leq K(x,y)\leq\frac{\Lambda}{|x-y|^{n+2\alpha}}\qquad\forall\ x,y\in\mathbb{R}^n,\ x\neq y$$

These correspond to linear elliptic equations of the type

$$-\operatorname{div}(A(x)Du)=f$$

where $A(x)$ is an elliptic matrix with measurable coefficients

Nonlinear nonlocal equations

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))[\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy = \int_{\mathbb{R}^n} f\varphi\,dx$$

where

$$|\Phi(t)| \leq \Lambda |t|\,, \qquad \Phi(t)t \geq t^2, \qquad \forall\ t \in \mathbb{R}$$

These correspond to linear elliptic equations of the type

$$-\mathrm{div}\, a(x, Du) = f$$

where $z \mapsto a(x,z)$ is strictly monotone with quadratic growth

The nonlocal p-Laplacean operator

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))[\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy = \int_{\mathbb{R}^n} f\varphi\,dx$$

where this time

$$\frac{1}{\Lambda|x-y|^{n+p\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+p\alpha}}$$

and

$$\Lambda^{-1}|t|^p \leq \Phi(t)t \leq \Lambda|t|^p$$

Nonlocal problems

We consider the fractional p-Laplacean

$$\begin{aligned}
&\langle -\mathcal{L}_p u, \varphi \rangle \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x)-u(y)|^{p-2}[u(x)-u(y)][\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy \\
&\qquad = \int_{\mathbb{R}^n} f\varphi\,dx
\end{aligned}$$

with

$$\frac{1}{\Lambda|x-y|^{n+p\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+p\alpha}}$$

and

$$p \geq 2$$

for simplicity

This arises when minimizing fractional energies of the type

$$v \mapsto \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(x) - u(y)|^p K(x, y)\, dx\, dy$$

Nonlocal problems

We consider the nonlocal Dirichlet problem

$$\begin{cases} -\mathcal{L}_p u = 0 \ \text{ in } \ \Omega \\ \qquad u = g \ \text{ on } \ \mathbb{R}^n \setminus \Omega \end{cases}$$

where

$$g \in W^{\alpha,p}(\mathbb{R}^n)$$

The Tail

$$\mathrm{Tail}(v; x_0, r) := \left[r^{p\alpha} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|v(x)|^{p-1}}{|x-x_0|^{n+p\alpha}}\, dx \right]^{1/(p-1)}$$

Observe that $W^{\alpha,p}(\mathbb{R}^n)$-functions have finite tail. We can consider the tail space

$$L^{p-1}_{p\alpha}(\mathbb{R}^n) := \Big\{ v \in L^{p-1}_{\mathrm{loc}}(\mathbb{R}^n) \ :$$

$$\mathrm{Tail}(v; z, r) < \infty \ \ \forall\, z \in \mathbb{R}^n, \forall\, r \in (0, \infty) \Big\}$$

and assume that

$$g \in W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{p\alpha}(\mathbb{R}^n)$$

Theorem (Di Castro & Kuusi & Palatucci, Ann. IHP 2014)

Let $v \in W^{\alpha,p}(\mathbb{R}^n)$ be a weak solution. Let $B_r(x_0) \subset \Omega$; then the following estimate holds:

$$\sup_{B_{r/2}(x_0)} |v| \leq c \left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p} + c\mathrm{Tail}(v; x_0, r/2)$$

General regularity theory

- Moreover, Di Castro & Kuusi & Palatucci also developed a remarkable regularity theory including local Hölder continuity of such solutions and Harnack inequality (JFA 2014).

General regularity theory

- Moreover, Di Castro & Kuusi & Palatucci also developed a remarkable regularity theory including local Hölder continuity of such solutions and Harnack inequality (JFA 2014).
- More recently, Cozzi (JFA 2017) has released a beautiful paper where such a regularity theory is extended to minimizers of general, non-differentiable functional depending on nonlocal derivatives such as for instance

$$w \mapsto \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^p}{|x - y|^{n+\alpha p}} \, dx \, dy + \int_{\mathbb{R}^n} F(w) \, dx$$

  where F is a non-differentiable integrand. In this case the associated Euler-Lagrange equation cannot be considered and a direct approach via fractional De Giorgi classes must be considered.

- The results of Cozzi extend the classical Giaquinta & Giusti's regularity theory for minimizers of non-differentiable functionals of the type

$$w \mapsto \int_{\Omega} F(x, w, Dw)\, dx$$

where again the main point is that no Euler-Lagrange equation can be considered

General regularity theory

- The results of Cozzi extend the classical Giaquinta & Giusti's regularity theory for minimizers of non-differentiable functionals of the type

$$w \mapsto \int_{\Omega} F(x, w, Dw)\, dx$$

  where again the main point is that no Euler-Lagrange equation can be considered
- The classical approach in this case is to prove a class of Caccioppoli type inequalities directly using minimality rather than using the Euler-Lagrange equation and deriving further regularity from those
- The higher gradient theory is still an open problem. Some results are in a recent paper of Brasco & Lindgren (Adv. Math. 2015)

SOLA (detailed definition in the nonlocal setting)

Solutions obtained via limiting approximations

$$\begin{cases} -\mathcal{L}_p u_j = \mu_j \text{ in } \ \Omega \\ \qquad u_j = g_j \text{ on } \ \mathbb{R}^n \setminus \Omega \,, \end{cases}$$

where u_j converges to u a.e. in $\mathbb{R}^n$ and locally in $L^q(\mathbb{R}^n)$.
The sequence $\{\mu_j\} \subset C_0^\infty(\mathbb{R}^n)$ converges to μ weakly in the sense of measures in Ω and moreover satisfies

$$\limsup_{j\to\infty} |\mu_j|(B) \leq |\mu|(\overline{B})$$

whenever B is a ball.
The sequence $\{g_j\} \subset C_0^\infty(\mathbb{R}^n)$ converges to g in the following sense: For all balls $B_r \equiv B_r(z)$ with center in z and radius $r > 0$, it holds that

$$g_j \to g \quad \text{in} \quad W^{\alpha,p}(B_r)\,, \qquad \text{and} \qquad \lim_j \operatorname{Tail}(g_j - g; z, r) = 0$$

Wolff potentials bounds

Theorem (Kuusi & Min. & Sire, CMP 2015)

Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, $g \in W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{p\alpha}(\mathbb{R}^n)$. Let u be a SOLA and assume that for a ball $B_r(x_0) \subset \Omega$ the Wolff potential $\mathbf{W}^{\mu}_{\alpha,p}(x_0, r)$ is finite.

Then x_0 is a Lebesgue point of u in the sense that there exists the precise representative of u at x_0

$$u(x_0) := \lim_{\varrho\to 0} (u)_{B_\varrho(x_0)} = \lim_{\varrho \to 0} \fint_{B_\varrho(x_0)} u\, dx$$

and the following estimate holds

$$|u(x_0)| \leq c\mathbf{W}^{\mu}_{\alpha,p}(x_0, r) + c \left(\fint_{B_r(x_0)} |u|^{p-1}\, dx \right)^{1/p-1} + c\mathrm{Tail}(u; x_0, r)$$

Comparison with the local case

In the case $-\text{div}\,(|Du|^{p-2}Du) = \mu$ we have

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B_R(x)} |u|^{p-1}\,dy \right)^{1/(p-1)}$$

where

$$\mathbf{W}^{\mu}_{1,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

In the fractional case we use

$$\mathbf{W}^{\mu}_{\alpha,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-p\alpha}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

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Let $\mu \in \mathcal{M}(\mathbb{R}^n)$, $g \in W^{\alpha,p}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^{p-1}_{p\alpha}(\mathbb{R}^n)$. Let u be a SOLA. If

$$\lim_{t \to 0} \sup_{x \in \Omega'} \mathbf{W}^{\mu}_{\alpha,p}(x,t) = 0\,,$$

then u is continuous in Ω'. In particular, this happens if

$$\mu \in L\left(\frac{n}{p\alpha}, \frac{1}{p-1}\right) \qquad \text{with} \quad p\alpha < n$$

or

$$\mu \in L^q\,, \qquad q > \frac{n}{p\alpha}\,.$$

We recall that

$$f \in L(q,\gamma)$$

iff

$$\int^{\infty} (\lambda^q |\{|f| > \lambda\}|)^{\gamma/q} \frac{d\lambda}{\lambda} < \infty$$

and

$$L\left(\frac{n}{p\alpha}, \frac{1}{p-1}\right) \subset L^{\frac{n}{p\alpha}} = L\left(\frac{n}{p\alpha}, \frac{n}{p\alpha}\right)$$

Part 3: Nonlocal self-improving properties

Classical self-improving properties

Theorem (Elcrat-Meyers, Giaquinta-Modica)

Let u be a weak solution to

$$-\text{div}\, a(x, Du) = f \in L^{2+\delta_0}$$

where

$$\frac{|z|^2}{\Lambda} \leq \langle a(x,z), z \rangle \qquad \textit{and} \qquad |a(x,z)| \leq \Lambda |z|$$

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Then

$$u \in W^{1,2} \Longrightarrow u \in W^{1,2+\delta}_{\mathrm{loc}}$$

for some $\delta > 0$ depending only on n, Λ, δ_0

The Gehring lemma with additional terms

Theorem (Gehring-Giaquinta-Modica)

Let $f \in L^p_{\rm loc}(\Omega)$ be such that

$$\left(\fint_{B/2} f^p\,dx\right)^{1/p} \lesssim \left(\fint_B f^q\,dx\right)^{1/q} + \left(\fint_B g^p\,dx\right)^{1/p}$$

for $q < p$, then

$$\left(\fint_{B/2} f^{p+\delta}\,dx\right)^{1/(p+\delta)} \lesssim \left(\fint_B f^q\,dx\right)^{1/q} + \left(\fint_B g^{p+\delta}\,dx\right)^{1/(p+\delta)}$$

Caccioppoli inequalities imply higher integrability

Theorem

Let $u \in W^{1,2}(\mathbb{R}^n)$ *such that for every ball* $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\fint_{B/2} |Du|^2 dx \lesssim \frac{1}{r^2} \fint_B |u(x) - (u)_B|^2 \, dx$$

holds; then there exists $\delta > 0$ *such that*

$$u \in W^{1,2+\delta}_{\mathrm{loc}}(\mathbb{R}^n)$$

Caccioppoli inequalities imply higher integrability

The proof is very simple: Sobolev-Poincaré yields

$$\left(\fint_{B/2} |Du|^2 dx\right)^{1/2} \lesssim \left(\fint_{B} |Du|^{2n/(n+2)} dx\right)^{(n+2)/2n}$$

and the assertion follows from Gehring lemma

No gradient oscillations control

Consider

$$(a(x)u_x)_x = 0$$

with

$$0 < \nu \leq a(x) \leq L$$

then

$$x \mapsto \int^x \frac{dt}{a(t)}$$

i.e. no gradient differentiability is possible when coefficients are just differentiable

Integrodifferential equations

We consider

$$\mathcal{E}_K(u,\varphi) = \int_{\mathbb{R}^n} f\varphi\,dx$$

For every test function $\varphi \in C_0^\infty(\mathbb{R}^n)$ where

$$\mathcal{E}_K(u,\varphi) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [u(x)-u(y)][\varphi(x)-\varphi(y)]K(x,y)\,dx\,dy$$

The Kernel satisfies

$$\frac{1}{\Lambda|x-y|^{n+2\alpha}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+2\alpha}}$$

for some $\Lambda \geq 1$

Integrodifferential equations

Energy solutions are initially considered in

$$u \in W^{\alpha,2}(\mathbb{R}^n)$$

The analogue of the Meyers property is now

$$u \in W^{\alpha,2+\delta}\,, \qquad \qquad \delta > 0$$

upon considering $f \in L^q$ for some $q > 2$

Fractional energies

For $\alpha \in (0,1)$

$$[u]_{\alpha,2}^2 := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy$$

The usual gradient can be obtained letting $\alpha \to 1$, but only after renormalisation, see the work of Bourgain & Brezis & Mironescu

A first result

Theorem (Bass & Ren, JFA 2013)

Define the α-gradient

$$\Gamma(x) := \left(\int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^2}{|x - y|^{n+2\alpha}} \, dy \right)^{1/2}$$

then

$$\Gamma \in L^{2+\delta}$$

This implies that $u \in W^{\alpha, 2+\delta}$

Self-improving property

Theorem (Kuusi & Min. & Sire, Analysis & PDE 2015)

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$$u \in W^{\alpha+\delta,2+\delta} \qquad \textit{for some } \delta > 0$$

This theorem has no analog in the local, classical case, where the improvement is only in the integrability scale

$$u \in W^{1,2+\delta}_{\mathrm{loc}}$$

Fractional improvement

Theorem (Schikorra, Math. Ann. 2016)

There exists a number

$$\delta_0 \equiv \delta_0(n, \alpha, \Lambda)$$

such that any $W^{\alpha-\delta_0, 2-\delta_0}$*-solution* u *to the equation*

$$\mathcal{E}_K(u, \varphi) = 0$$

is such that

$$u \in W^{\alpha+\delta_0, 2+\delta_0}_{\mathrm{loc}}(\mathbb{R}^n)$$

This extends to the nonlocal p-Laplacean as well. It is the nonlocal version of the classical theory of so-called very weak solutions valid for the local case

A sketch

A fractional approach to Gehring lemma

Caccioppoli inequalities imply higher integrability - local case

Theorem

Let $u \in W^{1,2}(\mathbb{R}^n)$ such that for every ball $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\fint_{B/2} |Du|^2 dx \lesssim \frac{1}{r^2} \fint_B |u(x) - (u)_B|^2 \, dx$$

holds; then there exists $\delta > 0$ such that

$$u \in W^{1,2+\delta}_{\text{loc}}(\mathbb{R}^n)$$

Caccioppoli inequalities imply higher integrability - nonlocal case

Theorem (Kuusi & Min. & Sire, Analysis & PDE 2015)

Let $u \in W^{\alpha,2}(\mathbb{R}^n)$ such that for every ball $B \equiv B(x_0, r) \subset \mathbb{R}^n$

$$\int_B \fint_B \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy \lesssim \frac{1}{r^{2\alpha}} \fint_B |u(x)-(u)_B|^2\,dx + \int_{\mathbb{R}^n\setminus B} \frac{|u(y)-(u)_B|}{|x_0-y|^{n+2\alpha}}\,dy \fint_B |u(x)-(u)_B|\,dx$$

holds; then there exists $\delta > 0$ such that

$$u \in W^{\alpha+\delta,2+\delta}_{\mathrm{loc}}(\mathbb{R}^n)$$

Key observation

- $u \in W^{1,2}$ means that $|Du|^2$ is integrable w.r.t. a **finite** measure (i.e. the Lebesgue measure)

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- $u \in W^{1,2}$ means that $|Du|^2$ is integrable w.r.t. a **finite** measure (i.e. the Lebesgue measure)
- $u \in W^{\alpha,2}$ means that

$$\left[\frac{|u(x)-u(y)|}{|x-y|^{\alpha}}\right]^2$$

is integrable w.r.t. an **infinite** set function, that is

$$E \to \int_E \frac{dx\, dy}{|x-y|^n}$$

there are therefore potentially more regularity properties to exploit in the above fractional difference quotient

Key idea: Dual pairs

To each u in $\mathbb{R}^{2n}$ and $\varepsilon < (0, 1-\alpha)$ we associate a function

$$U(x,y) := \frac{|u(x)-u(y)|}{|x-y|^{\alpha+\varepsilon}}$$

and a **doubling** measure

$$\mu(E) := \int_E \frac{dx\,dy}{|x-y|^{n-2\varepsilon}}$$

and note that they are in duality in the sense that

$$u \in W^{\alpha,2} \Longleftrightarrow U \in L^2(\mu)$$

- We translate the Caccioppoli inequality for u in a reverse Hölder inequality for U w.r.t. μ
- We prove a version of Gehring lemma for dual pairs (μ, U)
- The higher integrability of U turns into the higher differentiability of u
- All estimates heavily degenerate when $\alpha \to 1$ or $\alpha \to 0$

Assume $U \in L^{2+\delta}_{\text{loc}}$, this means that

$$\int_{B\times B} U^{2+\delta}\, d\mu = \int_B \int_B \frac{|u(x)-u(y)|^{2+\delta}}{|x-y|^{n+(2+\delta)\alpha+\varepsilon\delta}}\, dx\, dy < \infty$$

rewrite as follows:

$$\int_B \int_B \frac{|u(x)-u(y)|^{2+\delta}}{|x-y|^{n+(2+\delta)[\alpha+\varepsilon\delta/(2+\delta)]}}\, dx\, dy < \infty$$

and this means that

$$u \in W^{\alpha+\varepsilon\delta/(2+\delta),2+\delta}_{\text{loc}}(\mathbb{R}^n)$$

i.e. we have gained differentiability

The Gehring lemma for dual pairs (μ, U)

Theorem (Kuusi & Min. & Sire, Analysis & PDE 2015)

If (μ, U) *satisfiers*

$$\left(\fint_{\mathcal{B}} U^2 d\mu\right)^{1/2} \leq c(\sigma)\left(\fint_{\mathcal{B}} U^q\, d\mu\right)^{1/q} + \sigma \sum_{k=2}^{\infty} 2^{-k(\alpha-\varepsilon)}\left(\fint_{2^k\mathcal{B}} U^q\, d\mu\right)^{1/q}$$

where $q \in (1,2)$ *and for every choice of* $\mathcal{B} = B \times B$, *then*

$$U \in L^{2+\delta}_{\rm loc} \qquad \textit{for some} \quad \delta > 0$$

and

$$\left(\fint_{\mathcal{B}} U^{2+\delta}\, d\mu\right)^{1/(2+\delta)} \lesssim \sum_{k=1}^{\infty} 2^{-k(\alpha-\varepsilon)}\left(\fint_{2^k\mathcal{B}} U^2\, d\mu\right)^{1/2}$$

Part 4: Fractional tools in (local) nonlinear problems

First example: Limiting Calderón-Zygmund theory

- Here we consider measure data problems of the type

$$-\operatorname{div} a(Du) = \mu$$

where the model case is given by

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu\,, \qquad p > 2 - \frac{1}{n}$$

and μ is a Radon measure with finite total mass

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$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu\,, \qquad p > 2 - \frac{1}{n}$$

and μ is a Radon measure with finite total mass

- The standard existence theory is a by now classical achievement of Boccardo & Gallöuet (JFA 1989) and gives

$$Du \in L^q \qquad \forall\; q < \frac{n(p-1)}{n-1}$$

which is optimal in the scale of Lebesgue spaces (look at the fundamental solution)

Differentiability of Du for measure data - $p = 2$

- **You cannot have** $Du \in W^{1,1}$, already in the linear case

$$-\triangle u = \mu$$

(classical failure of Calderón-Zygmund theory in the limiting clase)

- **Full integrability** On the other hand we know that

$$Du \in L^q \qquad \forall\ q < \frac{n}{n-1}$$

i.e. optimal integrability vs total lack of differentiability of Du

Differentiability of *Du* for measure data - $p = 2$

Theorem (Min., Ann. SNS Pisa 2007)

$$Du \in W^{\sigma,1} \qquad \textit{for every } \sigma \in (0,1)$$

Differentiability of Du for measure data - $p = 2$

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This uses a sort of nonlinear analog of local Littlewood-Paley decomposition, inspired by the atomic decomposition characterization of fractional Sobolev spaces.

Differentiability of Du for $p \neq 2$

- **What should we expect?** We have

$$\frac{1}{|x|^{\beta}} \in W^{s,\gamma}(B) \Longleftrightarrow \beta < \frac{n}{\gamma} - s$$

We apply this fact to the fundamental solution

$$|Du| \approx \frac{1}{|x|^{\frac{n-1}{p-1}}}$$

with the natural choice $\gamma = p - 1$ (this maximizes the integrability parameter). This yields

$$s < \frac{1}{p-1}$$

and we expect

$$Du \in W^{s,p-1} \qquad \forall \;\; s < \frac{1}{p-1}$$

Indeed....

Theorem (Min., Ann. SNS Pisa 2007)

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1} \qquad \textit{for every } \varepsilon > 0$$

Recall that we are assuming $p \geq 2$ so that

$$\frac{1}{p-1} \leq 1$$

Sharpness

- Fractional Sobolev embedding theorem

$$W^{\sigma,q} \hookrightarrow L^{\frac{nq}{n-\sigma q}} \qquad \sigma q < n$$

- Therefore

$$Du \notin W^{\frac{1}{p-1},p-1}$$

- otherwise

$$Du \in L^{\frac{n(p-1)}{n-1}}$$

which does not hold in the case of the fundamental solution

Another linearization phenomenon

Theorem (Avelin & Kuusi & Min., Preprint 2016)

$$a(Du) \in W^{\sigma,1} \qquad \textit{for every } \sigma \in (0,1)$$

Another linearization phenomenon

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$$a(Du) \in W^{\sigma,1} \qquad \textit{for every } \sigma \in (0,1)$$

- Exactly the same phenomenon happens in the linear case $-\triangle u = -\mathrm{div}\, Du = \mu$ via fundamental solutions
- Complete linearization of the equation with respect to fractional differentiability
- A similar phenomenon will happen with respect to potential estimates

Another linearization phenomenon

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Another linearization phenomenon

Theorem (Avelin & Kuusi & Min., Preprint 2016)

$$a(Du) \in W^{\sigma,1} \qquad \textit{for every } \sigma \in (0,1)$$

With a related Caccioppoli type inequality

$$[a(Du)]_{\sigma,1;B_{R/2}} = \int_{B_{R/2}} \int_{B_{R/2}} \frac{|a(Du(x)) - a(Du(y))|}{|x-y|^{n+\sigma}} \, dx \, dy$$
$$\lesssim \frac{1}{R^{\sigma}} \int_{B_R} |a(Du)| \, dx + R^{1-\sigma} |\mu|(B_R)$$

Uniformization of singularities

Theorem (Avelin & Kuusi & Min., Preprint 2016)

For

$$p \geq 2 \qquad \textit{and} \qquad 0 \leq \gamma \leq p-2$$

we have that

$$|Du|^{\gamma} Du \in W^{\sigma\frac{\gamma+1}{p-1}, \frac{p-1}{\gamma+1}}_{\mathrm{loc}}(\Omega; \mathbb{R}^n) \qquad \textit{holds for every } \sigma \in (0,1)\ .$$

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- Increasing integrability decreases differentiability, and vice versa
- Cathes-up both the original theorems
- Conjectured in my paper on Ann. SNS Pisa 2007

Second example: double phase functionals

Theorem (Colombo-Min., ARMA 2015)

Let $u \in W^{1,p}(\Omega)$ be a bounded local minimiser of the functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad q \leq p + \alpha$$

then

Du is Hölder continuous

Proof strongly based on the idea of proving that the gradient of minima lies in suitable fractional Sobolev spaces

Third example: A fractional approach to potential estimates

- We consider equations with measure data

$$-\operatorname{div} a(Du) = \mu$$

- We take $p = 2$ and consider

$$\begin{cases} |a(z)| + |\partial a(z)||z| \leq L|z| \\ \nu^{-1}|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda \rangle \end{cases}$$

Gradient potential estimate for the nonlinear Poisson equation

Theorem (Min., JEMS 2011)

If u solves

$$-\mathrm{div}\, a(Du) = \mu$$

then

$$|D_\xi u(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c \fint_{B(x,R)} |D_\xi u|\, dx$$

for every $\xi \in \{1, \ldots, n\}$

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For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

- **Consider energy solutions to** $\operatorname{div} a(Du) = 0$ for $p = 2$
- **First prove** $Du \in W^{1,2}$
- **Then use that** $v = D_\xi u$ solves

$$\operatorname{div}(A(x)Dv) = 0 \qquad\qquad A(x) := a_z(Du(x))$$

- **The boundedness of** $D_\xi u$ follows by Standard DeGiorgi's theory
- **This is a consequence of Caccioppoli's inequalities of the type**

$$\int_{B_{R/2}} |D(D_\xi u - k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} |(D_\xi u - k)_+|^2 \, dy$$

where

$$(D_\xi u - k)_+ := \max\{D_\xi u - k, 0\}$$

Recall the definition

- **We have**

$$v \in W^{\sigma,1}(\Omega')$$

iff $v \in L^1(\Omega')$ and

$$[v]_{\sigma,1;\Omega'} = \int_{\Omega'} \int_{\Omega'} \frac{|v(x) - v(y)|}{|x-y|^{n+\sigma}} \, dx \, dy < \infty$$

There is a differentiability problem

For solutions to

$$-\text{div}\, a(Du) = \mu \quad \text{in general} \quad Du \notin W^{1,1}$$

but nevertheless it holds

Theorem (Min., Ann. SNS Pisa 2007)

$$Du \in W^{\sigma,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \qquad \textit{for every } \sigma \in (0,1)$$

This means that

$$[Du]_{\sigma,1;\Omega'} = \int_{\Omega'}\int_{\Omega'} \frac{|Du(x) - Du(y)|}{|x-y|^{n+\sigma}}\, dx\, dy < \infty$$

holds for every $\sigma \in (0,1)$, and every subdomain $\Omega' \Subset \Omega$

Step 1: A non-local Caccioppoli inequality

Theorem (Min., JEMS 2011)

Let

$$w = D_\xi u \quad \textit{with} \quad -\operatorname{div} a(Du) = \mu$$

where $\xi \in \{1, \ldots, n\}$ *then*

$$[(|w| - k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c}{R^\sigma} \int_{B_R} (|w| - k)_+ \, dy + \frac{cR|\mu|(B_R)}{R^\sigma}$$

holds for every $\sigma < 1/2$

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holds for every $\sigma < 1/2$

Compare with the usual one for $\operatorname{div} a(Du) = 0$, that is

$$[(w - k)_+]^2_{1,2;B_{R/2}} \equiv \int_{B_{R/2}} |D(w - k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} (w - k)^2_+ \, dy$$

Step 1: A non-local Caccioppoli inequality

- **This approach reveal the robustness of energy inequalities, which hold below the natural growth exponent 2, and for fractional order of differentiability, although the equation has integer order**
- **Classical VS fractional**

	classical	fractional
spaces	$L^2 - L^2$	$L^1 - L^1$
differentiability	$0 \longrightarrow 1$	$0 \longrightarrow \sigma$

Step 2: Fractional De Giorgi's iteration

Theorem (Min., JEMS 2011)

Let w be an L^1-function w satisfying the fractional Caccioppoli's inequality

$$[(|w|-k)_+]_{\sigma,1;B_{R/2}} \leq \frac{L}{R^\sigma}\int_{B_R}(|w|-k)_+\,dy + \frac{LR|\mu|(B_R)}{R^\sigma}$$

for some $\sigma > 0$ and every $k \geq 0$. Then it holds that

$$|w(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c\fint_{B(x,R)}|w|\,dy$$

for every Lebesgue point x of w