Scattering Amplitudes

LECTURE 1

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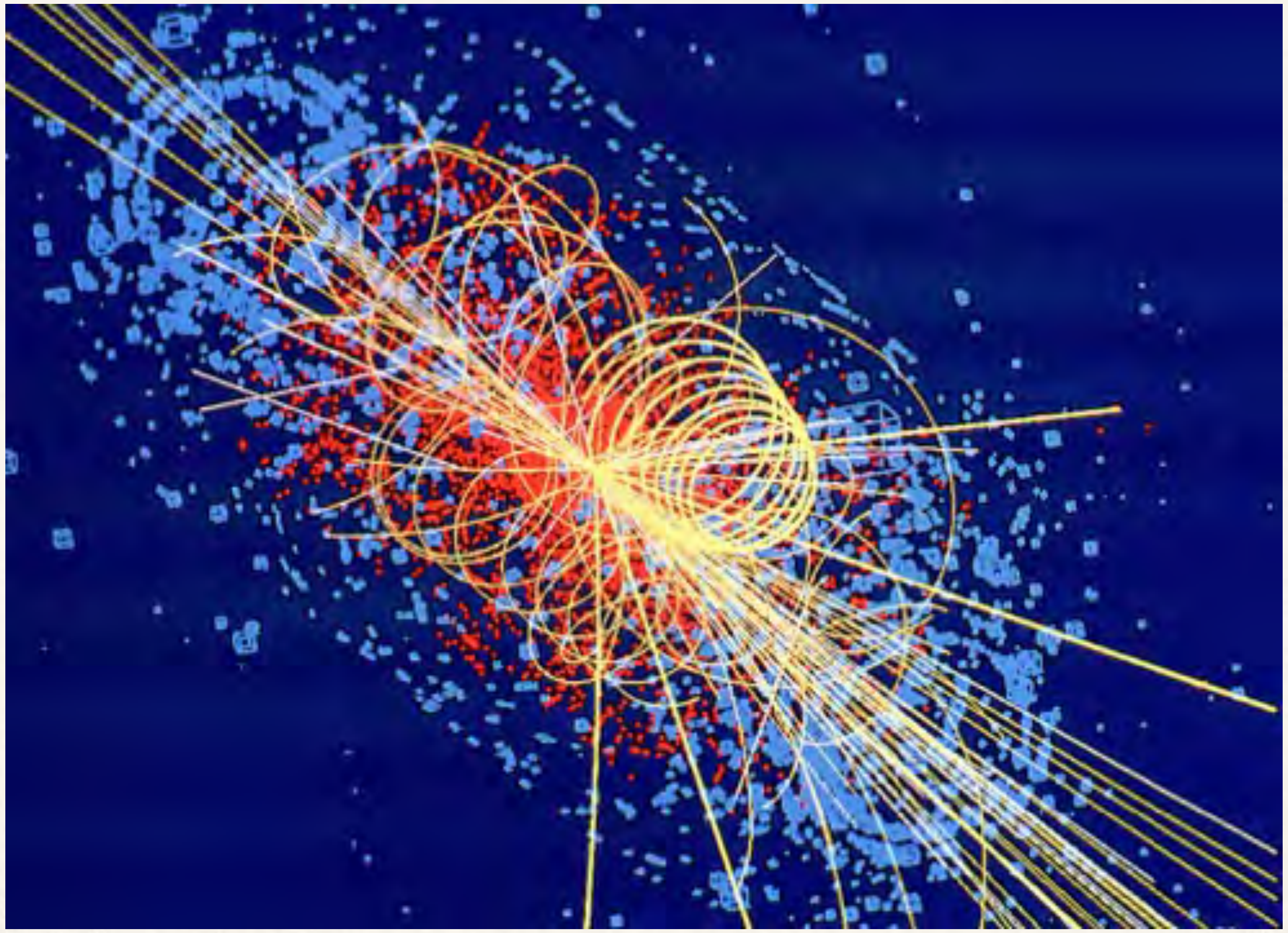
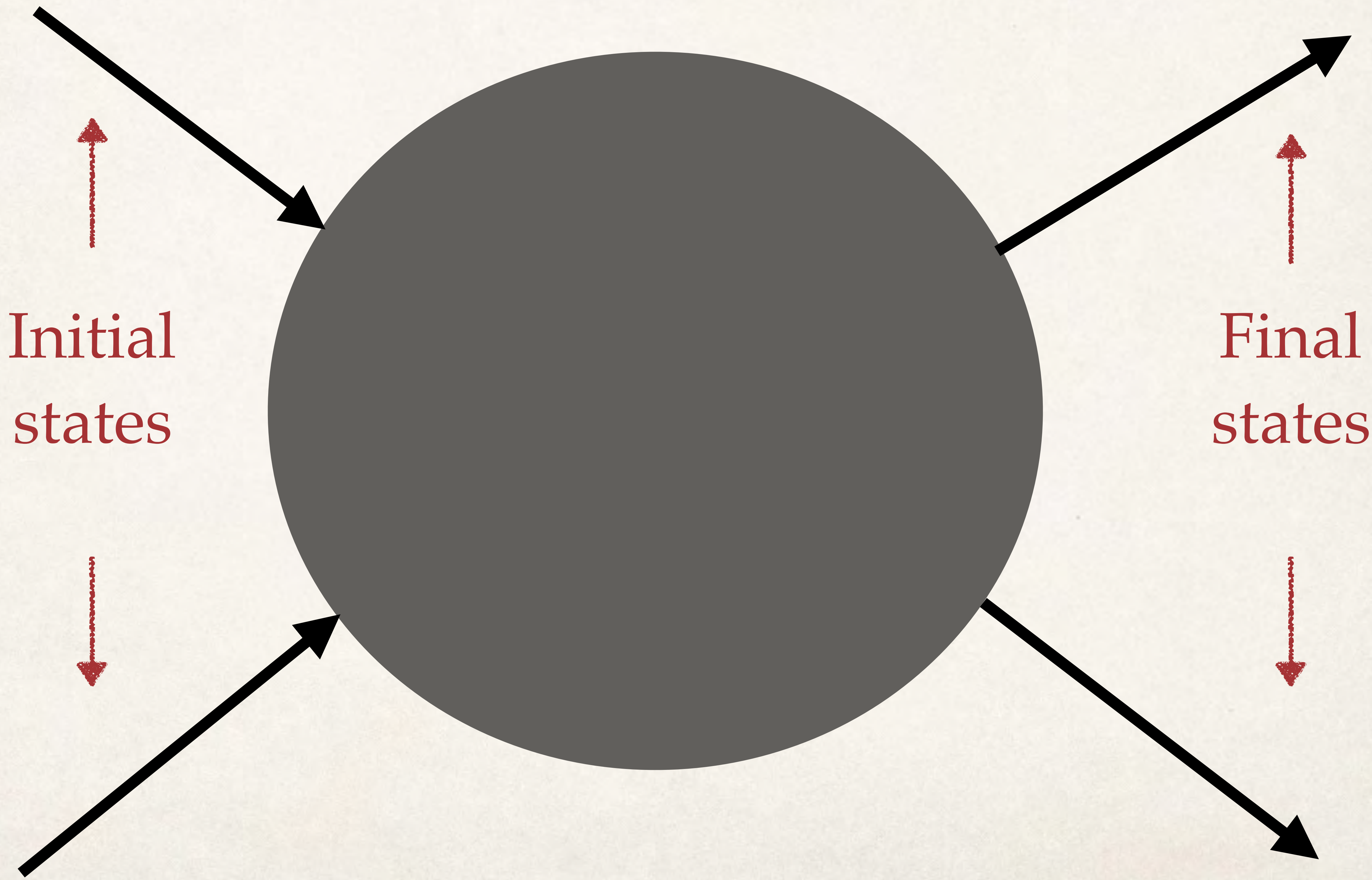
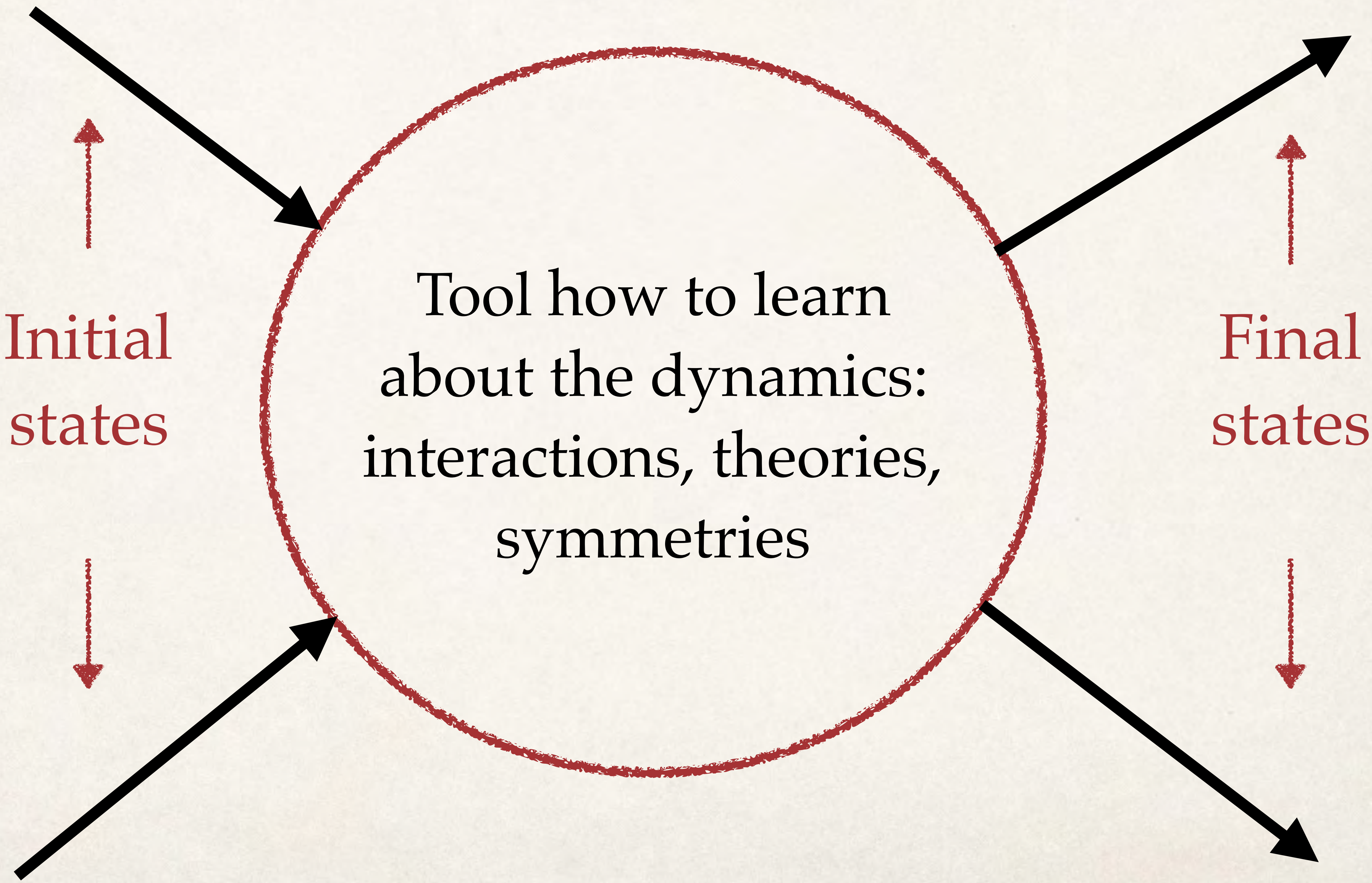
Particle experiments:
our probe to fundamental laws of Nature



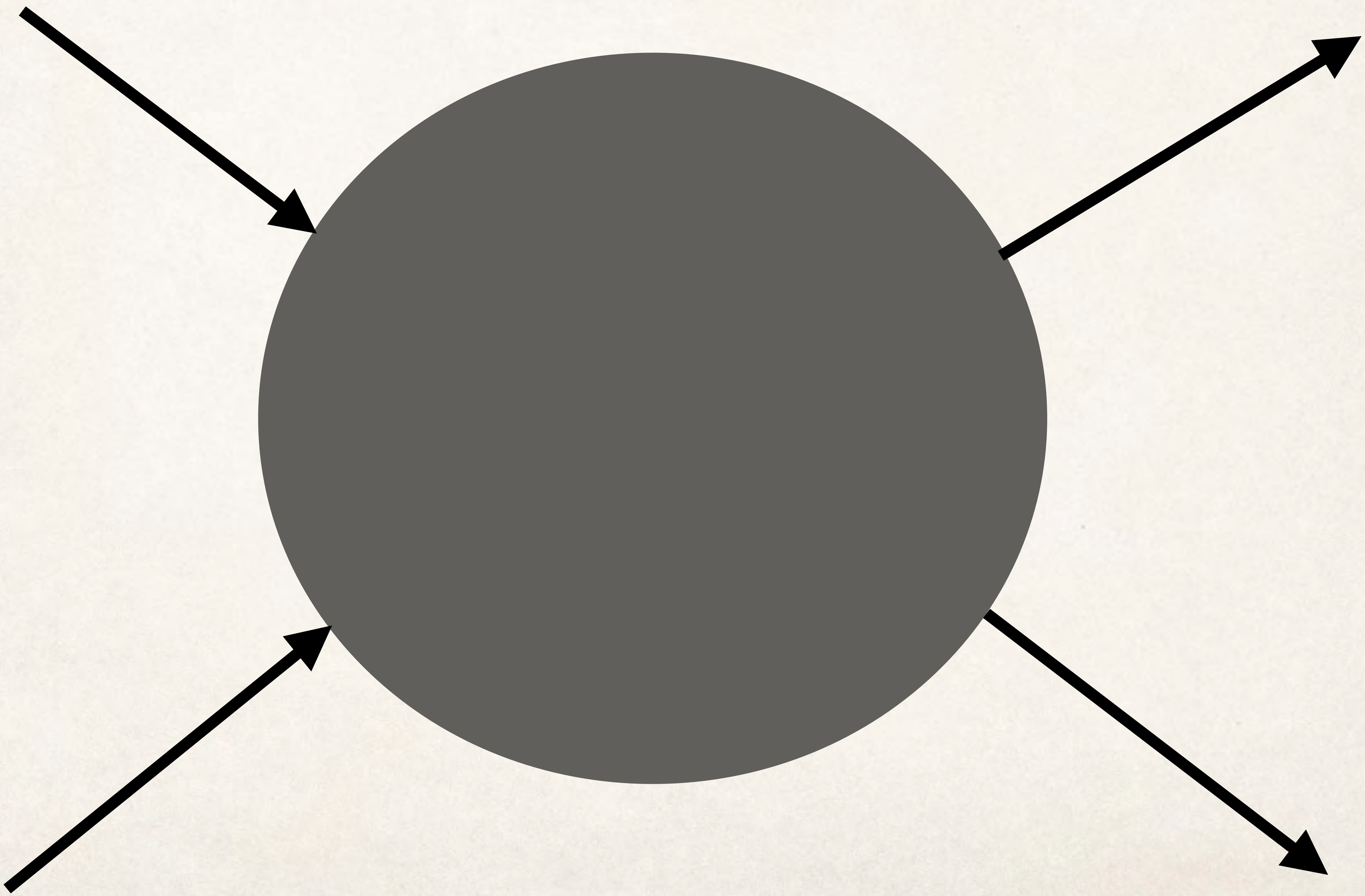
Theorist's perspective: scattering amplitude



Phenomenology



What does the blob really represent?



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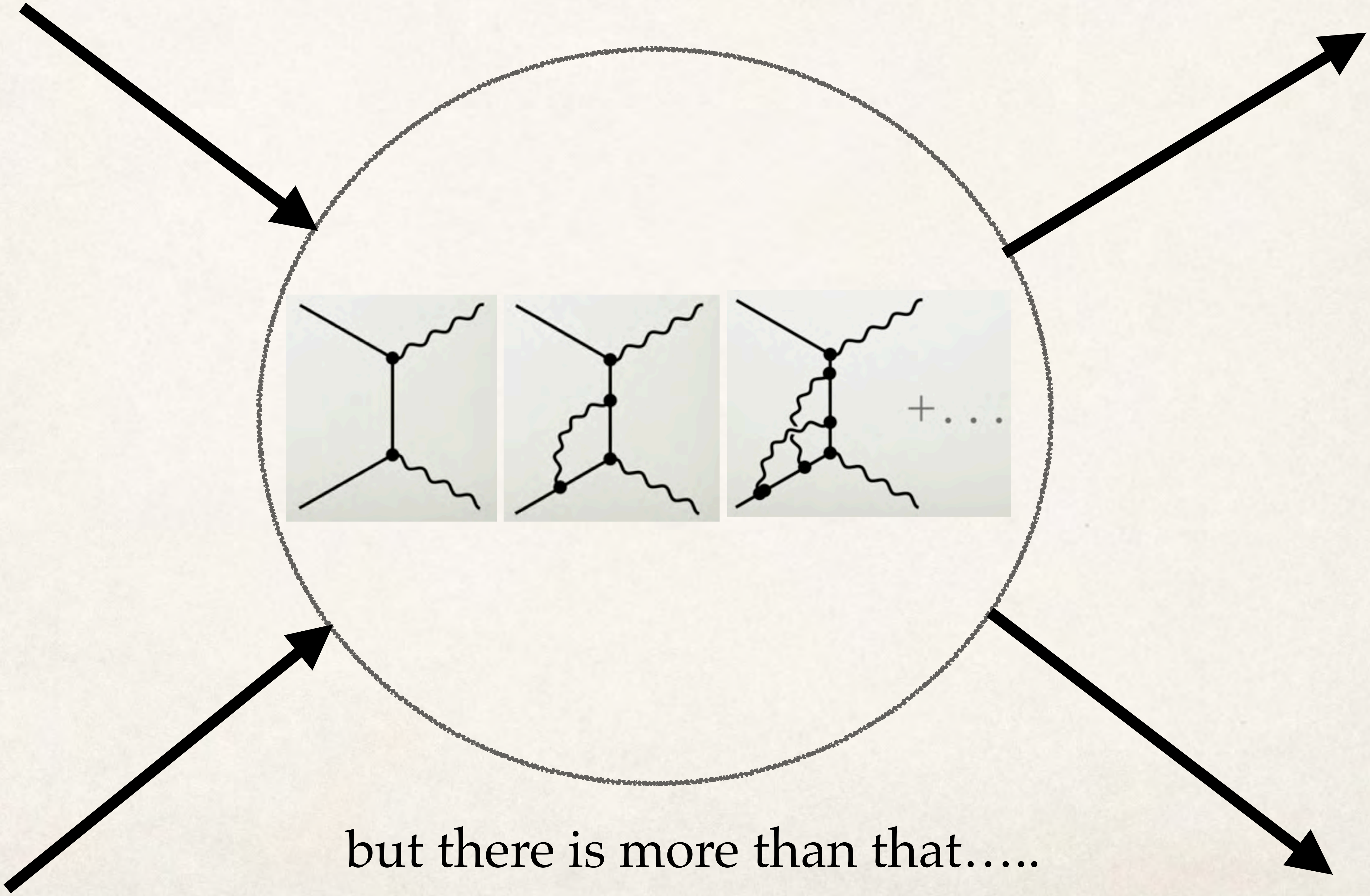
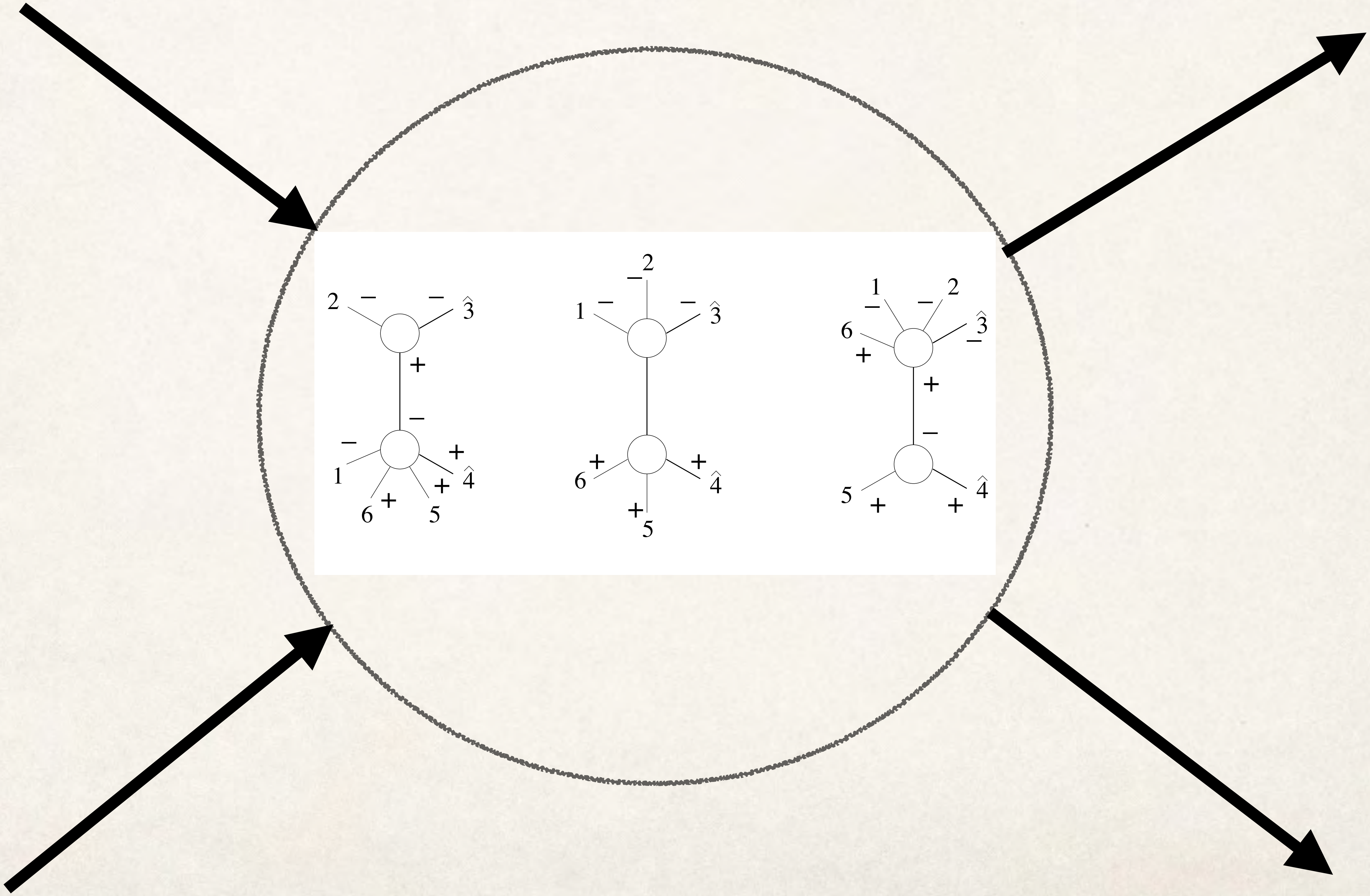
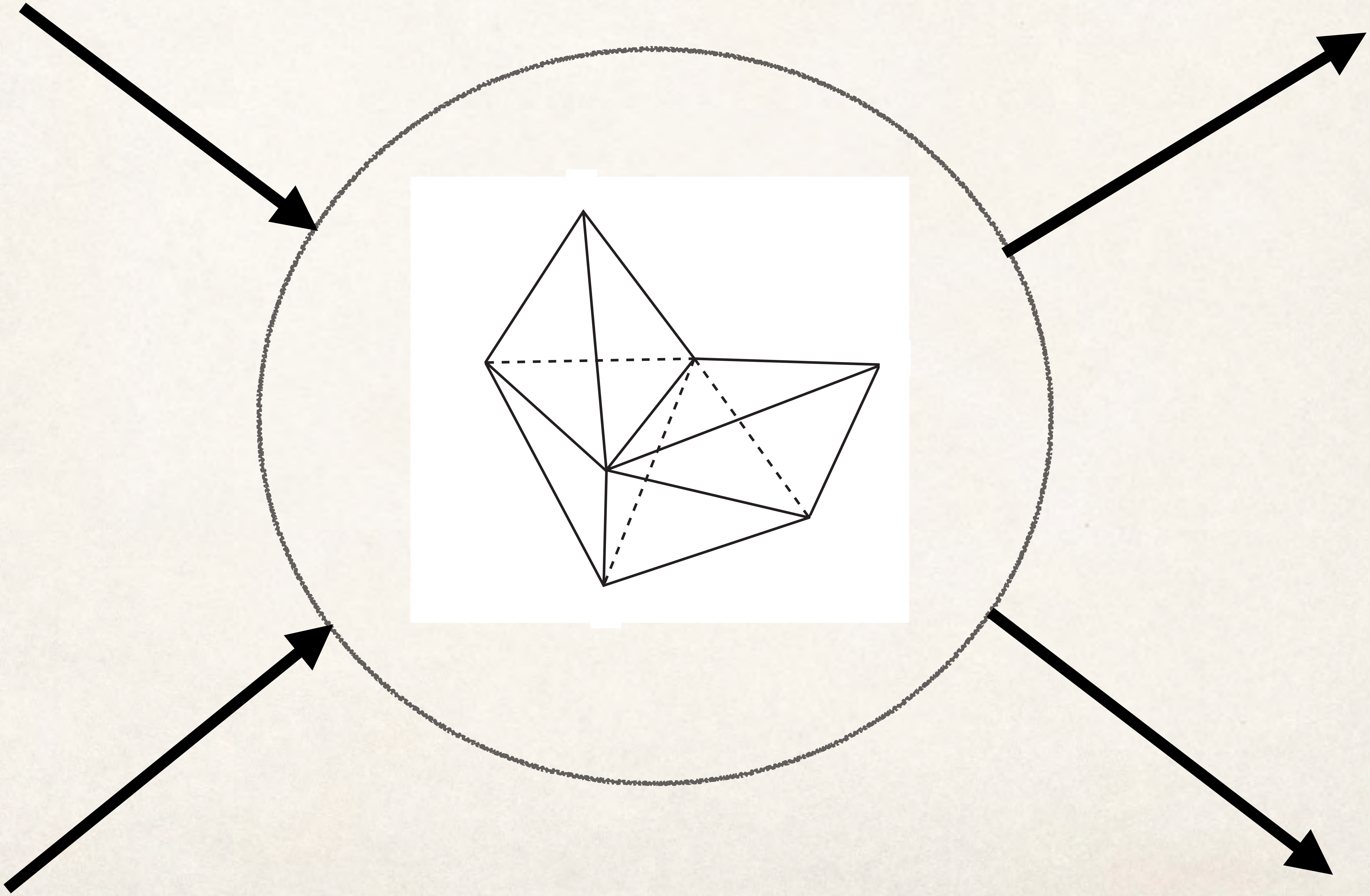


but there is more than that…..

It can be for example a sum of different pictures



And in a special case
even something more surprising



Overview of lectures

- Lecture 1: Review of scattering amplitudes
  - Motivation
  - On-shell amplitudes
  - Kinematics of massless particles
- Lecture 2: New methods for amplitudes
  - Recursion relations for tree-level amplitudes
  - Unitarity methods for loop amplitudes
  - On-shell diagrams
- Lecture 3: Geometric formulation
  - Toy model: N=4 SYM theory
  - Positive Grassmannian
  - Amplituhedron

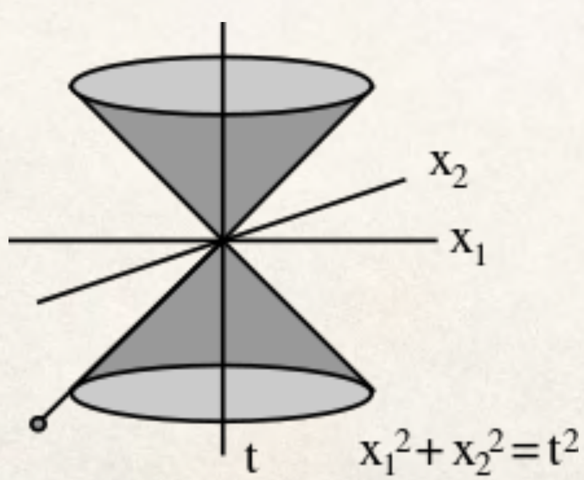
Motivation

Quantum Field Theory (QFT)

- Our theoretical framework to describe Nature

- Compatible with two principles

Special relativity	Quantum mechanics



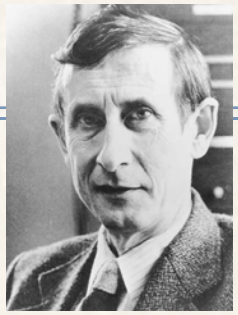
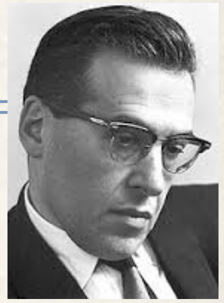
$$H(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

Perturbative QFT

(Dirac, Heisenberg, Pauli; Feynman, Dyson, Schwinger)



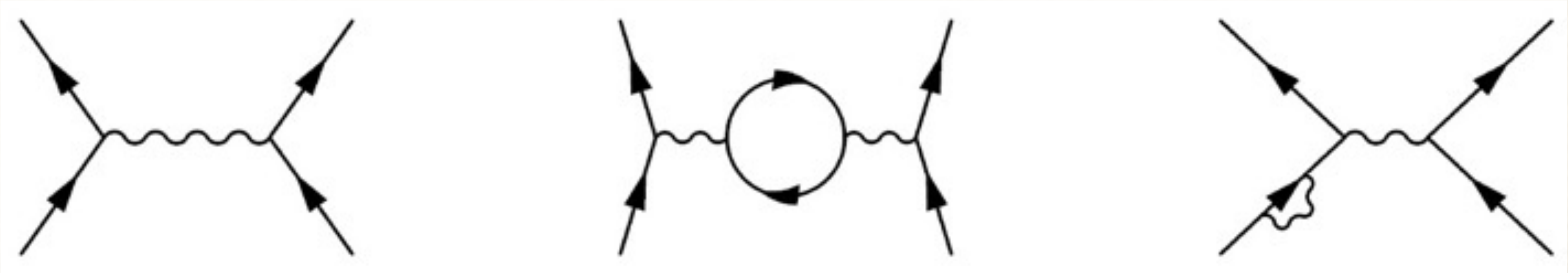


- Fields, Lagrangian, Path integral

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\not{D}\psi - m\overline{\psi}\psi \qquad \int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\overline{\psi}\; e^{iS(A,\psi,\overline{\psi},J)}$$

- Feynman diagrams: pictures of particle interactions
  Perturbative expansion: trees, loops



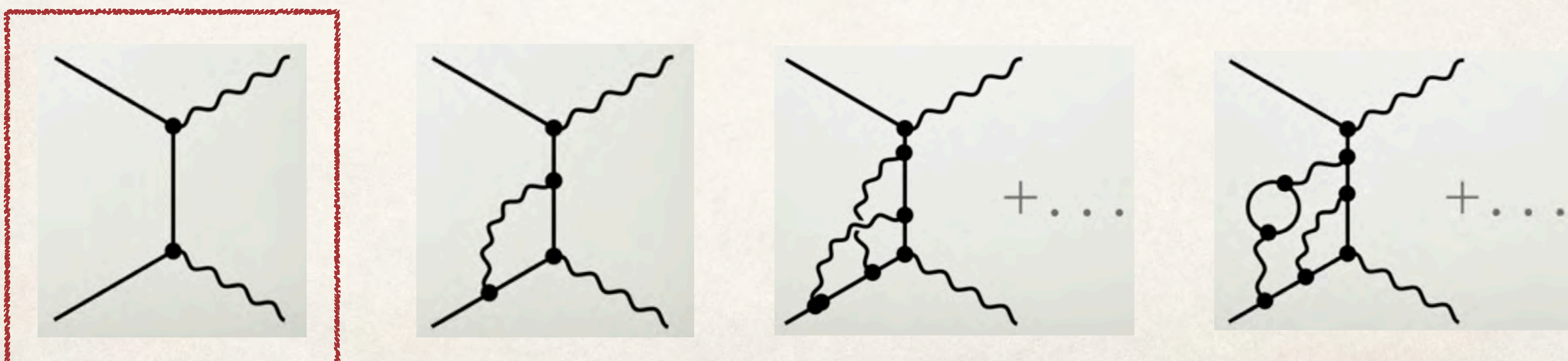
Great success of QFT

- QFT has passed countless tests in last 70 years
- Example: Magnetic dipole moment of electron

1928

Theory: $g_e = 2$

Experiment: $g_e \sim 2$



Great success of QFT

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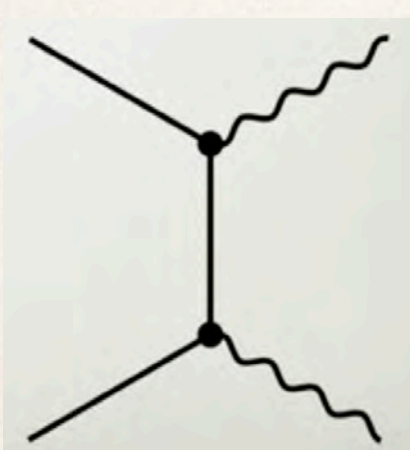
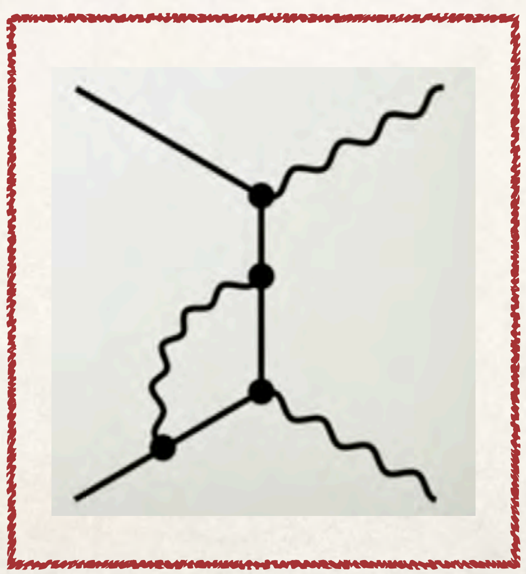
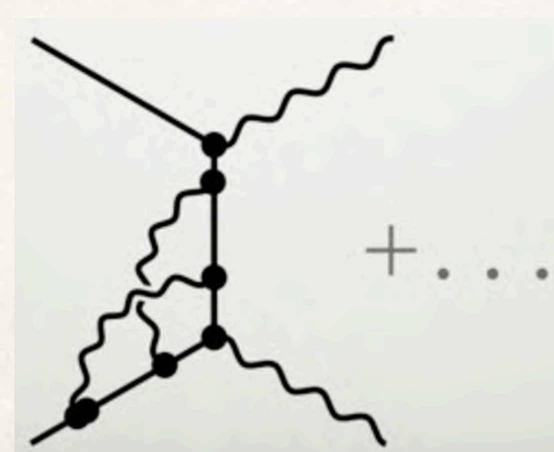
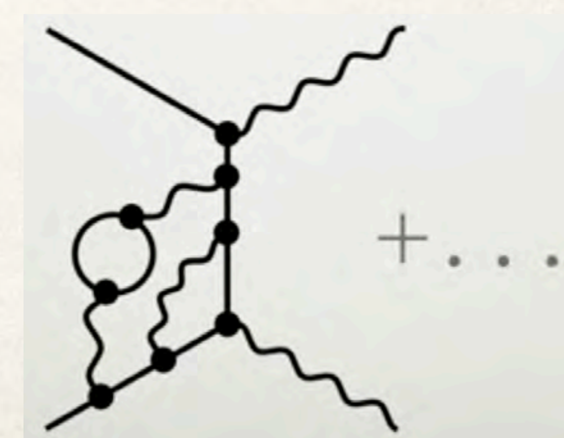
1947

Theory: $g_e = 2.00232$

Experiment: $g_e = 2.0023$

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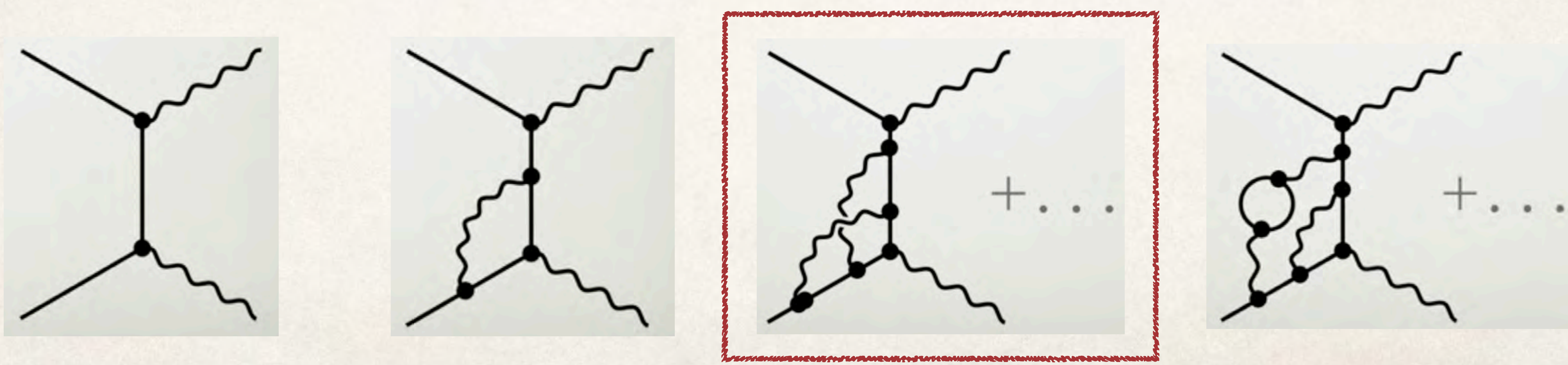
1957 Theory: $g_e = 2.0023193$

1972 Experiment: $g_e = 2.00231931$



Great success of QFT

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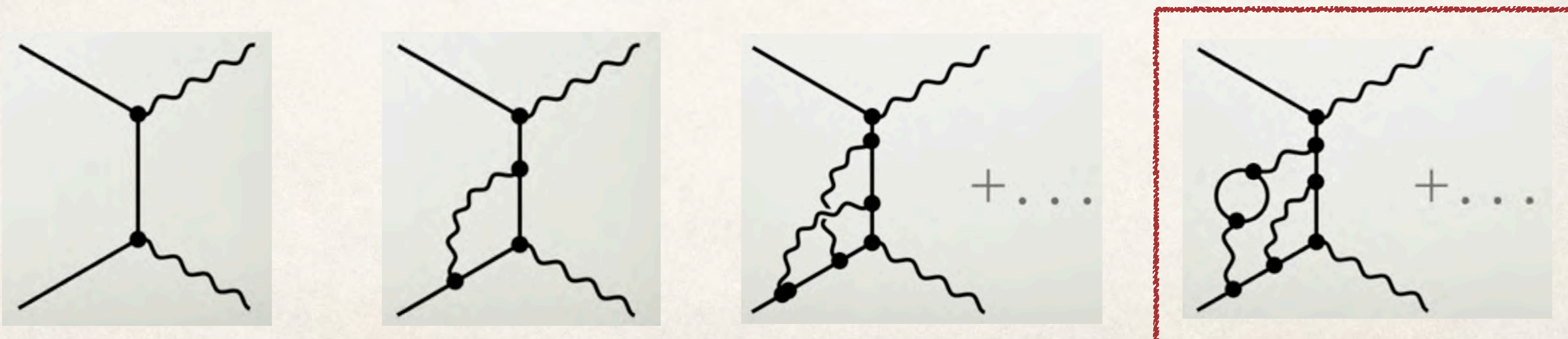
- Example: Magnetic dipole moment of electron

1990

Theory: $g_e = 2.0023193044$

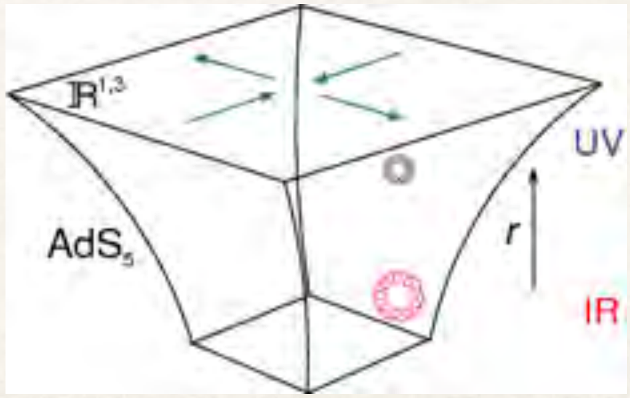
Experiment: $g_e = 2.00231930438$



Dualities

- At strong coupling: perturbative expansion breaks

- Surprises: dual to weakly coupled theory
  - Gauge-gauge dualities
    (Montonen-Olive 1977, Seiberg-Witten 1994)
  - Gauge-gravity duality
    (Maldacena 1997)



Incomplete picture

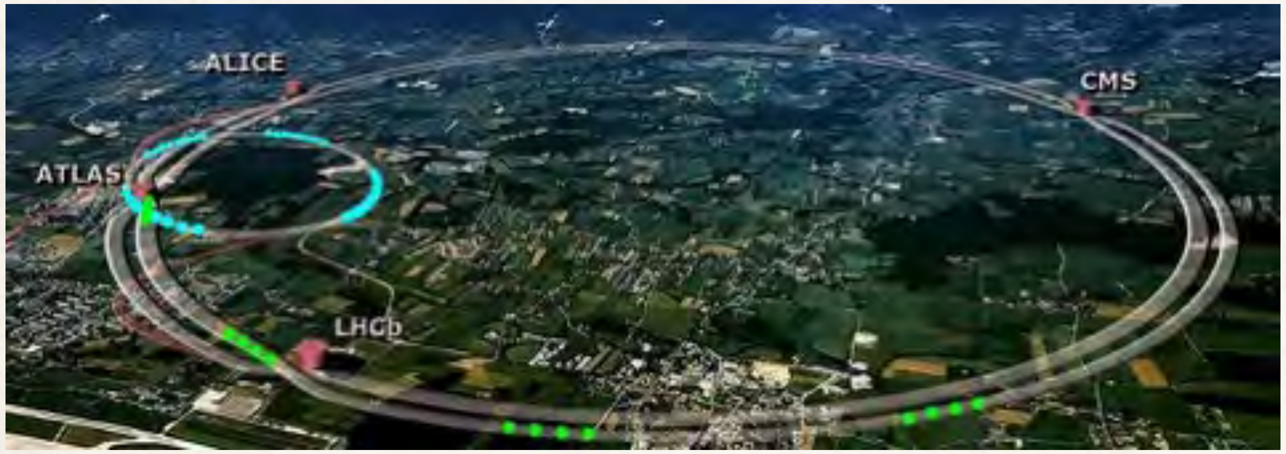
- Our picture of QFT is incomplete
- Also, tension with gravity and cosmology

> If there is a new way of thinking about QFT,
> it must be seen even at weak coupling

- Explicit evidence: scattering amplitudes

Colliders at high energies

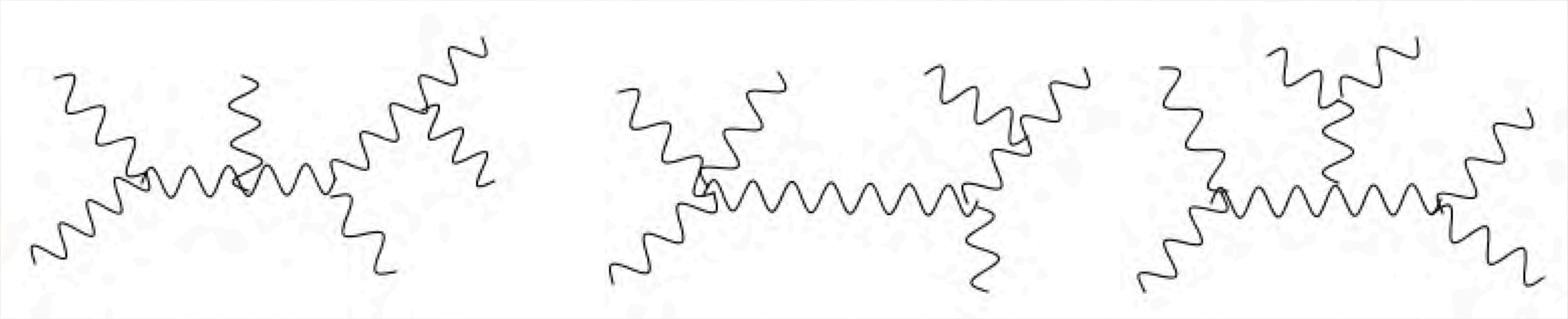
- Proton scattering at high energies



LHC - gluonic factory

- Needed: amplitudes of gluons for higher multiplicities

$$gg \to gg \dots g$$



Early 80s

- Status of the art: $gg \to ggg$

Brute force calculation
24 pages of result

$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$

New collider

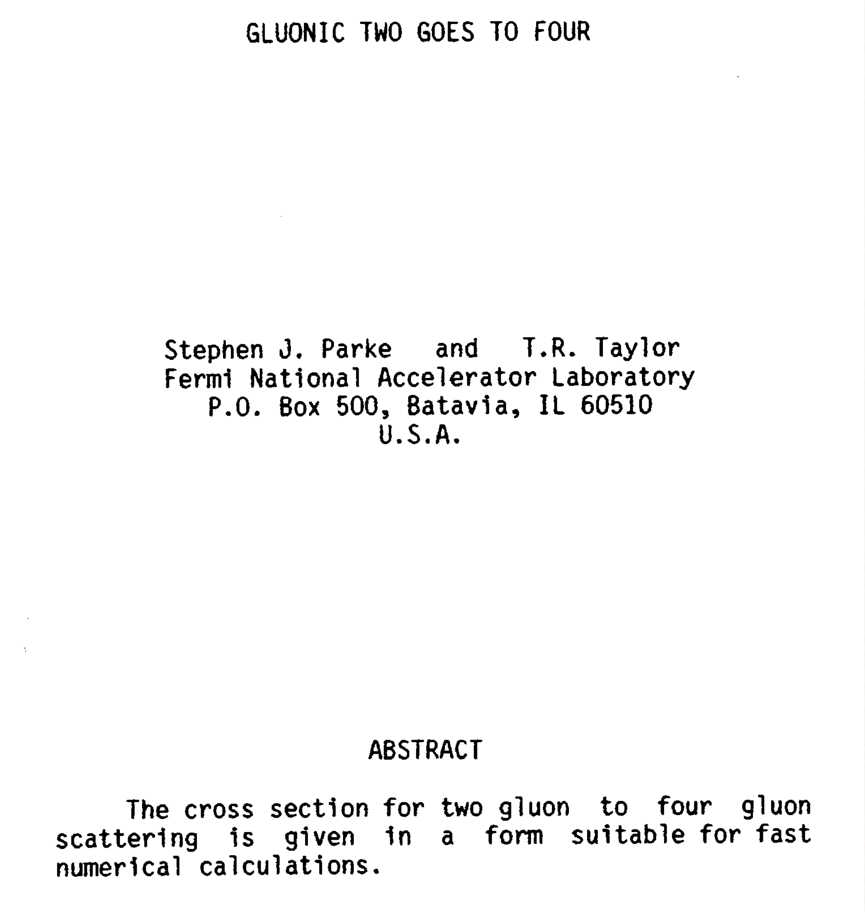
- 1983: Superconducting Super Collider approved
- Energy 40 TeV: many gluons!

- Demand for calculations, next on the list: $gg \to gggg$

Parke-Taylor formula

(Parke, Taylor 1985)

- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations
- 1985: Paper with 14 pages of result



GLUONIC TWO GOES TO FOUR

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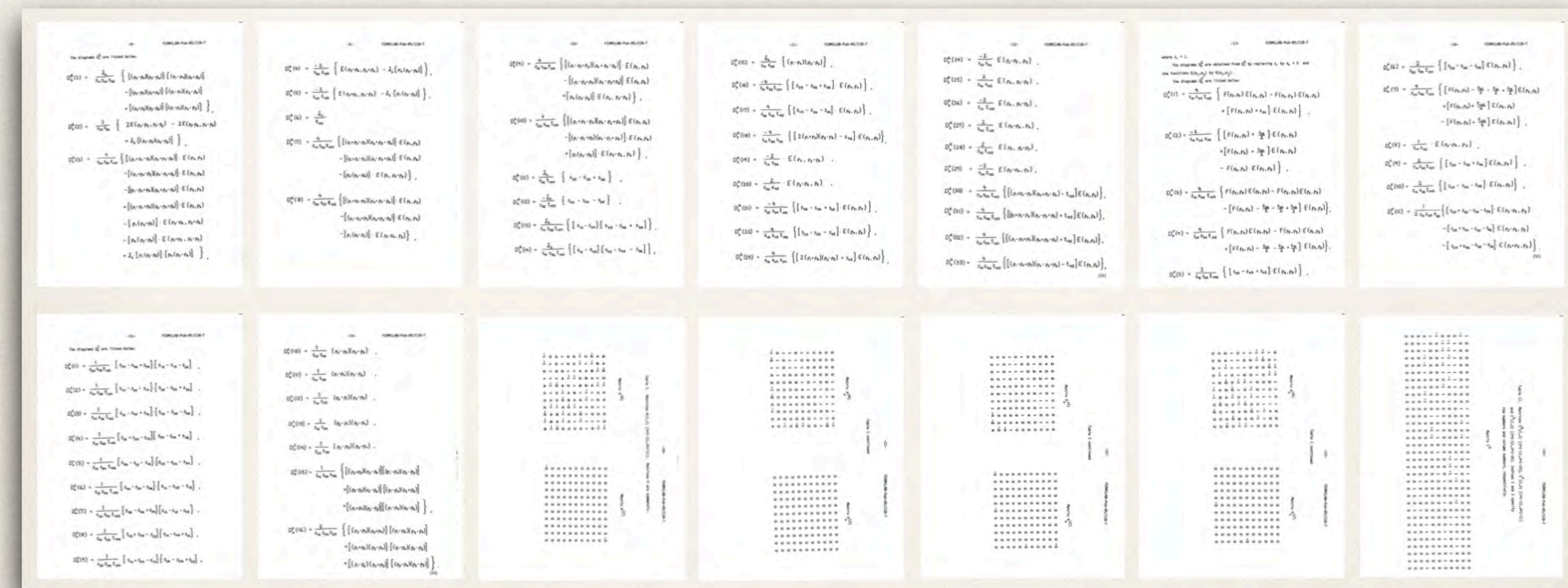
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.

Parke-Taylor formula

- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations



Parke-Taylor formula

Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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- Within a year they realized

$$\mathcal{M}_6 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}$$

Spinor-helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \tilde{\lambda}_{\dot{a}}$$
$$\langle 12 \rangle = \epsilon_{ab} \lambda^{(1)}_a \lambda^{(2)}_b$$
$$[12] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}^{(1)}_{\dot{a}} \tilde{\lambda}^{(2)}_{\dot{b}}$$

(Mangano, Parke, Xu 1987)

Parke-Taylor formula

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- Within a year they realized

$$\mathcal{M}_n = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\ldots\langle n1\rangle}$$

AN AMPLITUDE FOR n GLUON SCATTERING

STEPHEN J. PARKE and T. R. TAYLOR

Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510.

Problems with Feynman diagrams

- Particles on internal lines are not real
  - Individual diagrams not gauge invariant
- Obscure simplicity of the final answer
  - Most of the terms in each diagram cancels
- Lesson: work with gauge invariant quantities with fixed spin structure

Birth of amplitudes

- New field in theoretical particle physics

New methods and effective calculations

Uncovering new structures in QFT

"Road map"

Explicit calculation

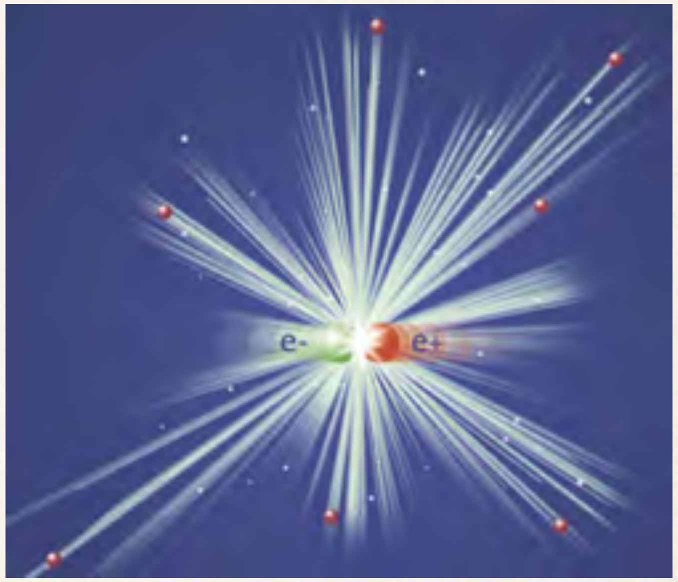
New structure discovered

New method which exploits it

What are scattering amplitudes

Scattering process

- Interaction of elementary particles
- Initial state $|i\rangle$ and final state $|f\rangle$
- Scattering amplitude $\mathcal{M}_{if} = \langle i|f\rangle$
- Example: $e^+e^- \to e^+e^-$ or $e^+e^- \to \gamma\gamma$ etc.
- Cross section: $\sigma = \int |\mathcal{M}|^2 \, d\Omega$ probability



Scattering amplitude in QFT

- Scattering amplitude depends on types of particles and their momenta

$$\mathcal{M}_{if} = F(p_i, s_i)$$

- Theoretical framework: calculated in some QFT

- Specified by Lagrangian: interactions and couplings

$$\mathcal{L} = \mathcal{L}(\mathcal{O}_j, g_k)$$

- Example: QED

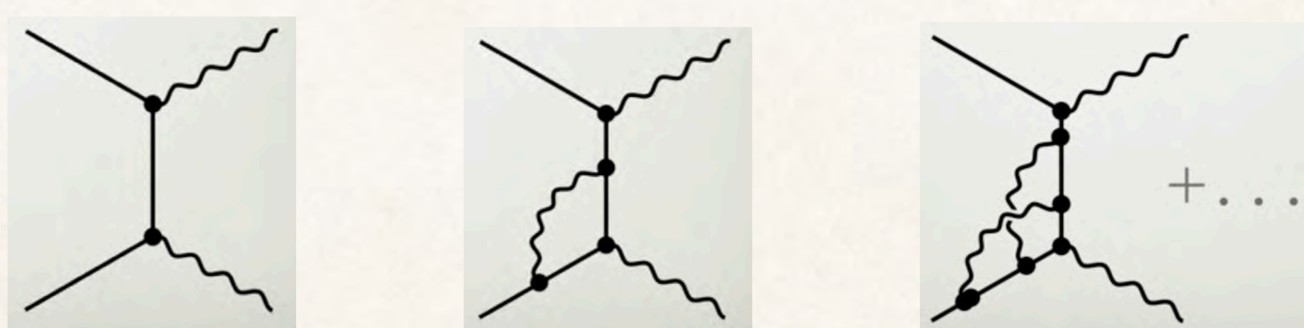
$$\mathcal{L}_{int} = e\,\overline{\psi}\gamma_\mu\psi A^\mu$$

Perturbation theory

- Weakly coupled theory

$$\mathcal{M} = \mathcal{M}_0 + g\,\mathcal{M}_1 + g^2\,\mathcal{M}_2 + g^3\,\mathcal{M}_3 + \ldots$$

- Representation in terms of Feynman diagrams

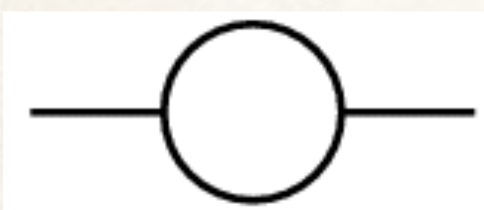


- Perturbative expansion = loop expansion

$$\mathcal{M} = \mathcal{M}^{tree} + \mathcal{M}^{1-loop} + \mathcal{M}^{2-loop} + \ldots$$

Divergencies

- Loop diagrams are generally UV divergent



$$\sim \int_{-\infty}^{\infty} \frac{d^4\ell}{(\ell^2+m^2)[(\ell+p)^2+m^2]} \sim \log\Lambda$$

- IR divergencies: physical effects, cancel in cross section

- Dimensional regularization: calculate integrals in $4+\epsilon$ dimensions

Divergencies $\sim \dfrac{1}{\epsilon^k}$

Renormalizable theories

- Absorb UV divergencies: counter terms
  - Finite number of them: renormalizable theory
  - Infinite number: non-renormalizable theory
- Mostly only renormalizable theories are interesting
- Exceptions: effective field theories

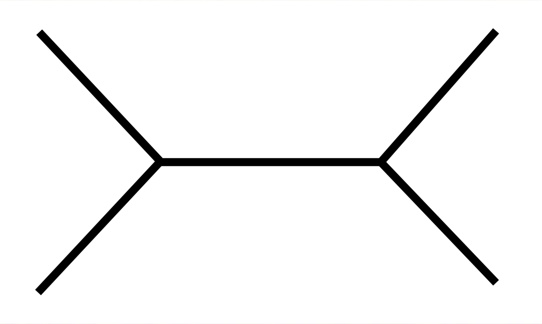
Example: Chiral perturbation theory - derivative expansion

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \dots$$

Different loop orders are mixed

Analytic structure of amplitudes

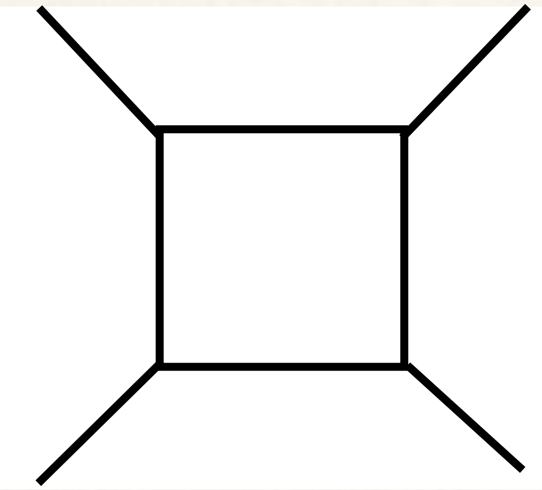
- Tree-level: rational functions



$$\sim \frac{g^2}{(p_1+p_2)^2}$$

Only poles

- Loops: polylogarithms and more complicated functions



$$\sim \log^2(s/t)$$

Branch cuts

Kinematics of massless particles

Massless particles

- Parameters of elementary particles of spin S
  - Spin $s = (-S, S)$
  - Mass m
  - Momentum p^μ

On-shell (physical) particle

$$p^2 = m^2$$

- Massless particle: $m = 0$ $p^2 = 0$

spin = helicity: only two extreme values $h = \{-S, S\}$

Example: photon $h = (+, -)$

$s = 0$ missing

Spin functions

- At high energies particles are massless

  Fundamental laws reveal there

- Spin degrees of freedom: spin function

  - s=0: Scalar - no degrees of freedom
  - s=1/2: Fermion - spinor u
  - s=1: Vector - polarization vector ϵ^{μ}
  - s=2: Tensor - polarization tensor $h^{\mu\nu}$

Spin functions

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Polarization vectors

- Spin 1 particle is described by vector ϵ^{μ}

2 degrees of freedom

4 degrees of freedom

- Null condition: $\epsilon \cdot \epsilon^{*} = 0$ 3 degrees of freedom left

- We further impose: $\epsilon \cdot p = 0$

Identification

$$\epsilon_{\mu} \sim \epsilon_{\mu} + \alpha p_{\mu}$$

Feynman diagrams depend on α

gauge dependence

Spinor helicity variables

- Standard SO(3,1) notation for momentum

$$p^\mu = (p_0, p_1, p_2, p_3) \qquad\qquad p_j \in \mathbb{R}$$

$$p^2 = p_0^2 + p_1^2 + p_2^2 - p_3^2$$

- We use SL(2,C) representation

$$p_{ab} = \sigma^\mu_{ab} p_\mu = \begin{pmatrix} p_0 + ip_1 & p_2 + p_3 \\ p_2 - p_3 & p_0 - ip_1 \end{pmatrix}$$

On-shell: $p^2 = \det(p_{ab}) = 0$

$$\boxed{\mathrm{Rank}\,(p_{ab}) = 1}$$

Spinor helicity variables

- We can then write $p_{ab} = \lambda_a \kappa_b$
- SL(2,C): dotted notation $\boxed{p_{a\dot{b}} = \lambda_a \widetilde{\lambda}_{\dot{b}}}$

  $\widetilde{\lambda}$ is complex conjugate of λ

- Little group transformation

$$\lambda \to t\lambda \qquad \widetilde{\lambda} \to \frac{1}{t}\widetilde{\lambda}$$

leaves momentum unchanged $p \to p$

3 degrees of freedom

Spinor helicity variables

- Momentum invariant $(p_1 + p_2)^2 = (p_1 \cdot p_2)$

$$p_1^\mu = \sigma^\mu_{a\dot{a}}\, \lambda_{1a}\widetilde{\lambda}_{1\dot{a}} \qquad p_2^\mu = \sigma^\mu_{b\dot{b}}\, \lambda_{2b}\widetilde{\lambda}_{2\dot{b}}$$

- Plugging for momenta

$$(p_1 \cdot p_2) = (\sigma^\mu_{a\dot{a}}\sigma_{\mu\, b\dot{b}})\, (\lambda_{1a}\lambda_{2b})(\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}})$$

$$\epsilon_{ab}\, \epsilon_{\dot{a}\dot{b}} \qquad = (\epsilon_{ab}\lambda_{1a}\lambda_{2b})(\epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}})$$

Define: $\boxed{\langle 12 \rangle \equiv \epsilon_{ab}\lambda_{1a}\lambda_{2b}}$ $\boxed{[12] \equiv \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}}}$

Invariant products

- Momentum invariant

$$s_{ij} = (p_i + p_j)^2 = \langle ij \rangle [ij]$$

Square brackets

Angle brackets $\langle ij \rangle = \epsilon_{ab} \lambda_{ia} \lambda_{jb}$ $\qquad$ $[ij] = \epsilon_{\dot{a}\dot{b}} \widetilde{\lambda}_{i\dot{a}} \widetilde{\lambda}_{j\dot{b}}$

- Antisymmetry

$$\langle 21 \rangle = -\langle 12 \rangle \qquad [21] = -[12]$$

- More momenta

$$(p_1 + p_2 + p_3)^2 = \langle 12 \rangle [12] + \langle 23 \rangle [23] + \langle 13 \rangle [13]$$

Invariant products

- Shouten identity

$$\langle 13\rangle\langle 24\rangle = \langle 12\rangle\langle 34\rangle + \langle 14\rangle\langle 23\rangle$$

- Mixed brackets

$$\langle 1|2+3|4] \equiv \langle 12\rangle[24] + \langle 13\rangle[34]$$

- Momentum conservation

$$\sum_{i=1}^{n} \lambda_{ia}\widetilde{\lambda}_{i\dot{a}} = 0$$

Non-trivial conditions:
Quadratic relation
between components

Polarization vectors

- Two polarization vectors

$$\epsilon_+^\mu = \sigma^\mu_{a\dot{a}} \frac{\eta_a \widetilde{\lambda}_{\dot{a}}}{\langle \eta \lambda \rangle} \qquad \epsilon_-^\mu = \sigma^\mu_{a\dot{a}} \frac{\lambda_a \widetilde{\eta}_{\dot{a}}}{[\widetilde{\eta}\, \widetilde{\lambda}]}$$

where $\eta, \widetilde{\eta}$ are auxiliary spinors

Note that

$$(\epsilon_+ \cdot \epsilon_-) = 1$$

- Freedom in choice of $\eta, \widetilde{\eta}$ corresponds to

$$\epsilon^\mu \sim \epsilon^\mu + \alpha\, p^\mu$$

- Gauge redundancy of Feynman diagrams

Scaling of amplitudes

- Consider some amplitude $A(-+--+\cdots-)$

$$A = (\epsilon_1 \epsilon_2 \ldots \epsilon_n) \cdot Q$$

depends only on momenta

- Little group scaling

$$\lambda \to t\lambda \qquad \widetilde{\lambda} \to \frac{1}{t}\widetilde{\lambda}$$

$$\longrightarrow$$

$$p \to p \qquad \epsilon_+ \to \frac{1}{t^2} \cdot \epsilon_+ \qquad \epsilon_- \to t^2 \cdot \epsilon_-$$

$$\longrightarrow$$

$$A(i^-) \to t^2 \cdot A(i^-) \qquad A(i^+) \to \frac{1}{t^2} \cdot A(i^+)$$

Back to Parke-Taylor formula

- Let us consider $\quad A(1^-2^-3^+4^+5^+6^+)$
- Scaling $\quad A\left(t\lambda_i, \frac{1}{t}\widetilde{\lambda}_i\right) = t^2 \cdot A(\lambda_i, \widetilde{\lambda}_i) \quad \text{for particles 1,2}$

$$A\left(t\lambda_i, \frac{1}{t}\widetilde{\lambda}_i\right) = \frac{1}{t^2} \cdot A(\lambda_i, \widetilde{\lambda}_i) \quad \text{for particles 3,4,5,6}$$

- Check for explicit expression

$$A_6 = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

If only $\langle ij\rangle$ allowed the form is unique

Helicity amplitudes

- In Yang-Mills theory we have + or - "gluons"

$$A_6(1^-2^-3^+4^+5^+6^+)$$

- We denote k: number of - helicity gluons

- Some amplitudes are zero

$$A_n(++ + \cdots +) = 0$$
$$A_n(- + + \cdots +) = 0$$
$$A_n(- - - \cdots -) = 0$$
$$A_n(+ - - \cdots -) = 0$$

First non-trivial: k=2

$$A_n(- - + \cdots +)$$

Parke-Taylor formula for tree level

Three point amplitudes

Three point kinematics

- Gauge invariant building blocks: on-shell amplitudes

$$p_1^2 = p_2^2 = p_3^2 = 0 \qquad\qquad p_1 + p_2 + p_3 = 0$$

- Plugging second equation into the first

$$(p_1 + p_2)^2 = (p_1 \cdot p_2) = 0$$

- Similarly we get for other pairs

$$(p_1 \cdot p_2) = (p_1 \cdot p_3) = (p_2 \cdot p_3) = 0$$

These momenta are very constrained!

Three point kinematics

- Use spinor helicity variables trivializes on-shell condition

$$p_1 = \lambda_1 \widetilde{\lambda}_1, \quad p_2 = \lambda_2 \widetilde{\lambda}_2, \quad p_3 = \lambda_3 \widetilde{\lambda}_3$$

- The mutual conditions then translate to

$$(p_1 \cdot p_2) = \langle 12 \rangle [12] = 0$$

- And similarly for other two pairs

$$(p_1 \cdot p_3) = \langle 13 \rangle [13] = 0 \qquad\qquad (p_2 \cdot p_3) = \langle 23 \rangle [23] = 0$$

Two solutions

- We want to solve conditions

$$\langle 12\rangle[12] = \langle 13\rangle[13] = \langle 23\rangle[23] = 0$$

- Solution 1: $\langle 12\rangle = 0$ which implies $\lambda_2 = \alpha\lambda_1$

  Then we also have $\langle 23\rangle = \alpha\langle 13\rangle$

  And we set $\langle 13\rangle = 0$ by demanding $\lambda_3 = \beta\lambda_1$

$$\boxed{\lambda_1 \sim \lambda_2 \sim \lambda_3}$$

Two solutions

- Solution 2: $[12] = [23] = [13] = 0$

$$\boxed{\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3}$$

- Let us take this solution

$$p_1 = \lambda_1 \widetilde{\lambda}_1, \quad p_2 = \alpha \lambda_2 \widetilde{\lambda}_1, \quad p_3 = (-\lambda_1 - \alpha\lambda_2)\widetilde{\lambda}_1$$

complex momenta

<u>No solution for real momenta</u>

Three point amplitudes

- Gauge theory: scattering of three gluons (not real)
- Building blocks: $\langle 12\rangle, \langle 23\rangle, \langle 13\rangle, [12], [23], [13]$
- Mass dimension: each term $\sim m$
- Three point amplitude $A_3 \sim \epsilon^3 p \sim p \sim m$

Three point amplitudes

- Two options

$$A_3^{(1)} = \langle 12\rangle^{a_1}\langle 13\rangle^{a_2}\langle 23\rangle^{a_3} \qquad A_3^{(2)} = [12]^{b_1}[13]^{b_2}[23]^{b_3}$$

- Apply to $A_3(1^-,2^-,3^+)$

$$\begin{aligned} A_3(t\lambda_1, t^{-1}\widetilde{\lambda}_1) &= t^{a_1+a_2}\cdot A_3 \\ A_3(t\lambda_2, t^{-1}\widetilde{\lambda}_2) &= t^{a_1+a_3}\cdot A_3 \\ A_3(t\lambda_3, t^{-1}\widetilde{\lambda}_3) &= t^{a_2+a_3}\cdot A_3 \end{aligned} \longrightarrow \begin{aligned} a_1+a_2 &= 2 \\ a_1+a_3 &= 2 \\ a_2+a_3 &= -2 \end{aligned} \longrightarrow \begin{aligned} a_1 &= 3 \\ a_2 &= -1 \\ a_3 &= -1 \end{aligned}$$

Three point amplitudes

- Similarly for $A_3(1^+, 2^+, 3^-)$
- Two fundamental amplitudes

$$\boxed{A_3(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}} \qquad \boxed{A_3(1^+, 2^+, 3^-) = \frac{[12]^3}{[13][23]}}$$

This is true to all orders: just kinematics

Three point amplitudes

- Collect all $(- - +)$ amplitudes

$$A_3(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}$$

$$A_3(1^-, 2^+, 3^-) = \frac{\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle}$$

$$A_3(1^+, 2^-, 3^-) = \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 13\rangle}$$

$$\boxed{\frac{\langle a\, b\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

where a,b are - helicity gluons

- Similarly for $(+ + -)$ amplitudes

$$\boxed{\frac{[ab]^4}{[12][23][31]}}$$

where a,b are + helicity gluons

Three point amplitudes

- Using similar analysis we find for gravity

$$\boxed{\frac{\langle ab\rangle^8}{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}} \qquad \boxed{\frac{[ab]^8}{[12]^2[23]^2[31]^2}}$$

where a,b are - helicity gravitons where a,b are + helicity gravitons

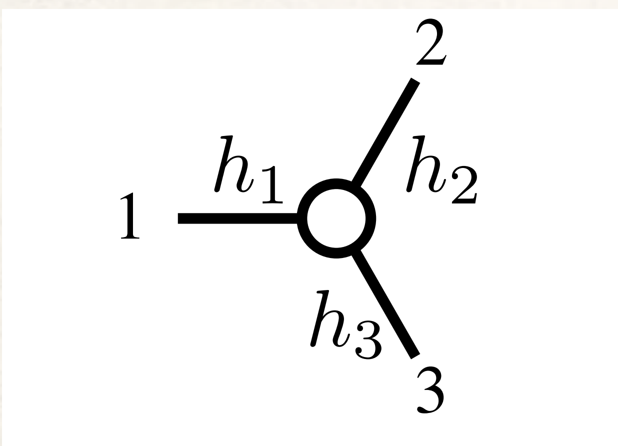
- Note that there is no three-point scattering

<u>They exist only for complex momenta</u>

- Important input into on-shell methods

General 3pt amplitudes

- Two solutions for 3pt kinematics

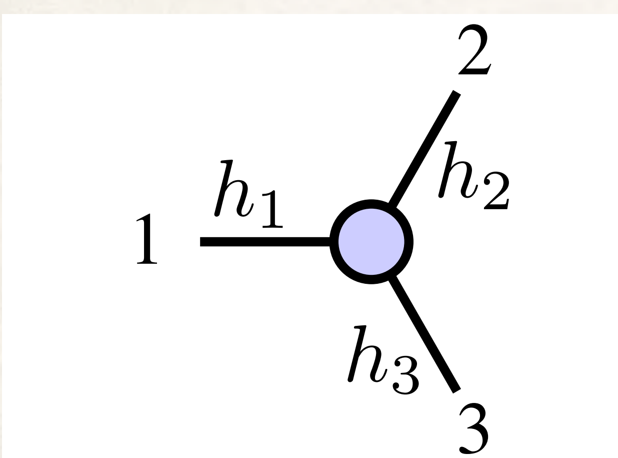


$$\lambda_1 \sim \lambda_2 \sim \lambda_3$$

Under the little group rescaling:

$$A_3(t\lambda_j, t^{-1}\widetilde{\lambda}_j) \sim t^{2h_j} \cdot A_3$$

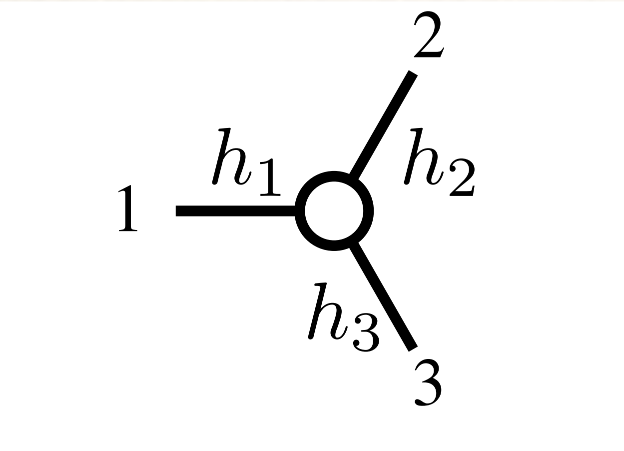
Solve the system of equations



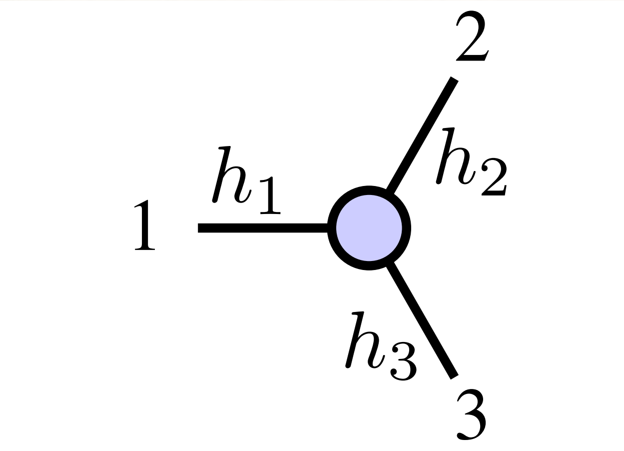
$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

General 3pt amplitudes

- Two solutions for amplitudes



$$A_3 = [12]^{-h_1-h_2+h_3}[23]^{-h_2-h_3+h_1}[31]^{-h_1-h_3+h_2}$$

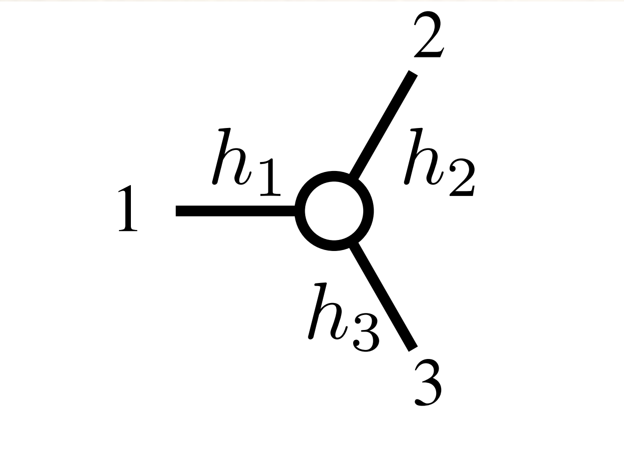


$$A_3 = \langle 12\rangle^{h_1+h_2-h_3}\langle 23\rangle^{h_2+h_3-h_1}\langle 31\rangle^{h_1+h_3-h_2}$$

Which one is correct?

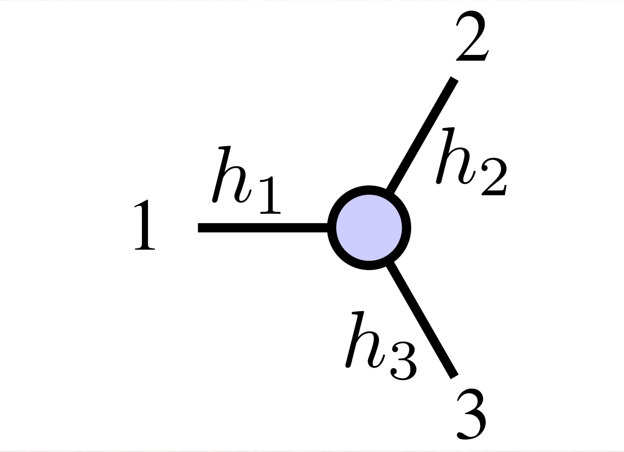
General 3pt amplitudes

- Two solutions for amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

$$h_1 + h_2 + h_3 \leq 0$$



$$A_3 = \langle 12\rangle^{-h_1-h_2+h_3}\langle 23\rangle^{+h_1-h_2-h_3}\langle 31\rangle^{-h_1+h_2-h_3}$$

$$h_1 + h_2 + h_3 \geq 0$$

Mass dimension must be positive!

All spins allowed

- Note that these formulas are valid for any spins
- For example for amplitude $A_3(1^0, 2^{1^+}, 3^{2^+})$

$$A_3 = \frac{\langle 23\rangle\langle 31\rangle^3}{\langle 12\rangle^3}$$

- But we can also do higher spins $A_3(1^{3^+}, 2^{5^+}, 3^{12^-})$

$$A_3 = \frac{\langle 23\rangle^{10}\langle 31\rangle^{14}}{\langle 12\rangle^{20}}$$

- Completely fixed just by kinematics!

Tree-level amplitudes

Feynman diagrams

- Yang-Mills Lagrangian

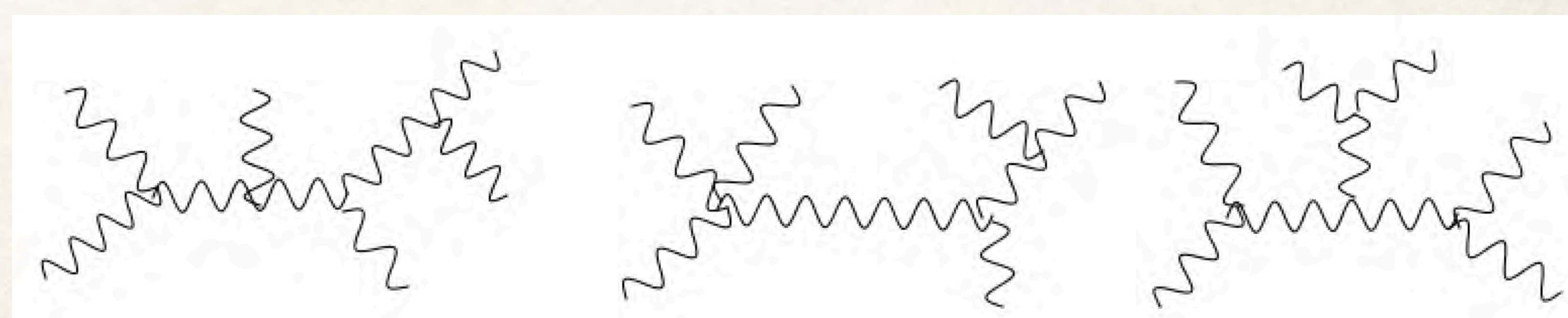
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \sim (\partial A)^2 + A^2\partial A + A^4$$

$$\sim f^{abc}\, g_{\mu\nu}p_\alpha \qquad\qquad \sim f^{abe}f^{cde}g_{\mu\nu}g_{\alpha\beta}$$

- Draw diagrams

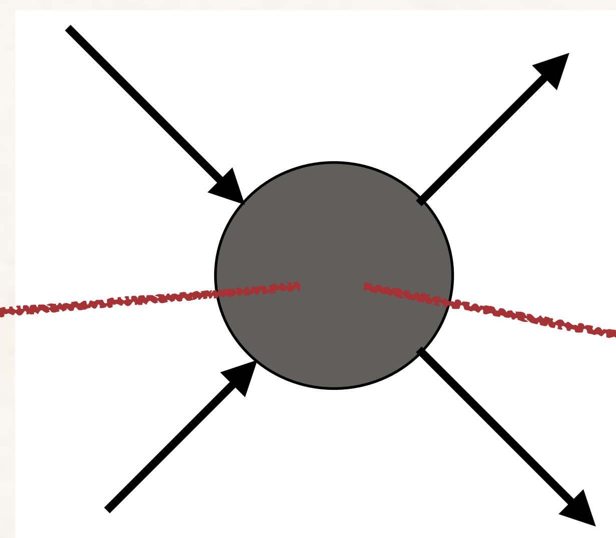
  Feynman rules

  Sum everything

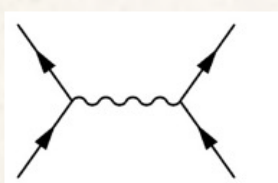


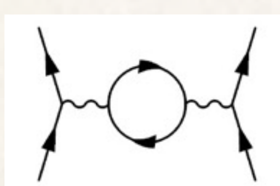
Change of strategy

What is the scattering amplitude?

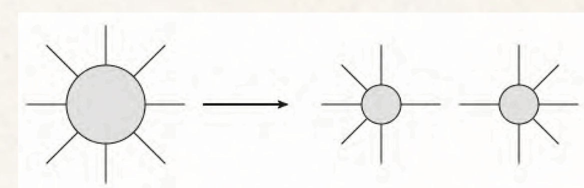


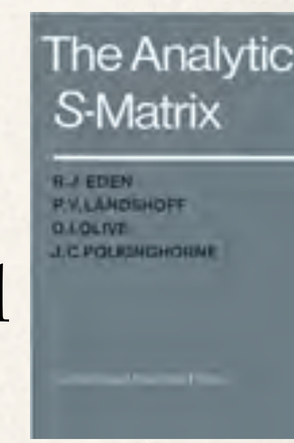
Feynman diagrams





Unique object fixed by physical properties





Was not successful

(1960s)

Modern methods use both:

- Calculate the amplitude directly
- Use perturbation theory

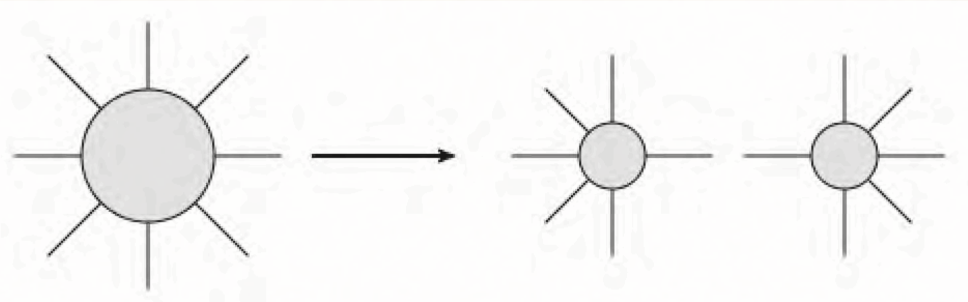
Locality and unitarity

- Only poles: Feynman propagators

Locality

$$\frac{1}{P^2} \quad \text{where} \quad P = \sum_{k \in \mathcal{P}} p_k$$

- On the pole



Unitarity

Feynman diagrams recombine on both sides into amplitudes

$$\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

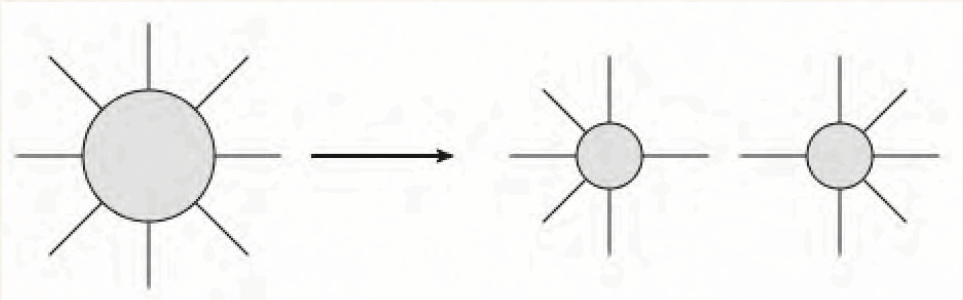
Factorization on the pole

On $P^2 = 0$ $$\boxed{\operatorname{Res} \mathcal{M} = \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R}$$

- For $P^2 = 0$ the internal leg: on-shell physical particle
- Both sub-amplitudes are on-shell, gauge invariant
- On-shell data: statement about on-shell quantities

On-shell constructibility

- Factorization of tree-level amplitudes



$$\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

- On-shell constructibility: factorizations fix the answer

- Write a proposal tree-level amplitude $\widetilde{\mathcal{M}}$
  - On-shell gauge invariant function, correct weights
  - It factorizes properly on all channels

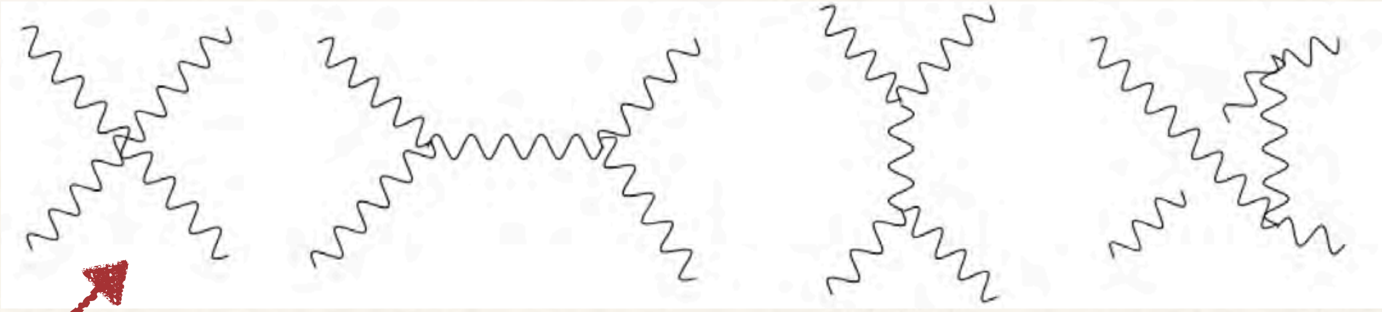
The amplitude is uniquely specified by these properties

On-shell constructibility

- This is obviously a theory specific statement
- Theories with contact terms might not be constructible
- Naively, this is false for Yang-Mills theory

$$gg \to gg$$

Four point amplitude



Contact term

On-shell constructibility

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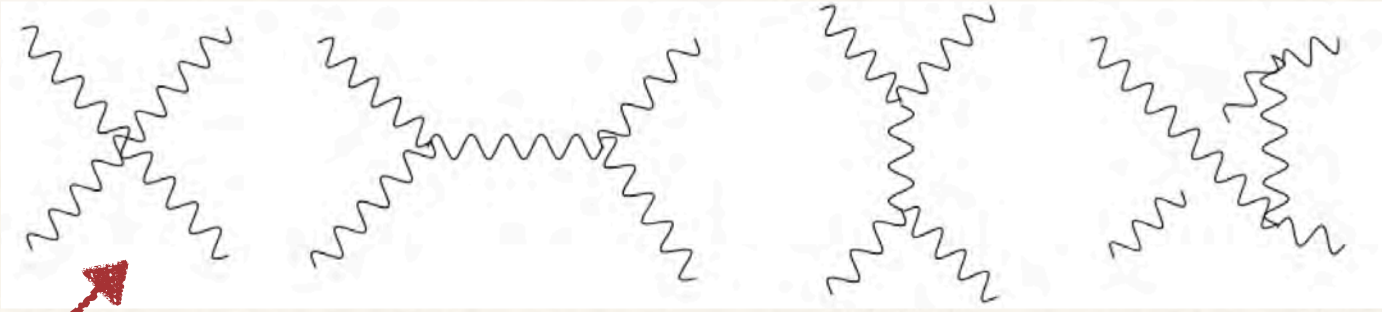
Four point amplitude



Contact term

Imposing gauge invariance fixes it

On-shell constructibility

- In gravity we have infinity tower of terms

$$\mathcal{L} \sim \sqrt{g}\, R \sim h^2 + h^3 + h^4 + \dots$$

- Only h^3 terms important, others fixed by diffeomorphism symmetry

- On-shell constructibility of Yang-Mills, GR, SM

Only function which factorizes properly on all poles is the amplitude.

Four point test

From 3pt to 4pt

- Three point amplitudes exist for all spins
- For 4pt amplitude: we have a powerful constraint

$$A_4 \xrightarrow[s=0]{} A_3 \frac{1}{s} A_3$$

This must be true on all channels

- This will immediately kill most of the possibilities
- We are left with spectrum of spins: $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

Three point of spin S

- I will discuss amplitudes of single spin S particle

- For 3pt amplitudes we get

$$A_3 = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^S \qquad A_3 = \left(\frac{[12]^3}{[23][31]}\right)^S$$

$(--+)$ minimal powercounting $(++-)$

- There exist also non-minimal amplitudes

$$A_3 = (\langle 12\rangle\langle 23\rangle\langle 31\rangle)^S \qquad A_3 = ([12][23][31])^S$$

$(---)$ $(+++)$

Four point amplitude

- Let us consider a 4pt amplitude of particular helicities

$$A_4(--++)$$

- Mandelstam variables:

$$\begin{aligned} s &= (p_1+p_2)^2 = \langle 12\rangle[12] = \langle 34\rangle[34] \\ t &= (p_1+p_4)^2 = \langle 14\rangle[14] = \langle 23\rangle[23] \\ u &= (p_1+p_3)^2 = \langle 13\rangle[13] = \langle 24\rangle[24] \end{aligned}$$

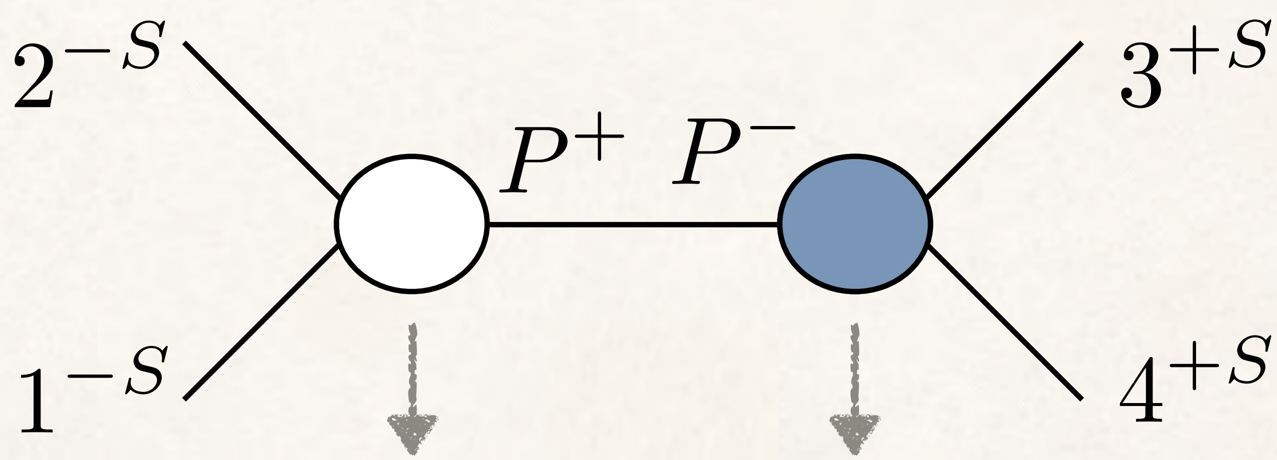
- One can show that the little group dictates:

$$A_4 = (\langle 12\rangle[34])^{2S} \cdot F(s,t)$$

- It must be consistent with factorizations

s-channel constraint

- The s-channel factorization dictates $P = 1 + 2 = -3 - 4$



$$A_4 \to \left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S \quad \text{on s=0}$$

Note: $s = \langle 12\rangle[12] = \langle 34\rangle[34]$

s-channel constraint

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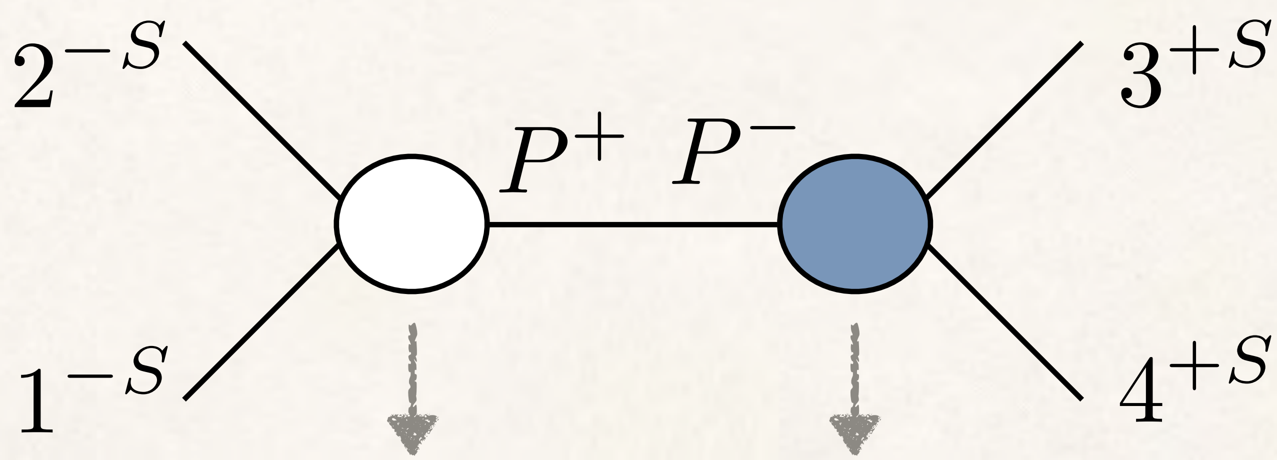

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s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get

$$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^3}{\langle 1P\rangle[3P]\langle 2P\rangle[4P]}\right)^S$$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

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- We get

$$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^2}{\langle 23\rangle[23]}\right)^S$$

s-channel constraint

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- We get $$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^2}{t}\right)^S$$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get

$$\underbrace{(\langle 12\rangle[34])^{2S}}_{\text{“Trivial” helicity factor}} \cdot \underbrace{\frac{1}{s\,t^S}}_{\text{Important piece}}$$

Note:
$t = -u$

Comparing channels

- On s-channel we got: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\frac{1}{s\,t^S}}$
- On t-channel we would get: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\frac{1}{t\,s^S}}$
- Require simple poles: $\frac{1}{s}, \frac{1}{t}, \frac{1}{u}$ and search for $F(s,t,u)$

Comparing channels

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- On t-channel we would get: $$A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\frac{1}{t\,s^S}}$$
- Require simple poles: $\frac{1}{s}, \frac{1}{t}, \frac{1}{u}$ and search for $F(s,t,u)$
- There are only two solutions:

$$F(s,t,u) = \frac{1}{s} + \frac{1}{t} + \frac{1}{u}$$

spin 0 (ϕ^3)

$$F(s,t,u) = \frac{1}{stu}$$

spin 2 (GR)

Where are gluons (spin-1)?

- Need to consider multiplet of particles

$$A_3 = \left(\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle} \right)^S f^{a_1 a_2 a_3} \qquad A_3 = \left(\frac{[12]^3}{[23][31]} \right)^S f^{a_1 a_2 a_3}$$

- The same check gives us S=1 and requires

$$f^{a_1 a_2 a_P} f^{a_3 a_4 a_P} + f^{a_1 a_4 a_P} f^{a_2 a_3 a_P} = f^{a_1 a_3 a_P} f^{a_2 a_4 a_P}$$

and the result corresponds to
SU(N) Yang-Mills theory

Power of 4pt check

- We can apply this check for cases with mixed particle content:
  - Spin >2 still not allowed
  - Spin 2 is special: only one particle and it couples universally to all other particles
  - We get various other constraints on interactions (of course all consistent with known theories)
- General principles very powerful

Thank you for attention!