Scattering Amplitudes

LECTURE 2

Jaroslav Trnka

Center for Quantum Mathematics and Physics (QMAP), UC Davis

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Review of Lecture 1

Spinor helicity variables

- Standard SO(3,1) notation for momentum

$$p^\mu = (p_0, p_1, p_2, p_3) \qquad p_j \in \mathbb{R}$$

$$p^2 = p_0^2 + p_1^2 + p_2^2 - p_3^2$$

- Matrix representation

$$p_{ab} = \sigma^\mu_{ab} p_\mu = \begin{pmatrix} p_0 + ip_1 & p_2 + p_3 \\ p_2 - p_3 & p_0 - ip_1 \end{pmatrix}$$

On-shell: $p^2 = \det(p_{ab}) = 0$

$$\boxed{\text{Rank}\,(p_{ab}) = 1}$$

Spinor helicity variables

- Rewrite the four component momentum

$$p_1^\mu = \sigma^\mu_{a\dot{a}}\,\lambda_{1a}\widetilde{\lambda}_{1\dot{a}}$$

- Little group scaling

$$\lambda \to t\lambda \qquad\qquad p \to p$$
$$\widetilde{\lambda} \to \frac{1}{t}\widetilde{\lambda}$$

- Invariants

$$\langle 12\rangle \equiv \epsilon_{ab}\lambda_{1a}\lambda_{2b} \qquad [12] \equiv \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}}$$
$$s_{12} = \langle 12\rangle[12]$$

Three point amplitudes

- Three point kinematics

$$p_1^2 = p_2^2 = p_3^2 = 0 \qquad p_1 + p_2 + p_3 = 0$$

- Two solutions:

$$\langle 12\rangle = \langle 23\rangle = \langle 13\rangle = 0 \qquad [12] = [23] = [13] = 0$$

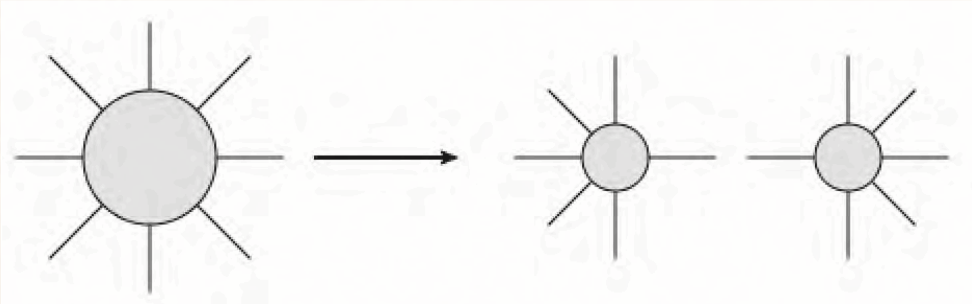
$$\lambda_1 \sim \lambda_2 \sim \lambda_3 \qquad \widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

$(+ + -)$ ↓ No solution for real momenta ↓ $(- - +)$

E.g. $\left(\frac{[12]^3}{[23][31]}\right)^S$ spin-S amplitudes $\left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^S$

Tree-level amplitudes

- Single function: locality and unitarity constraints



$$\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

- On-shell constructibility: amplitude fixed by poles

- Consistency of four point amplitude: only spins ≤ 2

Recursion relations

Tree level amplitudes

- Tree-level amplitude is a rational function of kinematics

$$A = \sum (\text{Feyn. diag}) = \frac{N}{\prod_j P_j^2}$$

momenta
polarization vectors

Feynman propagators

$$P_j = \sum_k p_k$$

- Only poles, no branch cuts

- Gauge invariant object: use spinor helicity variables

Reconstruction of the amplitude

- Amplitude on-shell constructible: fixed only from factorizations: try to reconstruct it

  "Integrate the relation" $\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$

- First guess: $\mathcal{M} = \sum_P \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$

Reconstruction of the amplitude

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  "Integrate the relation" $\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$

- First guess: $\mathcal{M} = \sum_P \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$ **WRONG**

  Overlapping factorization channels

- Solution: shift external momenta

Momentum shift

- Let us shift two external momenta

$$\lambda_1 \to \lambda_1 - z\lambda_2 \qquad \widetilde{\lambda}_1 \to \widetilde{\lambda}_1$$

$$\lambda_2 \to \lambda_2 \qquad \widetilde{\lambda}_2 \to \widetilde{\lambda}_2 + z\widetilde{\lambda}_1$$

- Momentum is conserved, stays on-shell

$$(\lambda_1 - z\lambda_2)\widetilde{\lambda}_1 + \lambda_2(\widetilde{\lambda}_2 + z\widetilde{\lambda}_1) = \lambda_1\widetilde{\lambda}_1 + \lambda_2\widetilde{\lambda}_2$$

- This corresponds to shifting

$$p_1, p_2, \epsilon_1, \epsilon_2$$

Shifted amplitude

- On-shell tree-level amplitude with shifted kinematics

$$A_n(z) = A(\hat{p}_1(z), \hat{p}_2(z), p_3, \ldots, p_n)$$

- Analytic structure

$$A_n(z) = \frac{N(z)}{\prod_j P_j(z)^2}$$

- Location of poles:

$$P_j(z) = P_j - z\lambda_2\widetilde{\lambda}_1 \quad \text{if} \quad p_1 \in P_j$$

$$P_j(z) = P_j + z\lambda_2\widetilde{\lambda}_1 \quad \text{if} \quad p_2 \in P_j$$

$$P_j(z) = P_j \qquad \text{otherwise}$$

Shifted amplitude

- On the pole if $p_1 \in P_j$

$$P_j(z)^2 = P_j^2 - 2z\langle 1|P_j|2] = 0$$

$$z = \frac{P_j^2}{2\langle 1|P_j|2]} \equiv z_j$$

- Shifted amplitude:

$$A_n(z) = \frac{N(z)}{\prod_j P_j(z)^2}$$

location of poles

Residue theorem

- Shifted amplitude $$A_n(z) = \frac{N(z)}{\prod_k (z - z_k)}$$
- Let us consider the contour integral

$$\int \frac{dz}{z} A_n(z) = 0$$

No pole at $z \to \infty$

- Original amplitude $A_n = A_n(z=0)$ ← Residue at $z = 0$
- Residue theorem: $$A_n + \sum_k \text{Res}\left(\frac{A_n(z)}{z}\right)\Bigg|_{z=z_k} = 0$$

Residue theorem

$$A_n = -\sum_k \text{Res}\left(\frac{A_n(z)}{z}\right)\Bigg|_{z=z_k}$$

Residue on the pole $P_j(z)^2 = 0$

- Unitarity of shifted tree-level amplitude

$$A_n(z) \xrightarrow[P_j(z)^2=0]{} A_L(z)\frac{1}{P_j(z)^2}A_R(z)$$

Residue theorem

$$A_n = -\sum_k \text{Res}\left(\frac{A_n(z)}{z}\right)\Bigg|_{z=z_k}$$

Residue on the pole $P_j(z)^2 = 2\langle 1|P_j|2](z_j - z) = 0$

- Unitarity of shifted tree-level amplitude $\quad z_j = \dfrac{P_j^2}{2\langle 1|P_j|2]}$

$$A_n(z) \xrightarrow[z=z_j]{} A_L(z_j)\frac{1}{2\langle 1|P_j|2]}A_R(z_j)$$

Residue theorem

$$A_n = -\sum_k \text{Res}\left(\frac{A_n(z)}{z}\right)\Bigg|_{z=z_k}$$

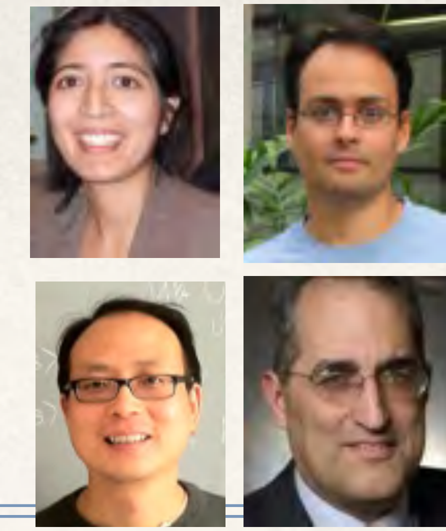
$$A_L(z_j)\frac{1}{2\langle 1|P_j|2]}A_R(z_j) \times \frac{2\langle 1|P_j|2]}{P_j^2} = A_L(z_j)\frac{1}{P_j^2}A_R(z_j)$$

<u>Final formula</u>

$$A_n = -\sum_j A_L(z_j)\frac{1}{P_j^2}A_R(z_j) \qquad z_j = \frac{P_j^2}{2\langle 1|P_j|2]}$$

BCFW recursion relations

(Britto, Cachazo, Feng, Witten, 2005)



$$A_n = -\sum_j A_L(z_j)\frac{1}{P_j^2}A_R(z_j)$$

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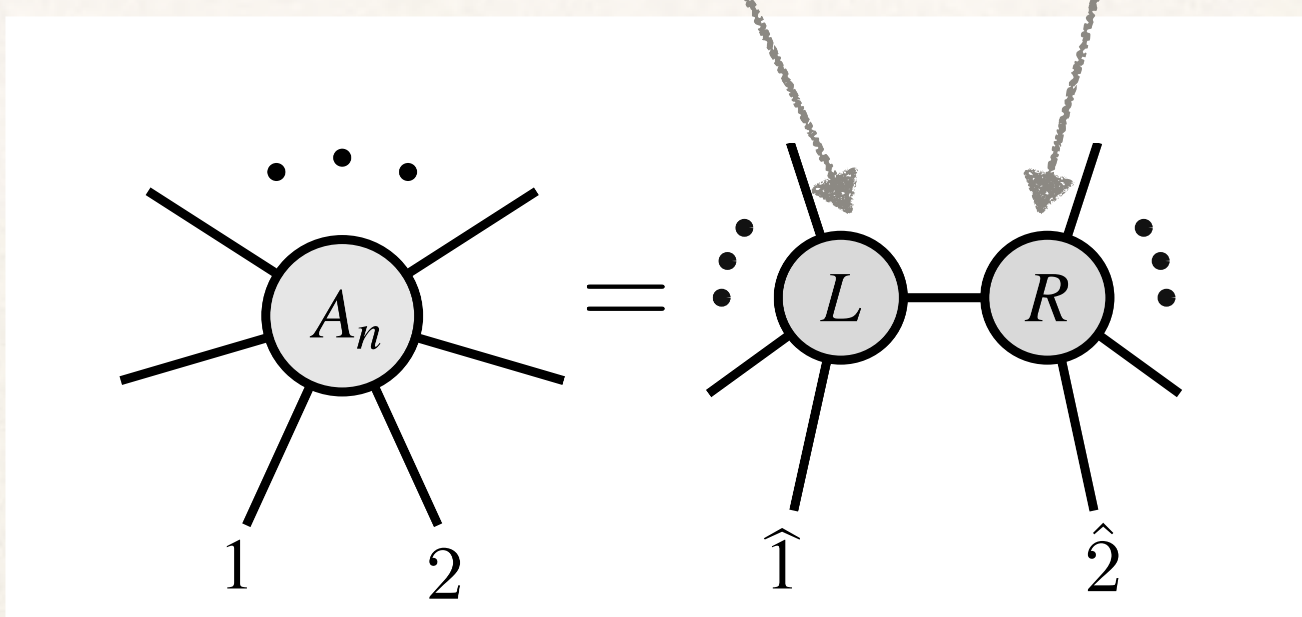

Chosen such that internal line is on-shell

Sum over all distributions of legs keeping 1,2 on different sides

Comment on applicability

- The crucial property is $A_n(z) \to 0$ for $z \to \infty$
- In Yang-Mills theory this is satisfied if

$$\lambda_1 \to \lambda_1 - z\lambda_2 \quad \longleftarrow \quad \text{Helicity } +$$

$$\widetilde{\lambda}_2 \to \widetilde{\lambda}_2 + z\widetilde{\lambda}_1 \quad \longleftarrow \quad \text{Helicity } -$$

- Same is true for Einstein gravity, and many others
- This means that amplitudes in these theories are fully specified by residues on their poles

Generalizations

- In Standard Model and other theories more general recursion relations needed: shift more momenta

- Include masses: go back to momenta

$$p_1 \to p_1 + zq \qquad q^2 = (p_1 \cdot q) = (p_2 \cdot q) = 0$$

$$p_2 \to p_2 - zq$$

Shifted momenta on-shell,
q completely fixed

- Extension to effective field theories

(Cheung, Kampf, Novotny, JT, 2015)

Example: amplitudes of gluons

Color decomposition

- Sum of Feynman diagrams in Yang-Mills

$$\mathcal{M} = \sum_{FD} (\text{Color}) \times (\text{Kinematics})$$

Polarization vectors
Gauge dependent

- Color factors

$$\text{Tr}(T^{a_1} T^{a_2} T^{a_3} \dots T^{a_n})$$

- Decomposition

$$\mathcal{M} = \sum_{\sigma} \text{Tr}(T^{\sigma_1} T^{\sigma_2} T^{\sigma_3} \dots T^{\sigma_n})\, A(123 \dots n)$$

Color decomposition

- Sum of Feynman diagrams in Yang-Mills

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Polarization vectors
Gauge dependent

- Color factors

$$\text{Tr}(T^{a_1} T^{a_2} T^{a_3} \ldots T^{a_n})$$

- Decomposition

$$\boxed{\mathcal{M} = \sum_{\sigma} \text{Tr}(T^{\sigma_1} T^{\sigma_2} T^{\sigma_3} \ldots T^{\sigma_n}) \boxed{A(123\ldots n)}}$$

Color ordered amplitude

$$A(123\ldots n)$$

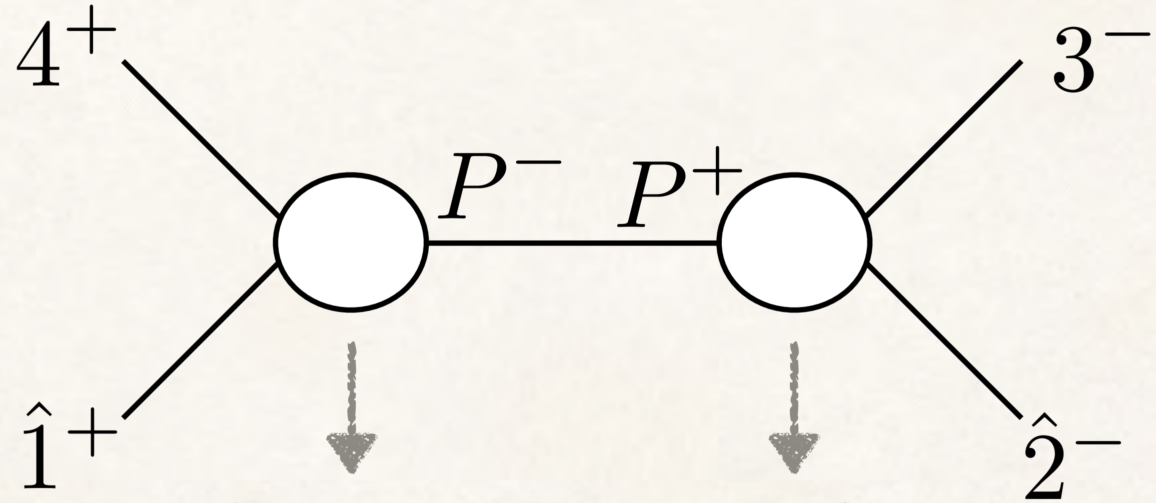
Particles are ordered, other orderings: permutations

Gauge invariant

- This is a key object of our interest
- Consider:
  - All particles massless and on-shell
  - All momenta incoming
  - Helicities fixed

Example 1: 4pt amplitude

- Let us consider amplitude of gluons $A_4(1^+2^-3^-4^+)$



Only one term contributes

$$\frac{[\hat{1}4]^3}{[\hat{1}P][4P]} \quad \frac{1}{s_{23}} \quad \frac{\langle\hat{2}3\rangle^3}{\langle\hat{2}P\rangle\langle 3P\rangle}$$

$$\hat{\lambda}_1 = \lambda_1 - z\lambda_2$$
$$\hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 + z\tilde{\lambda}_1$$

z takes the value when P is on-shell momentum

Example 1: 4pt amplitude

- Let us consider amplitude of gluons $A_4(1^+2^-3^-4^+)$

$$P^2 = \langle \hat{1}4\rangle[14] = 0$$

$$\langle \hat{1}4\rangle = \langle 14\rangle - z\langle 24\rangle = 0 \quad \rightarrow \quad z = \frac{\langle 14\rangle}{\langle 24\rangle}$$

We can now rewrite

Shouten identity

$$\hat{\lambda}_1 = \lambda_1 - z\lambda_2 = \lambda_1 - \frac{\langle 14\rangle}{\langle 24\rangle}\lambda_2 = \frac{\langle 12\rangle}{\langle 24\rangle}\lambda_4$$

$$\widetilde{\lambda}_2 = \widetilde{\lambda}_2 + z\widetilde{\lambda}_1 = \frac{[12]}{[13]}\widetilde{\lambda}_3$$

Use of momentum conservation

Example 1: 4pt amplitude

- Let us consider amplitude of gluons $A_4(1^+2^-3^-4^+)$

$$\hat{\lambda}_1 = \frac{\langle 12\rangle}{\langle 24\rangle}\lambda_4$$

Calculate on-shell momentum P

$$P = \hat{\lambda}_1\widetilde{\lambda}_1 + \lambda_4\widetilde{\lambda}_4 = \lambda_4\left(\frac{\langle 12\rangle}{\langle 24\rangle}\widetilde{\lambda}_1 + \widetilde{\lambda}_4\right)$$

$$\lambda_P = \lambda_4 \qquad \widetilde{\lambda}_P = \frac{\langle 23\rangle}{\langle 24\rangle}\widetilde{\lambda}_3$$

Example 1: 4pt amplitude

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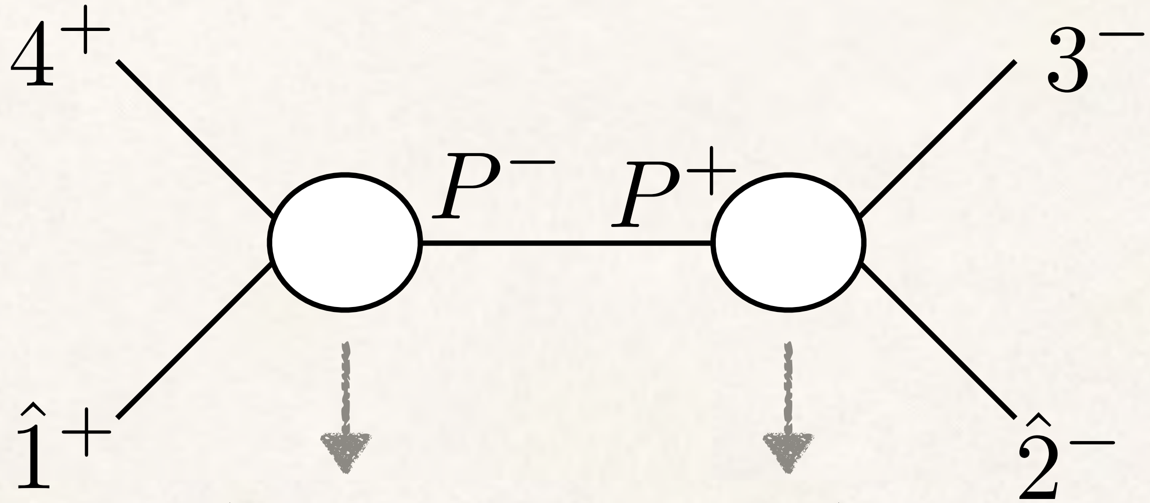

$$\frac{[\hat{1}4]^3}{[\hat{1}P][4P]} \quad \frac{1}{s_{23}} \quad \frac{\langle\hat{2}3\rangle^3}{\langle\hat{2}P\rangle\langle 3P\rangle}$$

$$\hat{\lambda}_1 = \frac{\langle 12\rangle}{\langle 24\rangle}\lambda_4$$

$$\hat{\tilde{\lambda}}_2 = \frac{[12]}{[13]}\tilde{\lambda}_3$$

$$\lambda_P = \lambda_4$$

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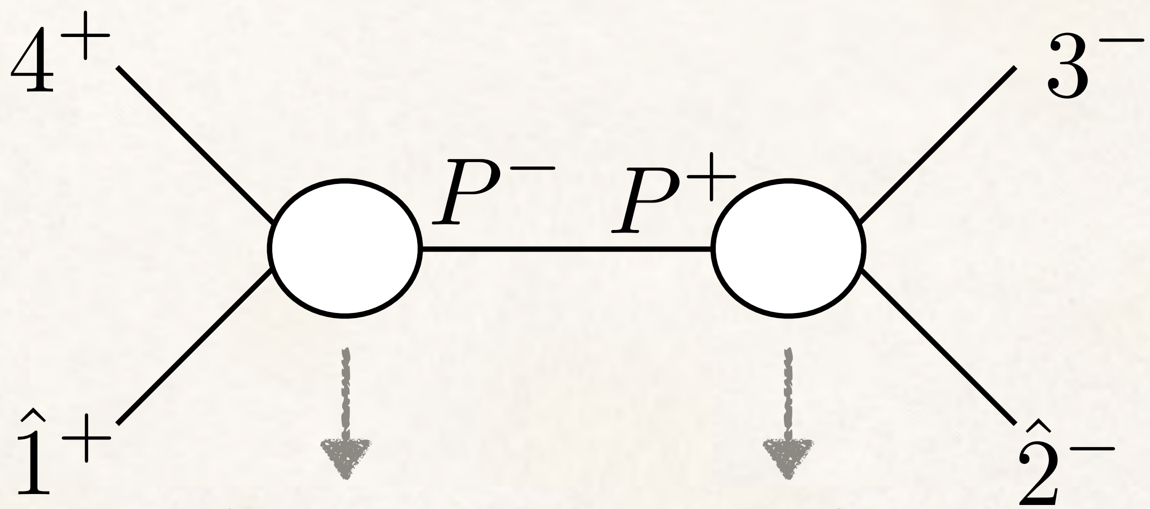

$$\frac{[\hat{1}4]^3}{[\hat{1}P][4P]}\;\frac{1}{s_{23}}\;\frac{\langle\hat{2}3\rangle^3}{\langle\hat{2}P\rangle\langle 3P\rangle}$$

$$= \frac{\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

$$\hat{\lambda}_1 = \frac{\langle 12\rangle}{\langle 24\rangle}\lambda_4$$

$$\hat{\tilde{\lambda}}_2 = \frac{[12]}{[13]}\tilde{\lambda}_3$$

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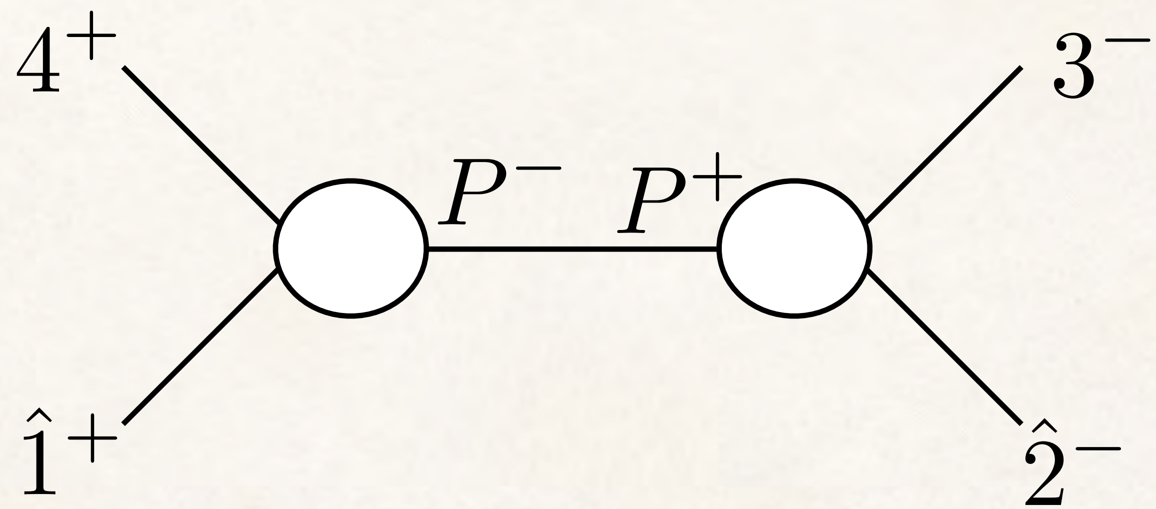
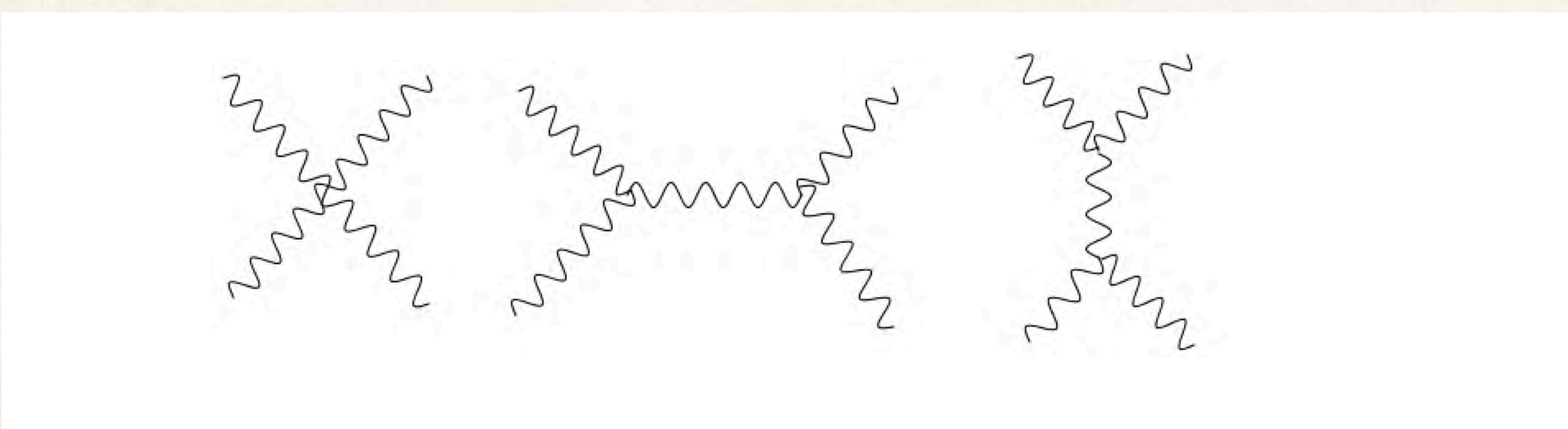
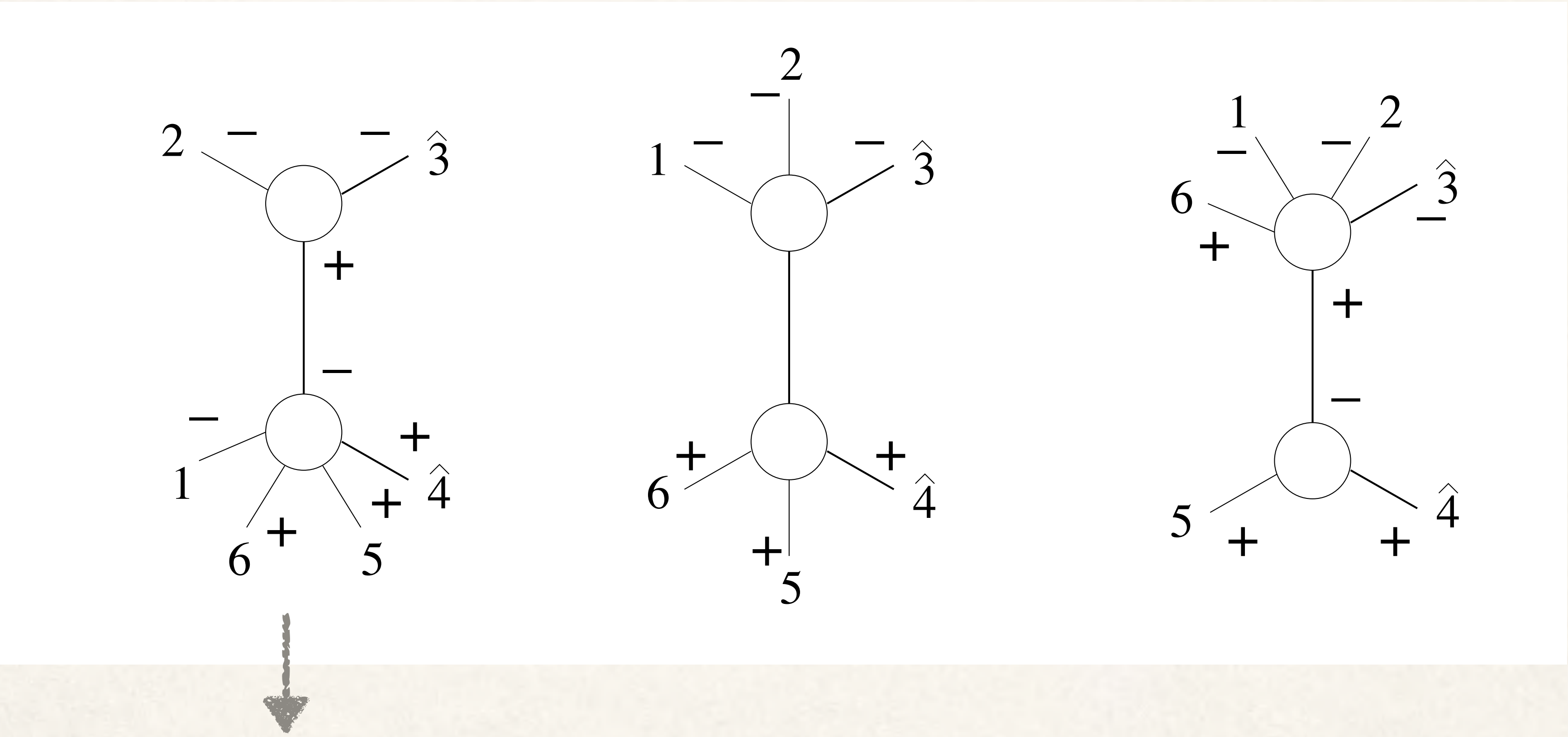


One gauge invariant object equivalent to three Feynman diagrams



Example 2: 6pt amplitude

- Let us consider $A_6(1^-2^-3^-4^+5^+6^+)$ and shift legs 3,4



vs
220 Feynman
diagrams

$$\frac{\langle 1|2+3|4]^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{234}\langle 5|3+4|2]}$$

$$\langle 1|2+3|4] = \langle 12\rangle[24] + \langle 13\rangle[34]$$

Example 2: 6pt amplitude

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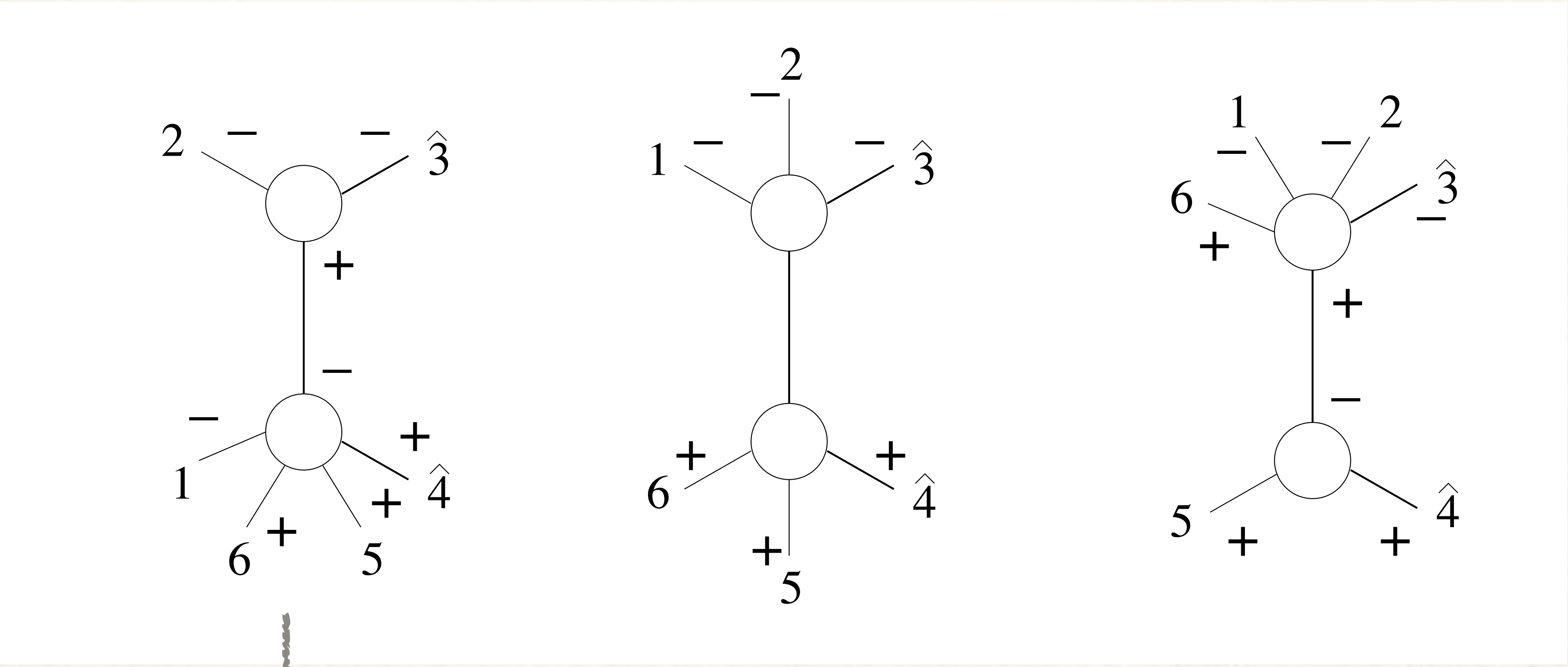


$$\frac{\langle 1|2+3|4]^3}{[23][34]\langle 56\rangle\langle 61\rangle s_{234}\langle 5|3+4|2]}$$

Spurious pole

Remark on BCFW

- Extremely efficient (3 vs 220 for 6pt, 20 vs 34300 for 8pt)
- Terms in BCFW recursion relations
  - Gauge invariant
  - Spurious poles
- Amplitude = sum of these terms dictated by unitarity
- Note: not all factorization channels are present

  when 1,2 are on the same side

Unitarity methods

One-loop amplitudes

- Sum of Feynman diagrams

$$\mathcal{M}^{1-loop} = \sum_j \int d\mathcal{I}_j \quad \text{where} \quad d\mathcal{I}_j = d^4\ell\,\mathcal{I}_j$$

- Re-express as basis of canonical integrals

Rational

$$\mathcal{M}^{1-loop} = \sum_j a_j \int d\mathcal{I}_j^{(4)} + \sum_j b_j \int d\mathcal{I}_j^{(3)} + \sum_j c_j \int d\mathcal{I}_j^{(2)} + \mathcal{R}$$

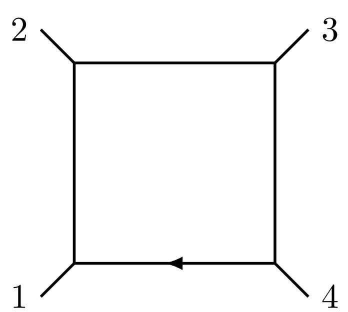
Box

Triangle

Bubble

One loop amplitudes

- Box integral



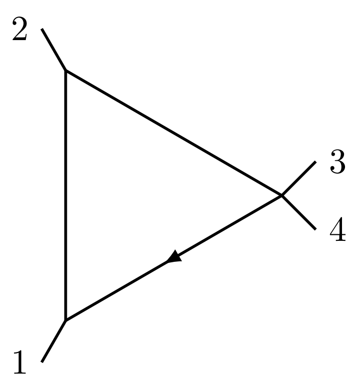
$$I = \frac{d^4\ell \; st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

Tadpoles and other integrals

↓

Vanish in dim reg

- Triangle and box integrals



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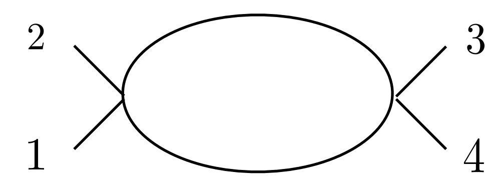
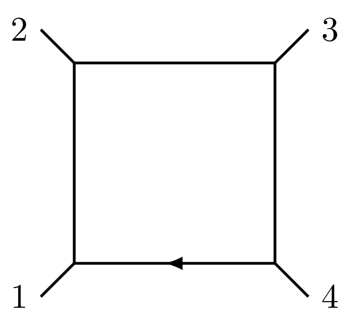


$$I = \frac{d^4\ell \, s}{\ell^2(\ell+k_1+k_2)^2}$$

One loop amplitudes

- Box integral



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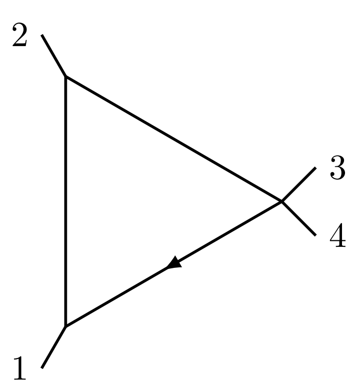
Tadpoles and other integrals

Vanish in dim reg

- Triangle and box integrals



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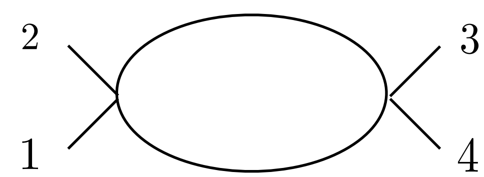


$$I = \frac{d^4\ell \, s}{\ell^2(\ell+k_1+k_2)^2}$$

UV divergent

(Super) Yang Mills amplitudes

- One-loop expansion in SYM theory

$$\mathcal{M} = \sum \text{Boxes} + \sum \text{Triangle} + \sum \text{Bubble} + \text{Rational}$$

Pure Yang-Mills (massless QCD)

(Super) Yang Mills amplitudes

- One-loop expansion in SYM theory

$$\mathcal{M} = \sum \text{Boxes} + \sum \text{Triangle} + \sum \text{Bubble} + \text{Rational}$$

N=1 and N=2 Super Yang-Mills

(Super) Yang Mills amplitudes

- One-loop expansion in SYM theory

$$\mathcal{M} = \sum \text{Boxes} + \sum \text{Triangle} + \sum \text{Bubble} + \text{Rational}$$

N=4 Super Yang-Mills

- Note that it is UV finite at 1-loop, but also all loops

One loop expansion

- One-loop expansion

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

One loop expansion

- One-loop expansion

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

How to calculate these coefficients?

How to calculate this function?

One loop expansion

- One-loop expansion

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

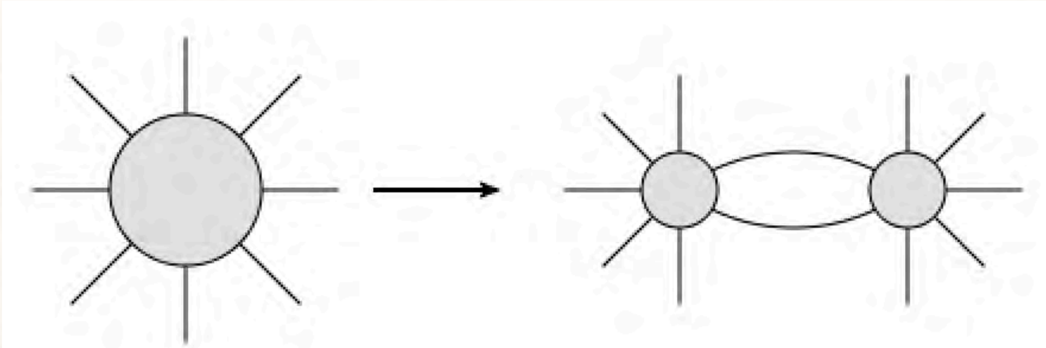
How to calculate these coefficients?

How to calculate this function?

Unitarity methods

One loop unitarity

- Analogue of tree-level unitarity at one-loop



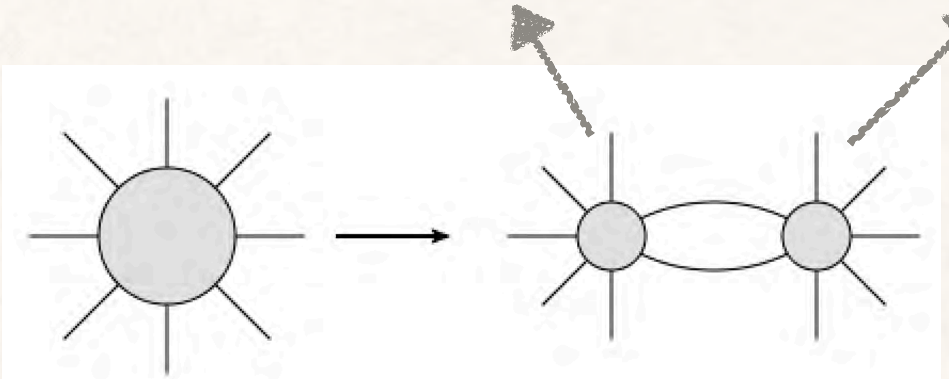
$$\mathcal{M}^{1-loop} \xrightarrow[\ell^2=(\ell+Q)^2=0]{} \mathcal{M}_L^{tree} \frac{1}{\ell^2(\ell+Q)^2} \mathcal{M}_R^{tree}$$

Unitarity cut

- In general $\quad \text{Cut} \leftrightarrow \ell^2 = 0$

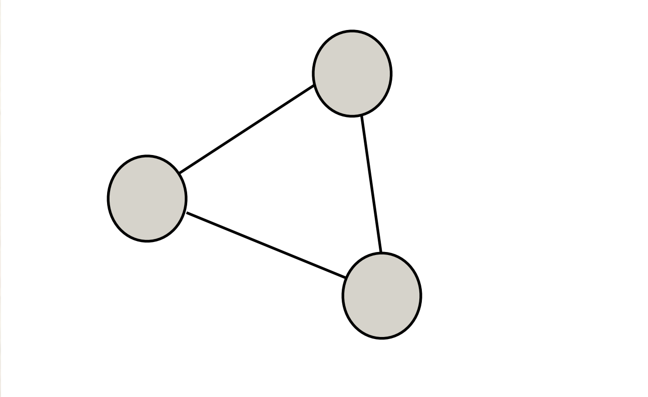
One-loop unitarity

- Higher cuts



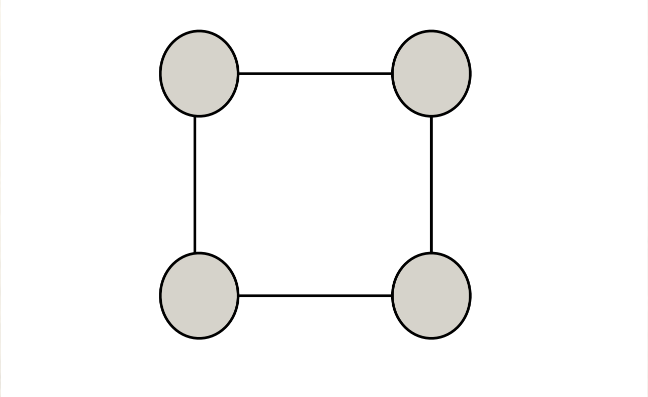
Triple cut

$$\ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = 0$$



Quadruple cut

$$\ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = (\ell + Q_3)^2 = 0$$



Fixing coefficients

- Perform cut on both side of equation

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

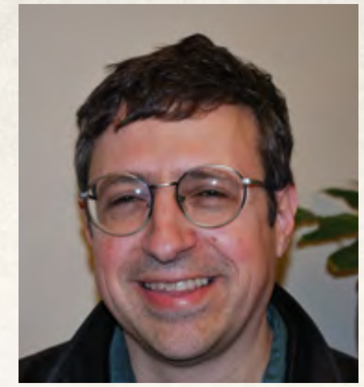
Product of trees   Linear combination of coefficients

- Example: Quadruple cut - only one box contributes

$$\mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\mathcal{M}_3^{tree}\mathcal{M}_4^{tree} = a_j$$

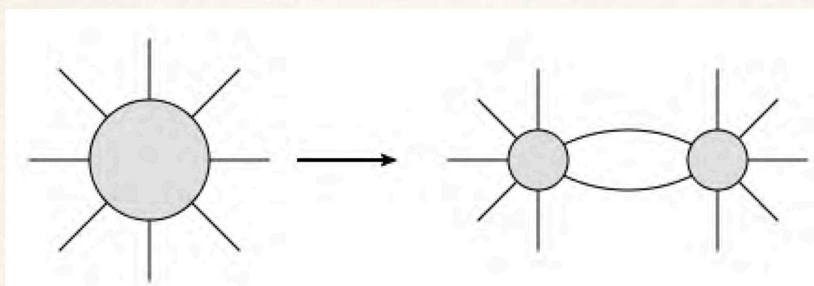
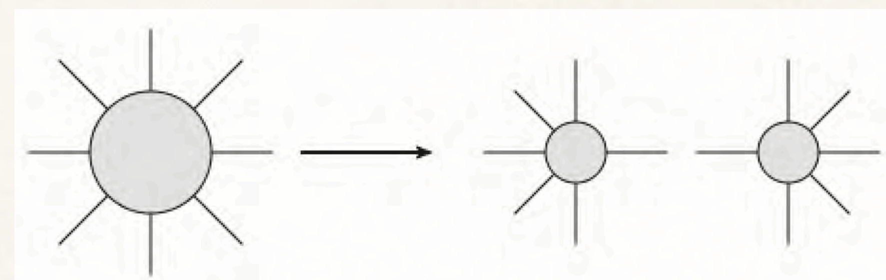
- All coefficients a_j, b_j, c_j can be obtained

Unitarity methods

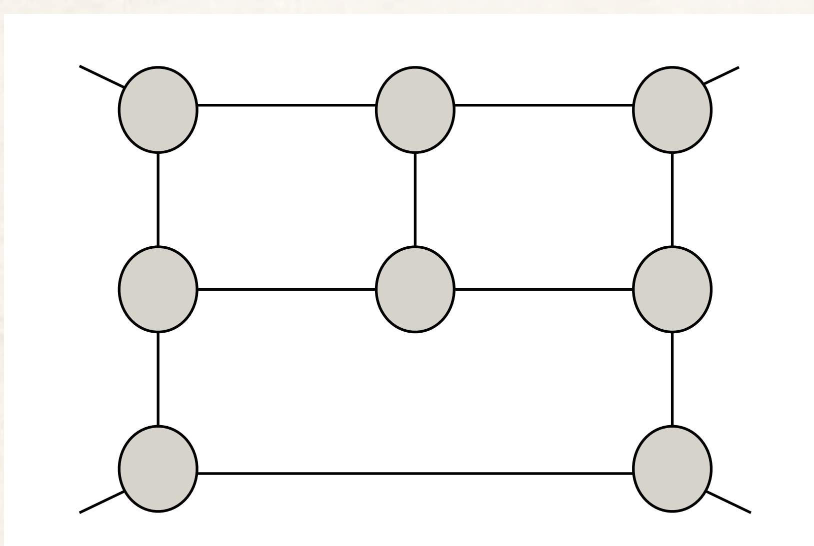


(Bern, Dixon, Kosower)

- We can iterate both types of cuts

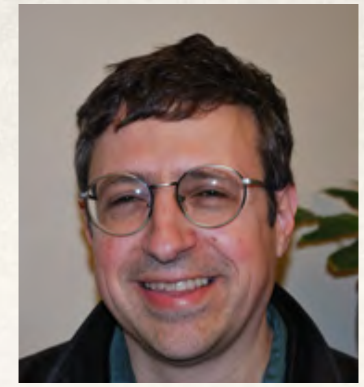



- Stop when all propagators are cut: maximal cut



Product of 3pt on-shell amplitudes

Unitarity methods



(Bern, Dixon, Kosower)

- Expansion of the amplitude

$$\mathcal{M}^{\ell-loop} = \sum_j a_j \int d\mathcal{I}_j$$

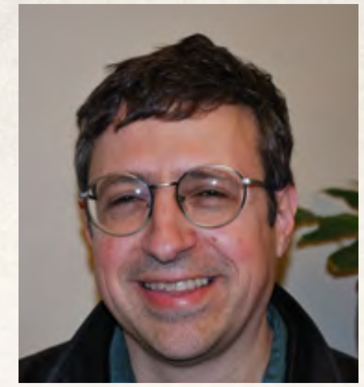
Cuts give product of trees

Linear combinations of coefficients a_j

- Very successful method for loop amplitudes in different theories

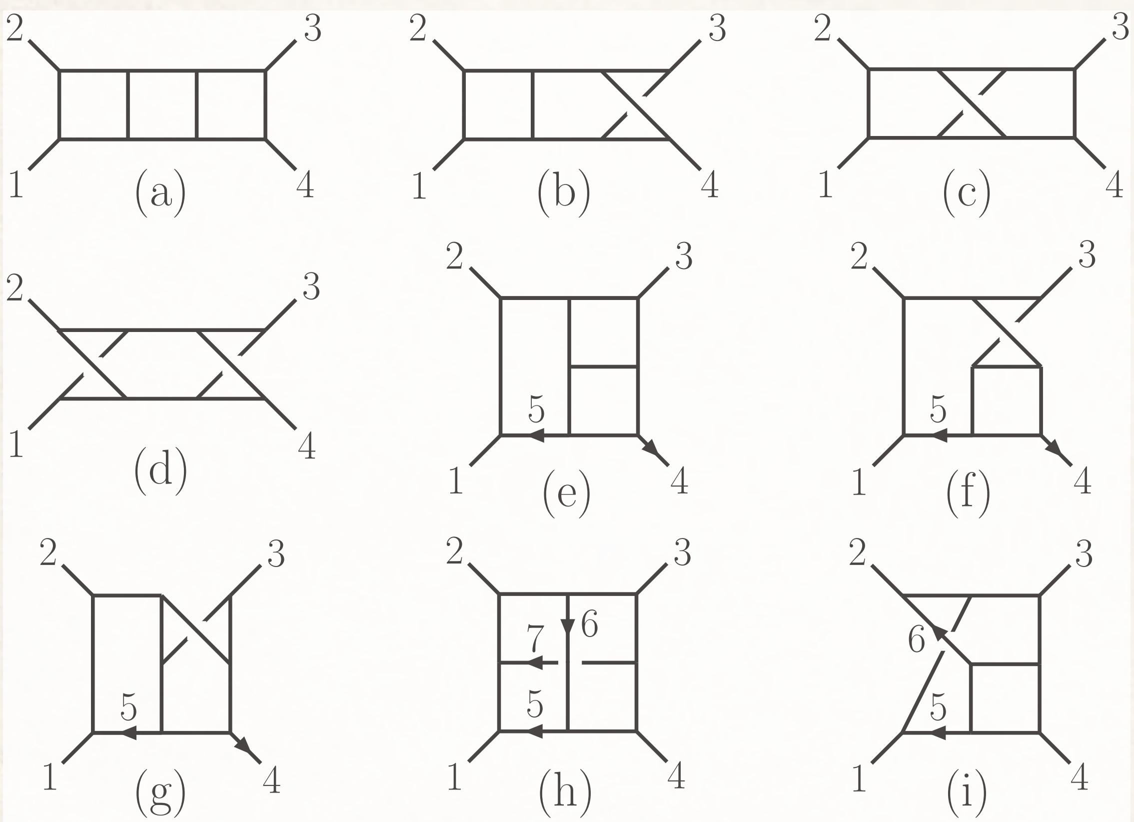
- Practical problems:
  - Find basis of integrals
  - Solve (long) system of equations

Unitarity methods

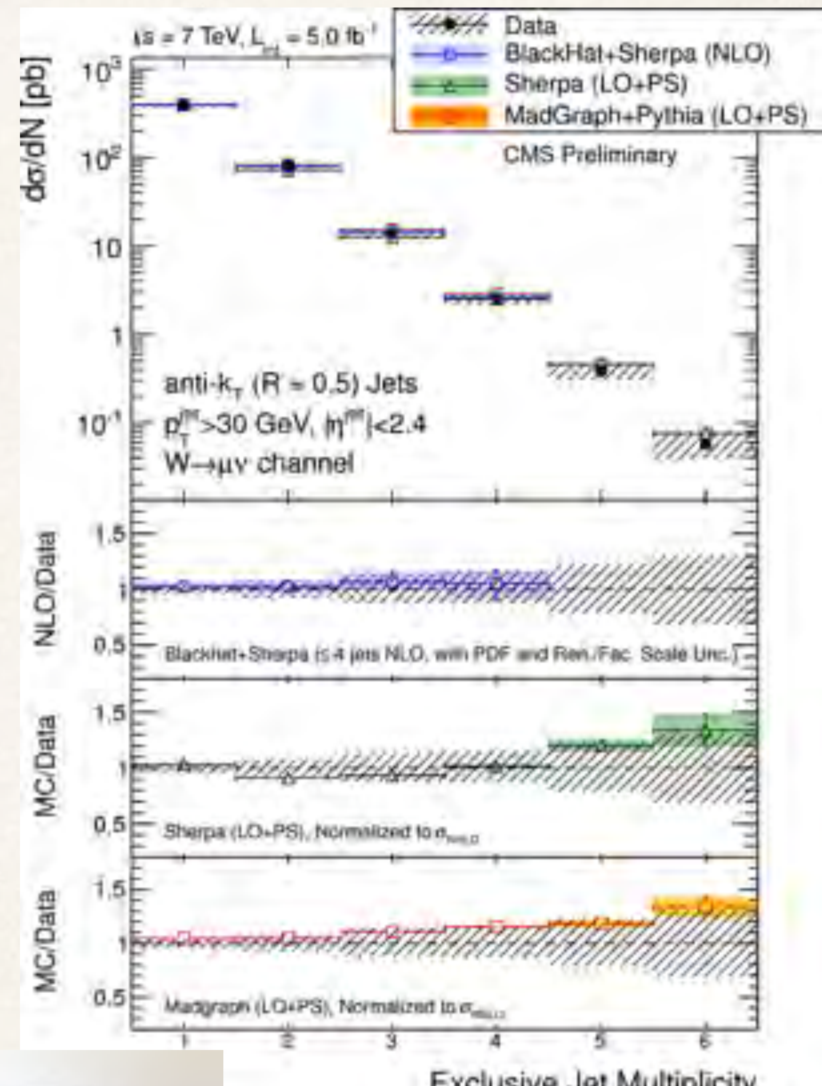


(Bern, Dixon, Kosower)

- Results in susy theories and QCD



Basis of integrals for 3-loop amplitudes in N=4 SYM and N=8 SUGRA



Black Hat

On-shell good, off-shell bad

- Feynman diagrams: off-shell objects

Off-shell objects

- Unitarity methods: $\mathrm{Cut}[\mathcal{M}] = \mathrm{Cut}[\text{Basis of integrals}]$

On-shell objects

Locality
Unitarity

- Recursion relations

$$\mathcal{M} \sim \mathcal{M}_L\, \mathcal{M}_R$$

On-shell objects

Locality lost
Unitarity

- Next direction: loosing manifest locality and unitarity

On-shell diagrams

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)

Atoms of amplitudes

What are natural gauge invariant objects?

Atoms of amplitudes

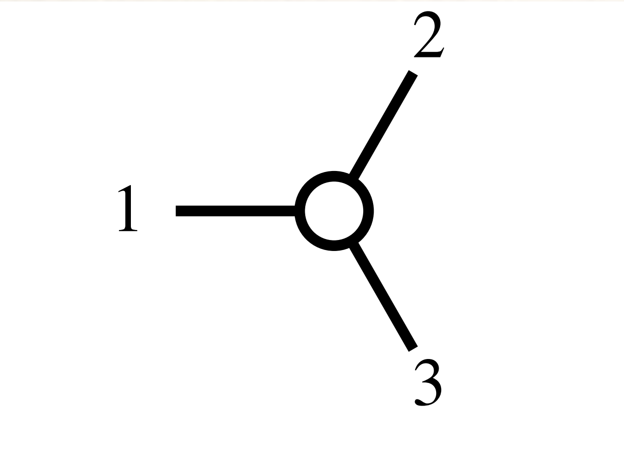
What are natural gauge invariant objects?

Scattering amplitudes

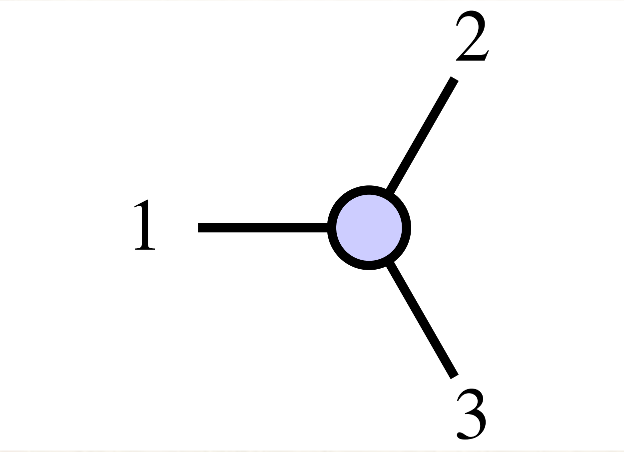
- Recursion relations, unitarity methods: products of amplitudes
- Iterative procedure: reduces to elementary amplitudes
- In most interesting theories these are three point

Three point kinematics

- Two options



$$\lambda_1 \sim \lambda_2 \sim \lambda_3$$



$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

<u>Spinor helicity variables</u>

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \widetilde{\lambda}_{\dot{a}}$$

$$\langle 12 \rangle = \epsilon_{ab} \lambda_{1a} \lambda_{2b}$$

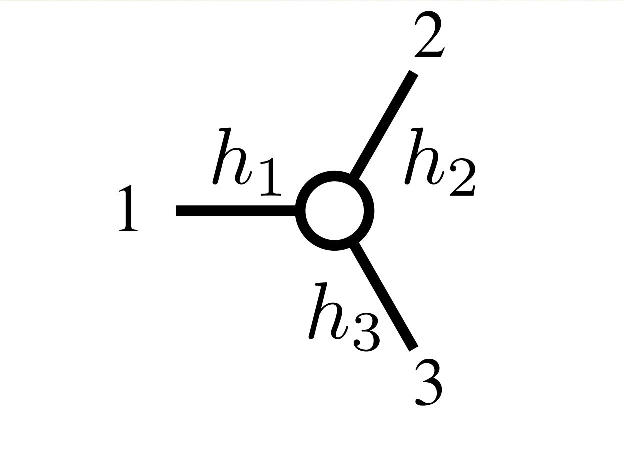
$$[12] = \epsilon_{\dot{a}\dot{b}} \lambda_{1\dot{a}} \lambda_{2\dot{b}}$$

Two solutions for 3pt kinematics

$$p_1^2 = p_2^2 = p_3^2 = (p_1 + p_2 + p_3) = 0$$

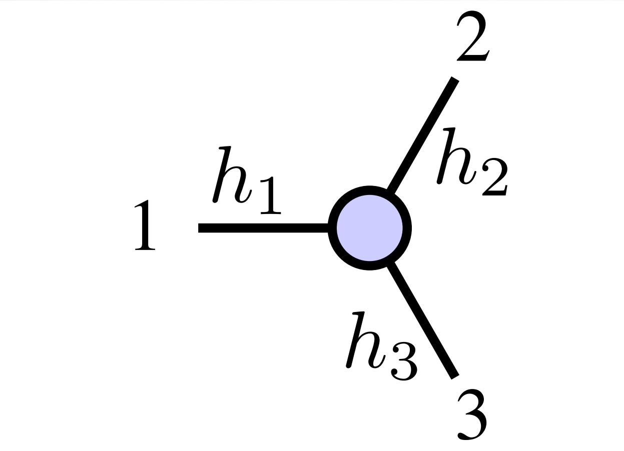
Three point amplitudes

- Two solutions for amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

$$h_1 + h_2 + h_3 \geq 0$$



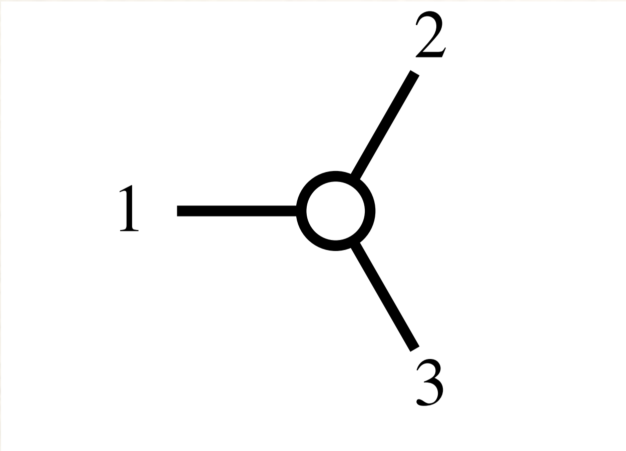
$$A_3 = \langle 12\rangle^{-h_1-h_2+h_3}\langle 23\rangle^{+h_1-h_2-h_3}\langle 31\rangle^{-h_1+h_2-h_3}$$

$$h_1 + h_2 + h_3 \leq 0$$

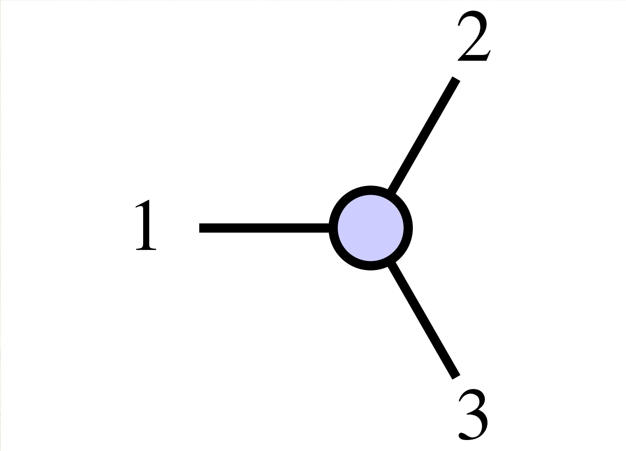
Supersymmetry: amplitudes of super-fields
(all component fields included)

Three point amplitudes

- In N=4 SYM: no need to specify helicities



$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2+[12]\widetilde{\eta}_3)}{[12][23][31]}$$



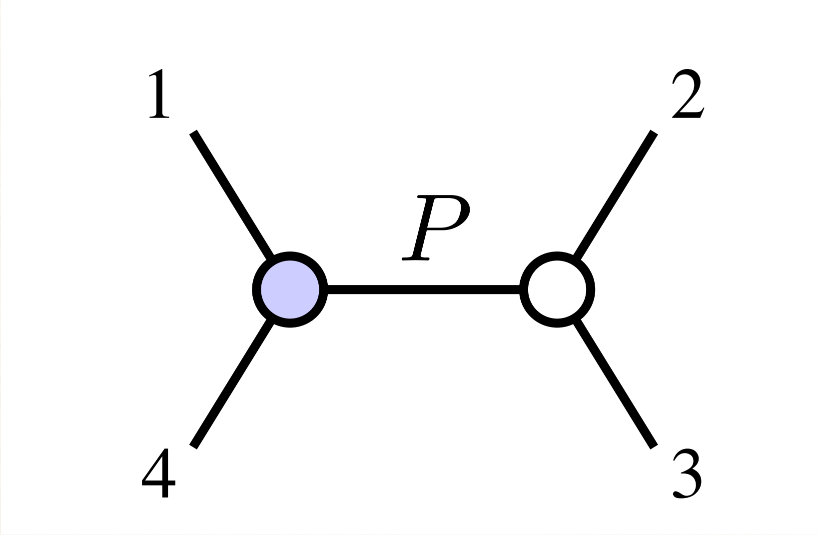
$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Easy book-keeping

Fully fixed in any QFT up to coupling

Gluing three point amplitudes

- Let us build a diagram



Multiply two three point amplitudes

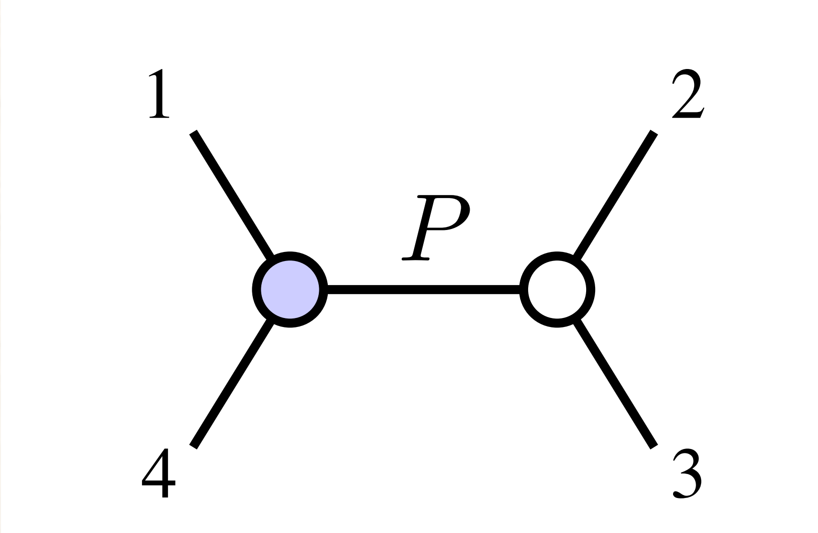
$$= \mathcal{A}_3^{(2)}(14P) \times \mathcal{A}_3^{(1)}(P23)$$

$$= \frac{\delta^4(p_1+p_4+P)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_4\widetilde{\eta}_4+\lambda_P\widetilde{\eta}_P)}{\langle 14\rangle\langle 4P\rangle\langle P1\rangle} \times \frac{\delta^4(p_2+p_3-P)\delta^4(\widetilde{\eta}_P[23]+\widetilde{\eta}_2[3P]+\widetilde{\eta}_3[P2]}{[23][3P][P2]}$$

also $\lambda_P \sim \lambda_2 \sim \lambda_3$ and $\widetilde{\lambda}_1 \sim \widetilde{\lambda}_4 \sim \widetilde{\lambda}_P$

Gluing three point amplitudes

- Let us build a diagram



Multiply two three point amplitudes

$$= \mathcal{A}_3^{(2)}(14P) \times \mathcal{A}_3^{(1)}(P23)$$

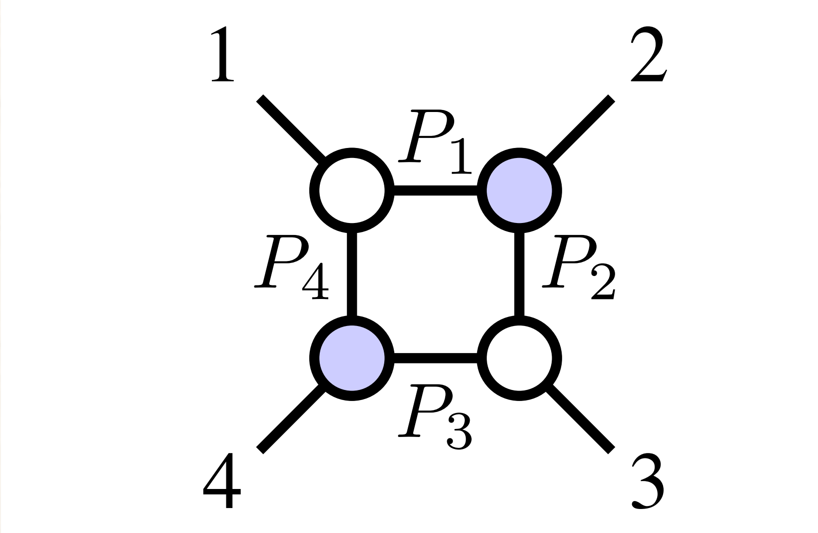
$$= \frac{\delta^4(p_1+p_2+p_3+p_4)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3+\lambda_4\widetilde{\eta}_4)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \delta((p_2+p_3)^2)$$

$$= \mathcal{A}_4^{(2)}(1234) \times \delta((p_2+p_3)^2)$$

Four point tree level amplitude on factorization channel

Gluing three point amplitudes

- Let us build a diagram

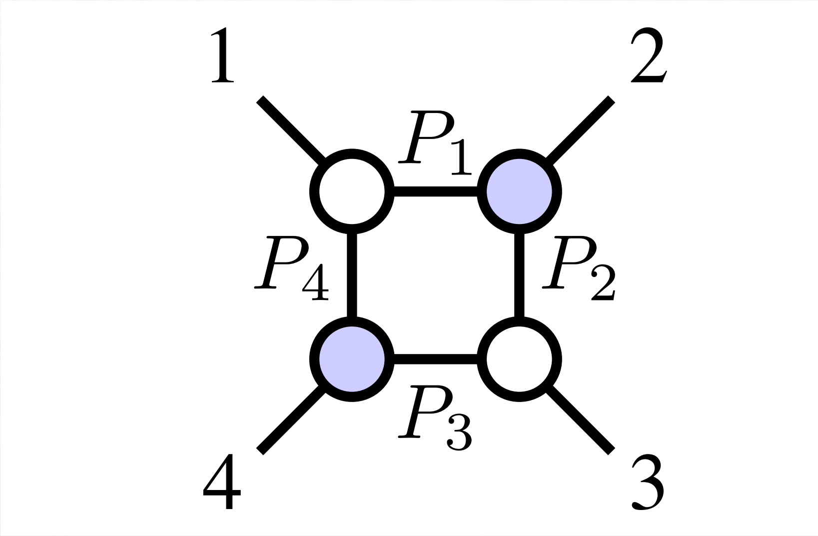


Multiply four three point amplitudes

$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

Gluing three point amplitudes

- Let us build a diagram



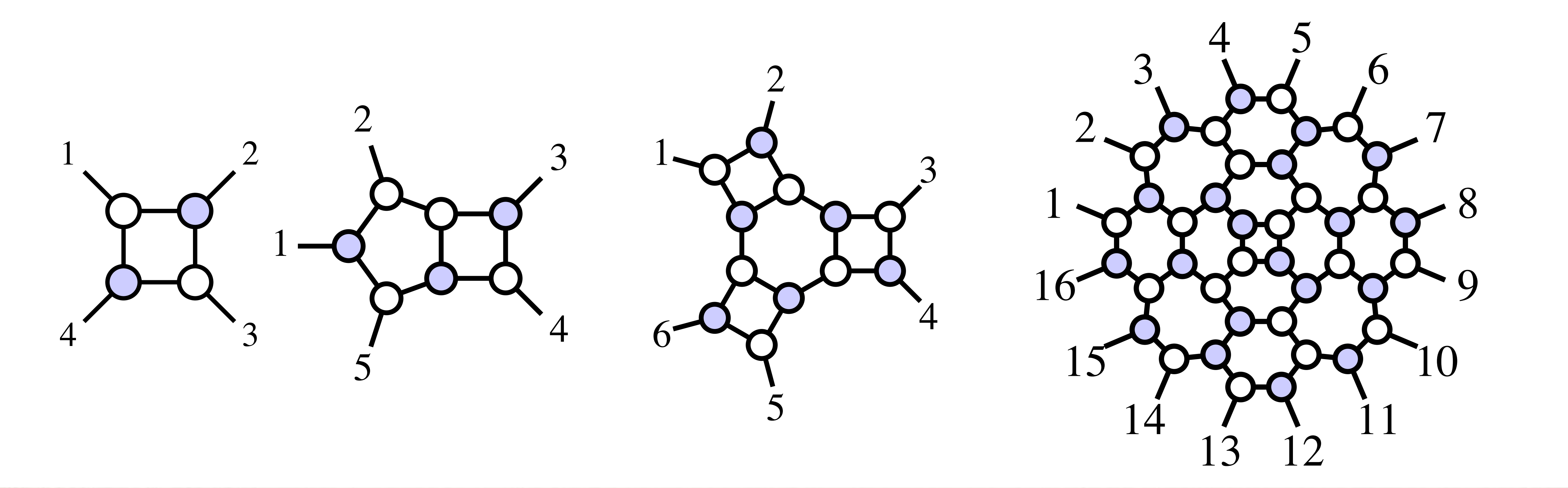
Multiply four three point amplitudes

$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

$$= \mathcal{A}_4(1234)$$

On-shell diagrams

- Draw arbitrary graph with three point vertices

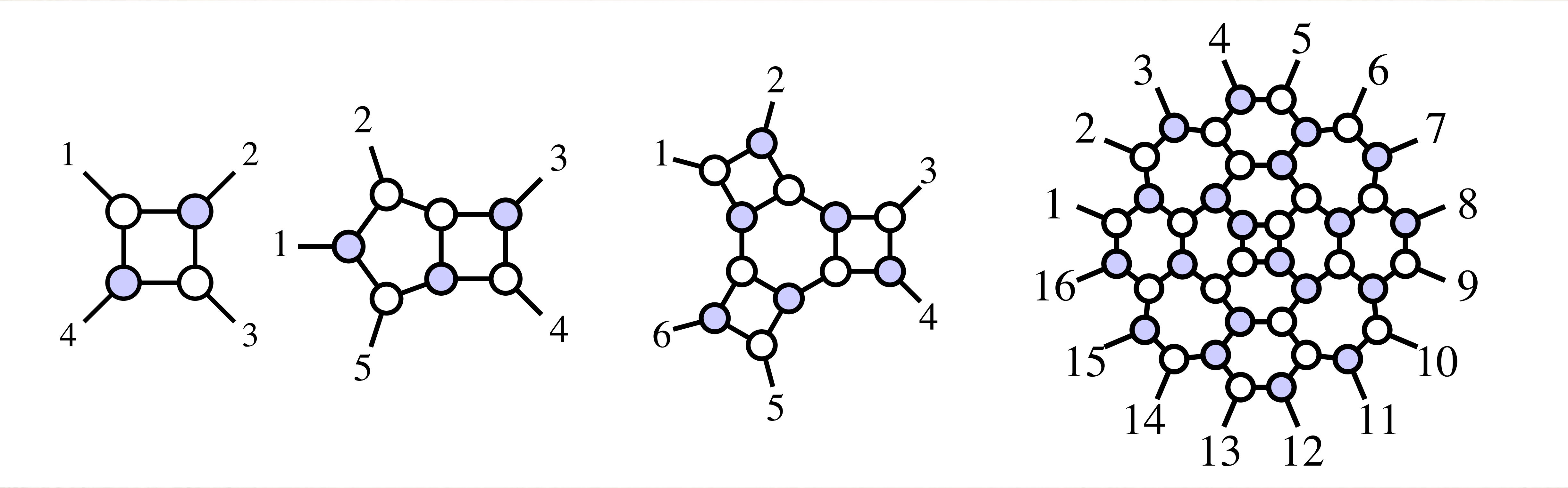


Products of three point amplitudes

$$\begin{cases} P > 4L & \text{Extra delta functions} \\ P = 4L & \text{Function of external data only} \\ P < 4L & \text{Unfixed parameters (forms)} \end{cases}$$

On-shell diagrams

- Draw arbitrary graph with three point vertices



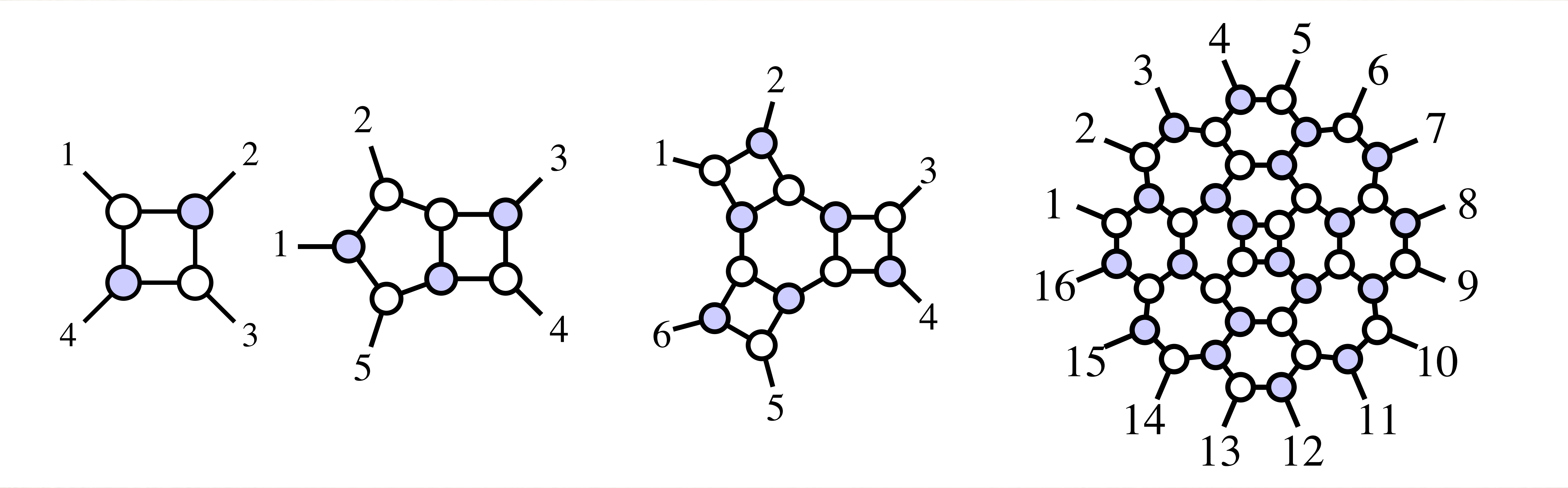
On-shell diagrams with $P \leq 4L$ are cuts of the amplitude

- Parametrized by n, k

$$k = 2B + W - P$$

On-shell diagrams

- Draw arbitrary graph with three point vertices

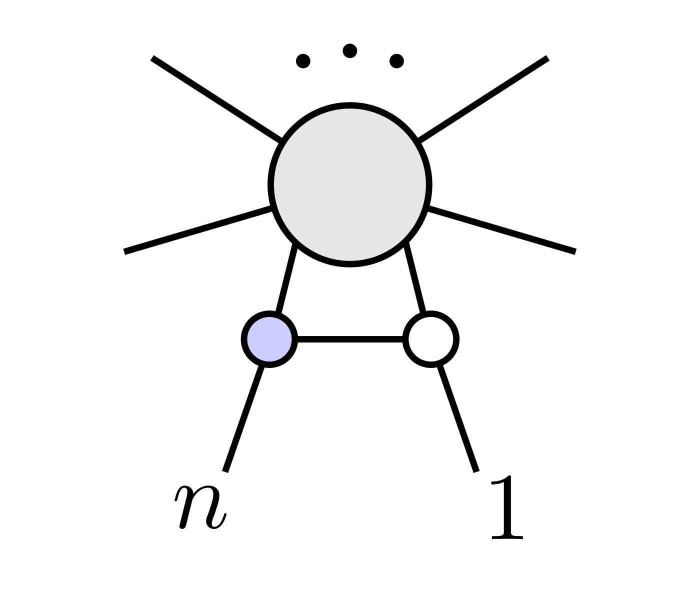


Question: Can we build amplitude from on-shell diagrams?

Recursion relations

BCFW shift

- Consider following diagram



One more loop

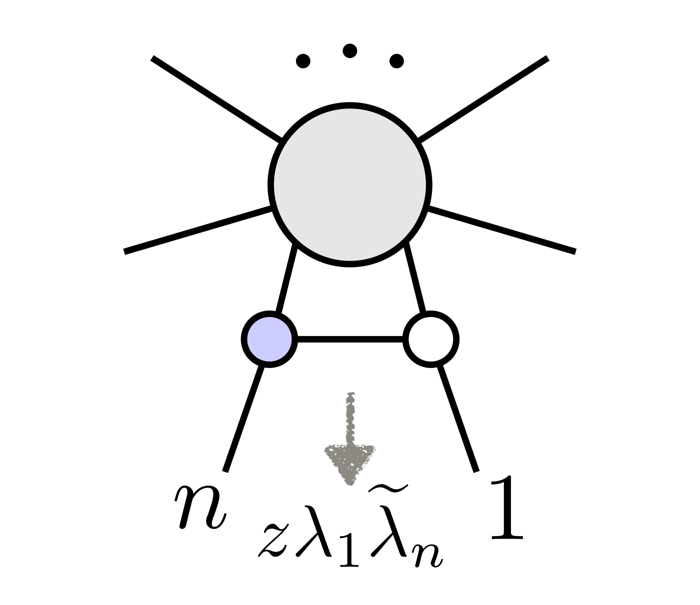
Three more on-shell conditions

↓

Adding one parameter

BCFW shift

- Consider following diagram



One more loop

Three more on-shell conditions

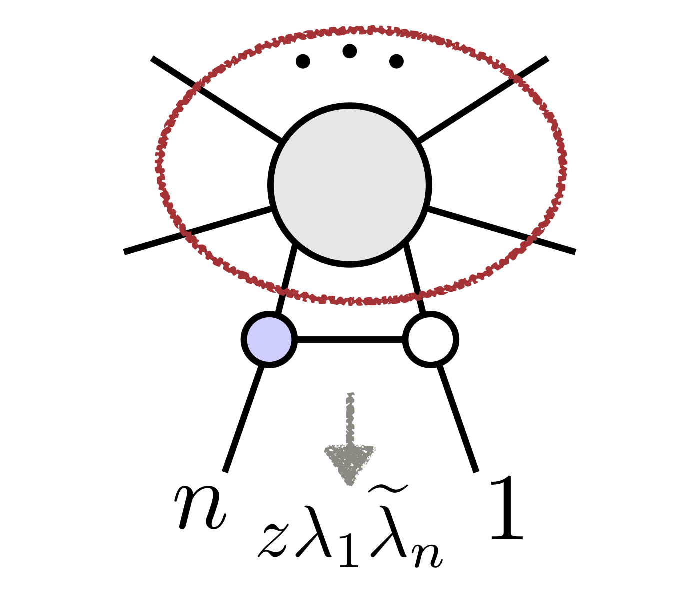
↓

Adding one parameter

New formula: $$K_1(z) = \frac{dz}{z} K_0(z)$$

BCFW shift

- Consider following diagram



One more loop

Three more on-shell conditions

↓

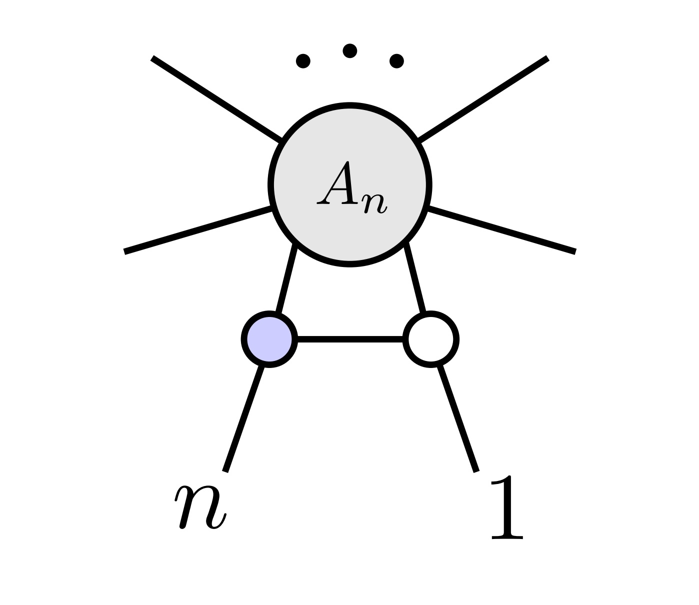
Adding one parameter

New formula: $$K_1(z) = \frac{dz}{z} K_0(z)$$ → Old on-shell diagram with shift

$$\lambda_n \to \lambda_n + z\lambda_1$$
$$\tilde{\lambda}_1 \to \tilde{\lambda}_n - z\tilde{\lambda}_1$$

BCFW recursion relations

- Suppose the blob is the amplitude



Shifted amplitude $\lambda_n \to \lambda_n + z\lambda_1$

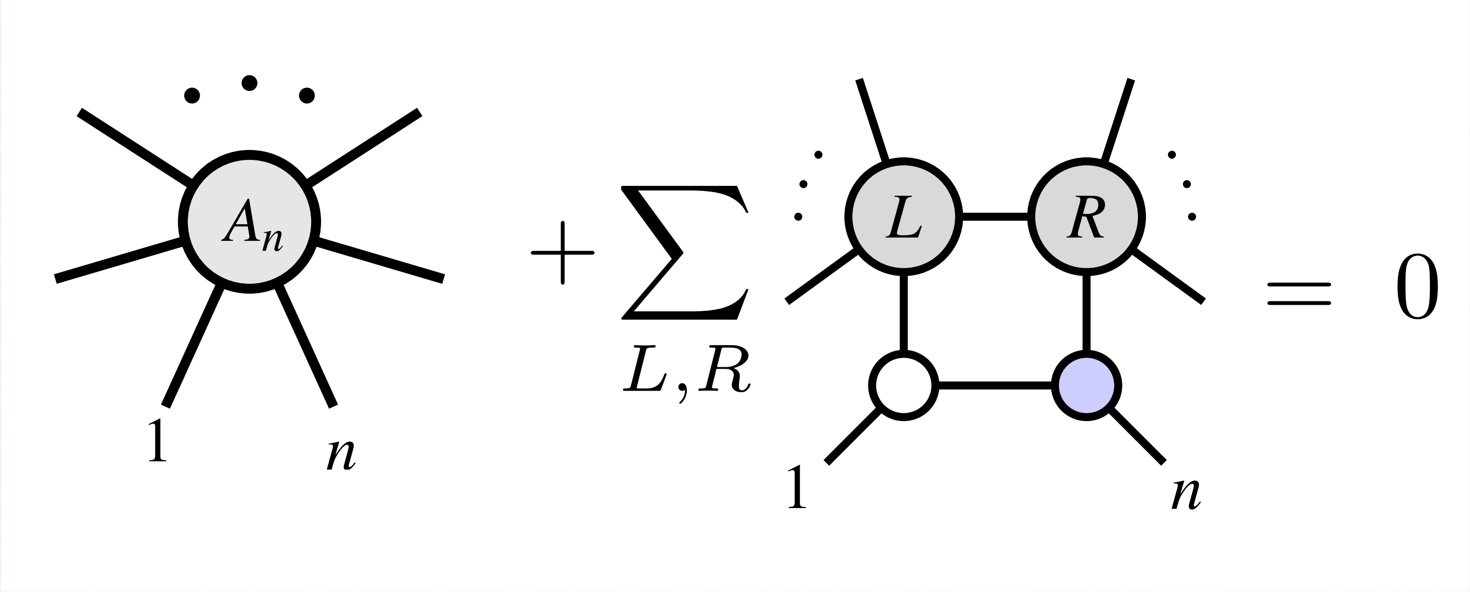
$= \mathcal{A}_n(z)$ $\qquad \widetilde{\lambda}_1 \to \widetilde{\lambda}_n - z\widetilde{\lambda}_1$

- Cauchy formula $\quad \partial\, \mathcal{A}_n(z) = 0$

Take the residue on $z = z_k$ $\quad \leftrightarrow \quad$ Erase an edge in the diagram

BCFW recursion relations

- Recursion relations for amplitude

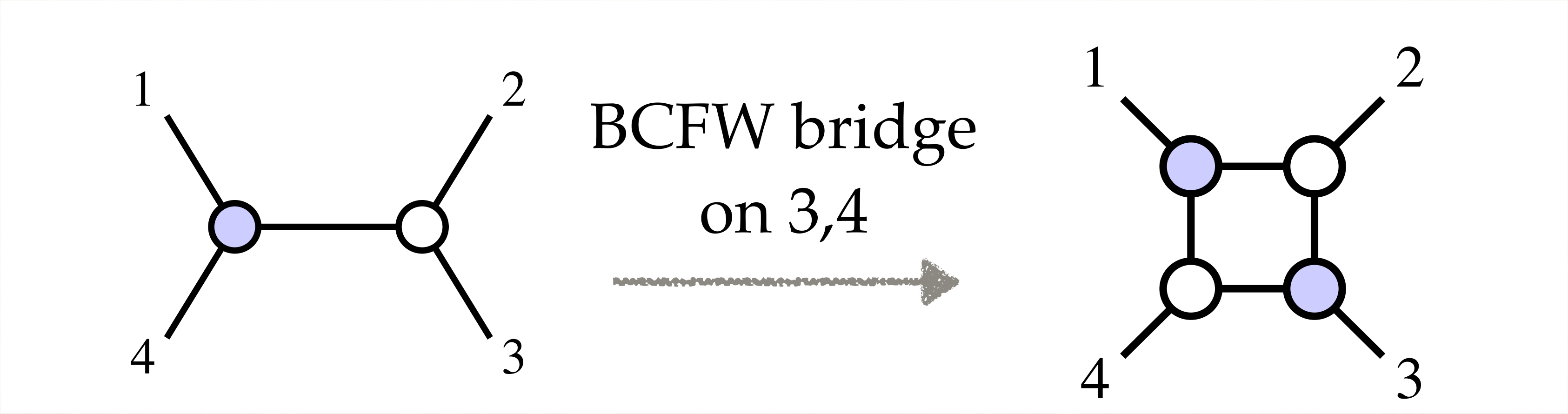


- Tree-level amplitude = sum of on-shell diagrams

- Term-by-term identical to terms in BCFW recursion

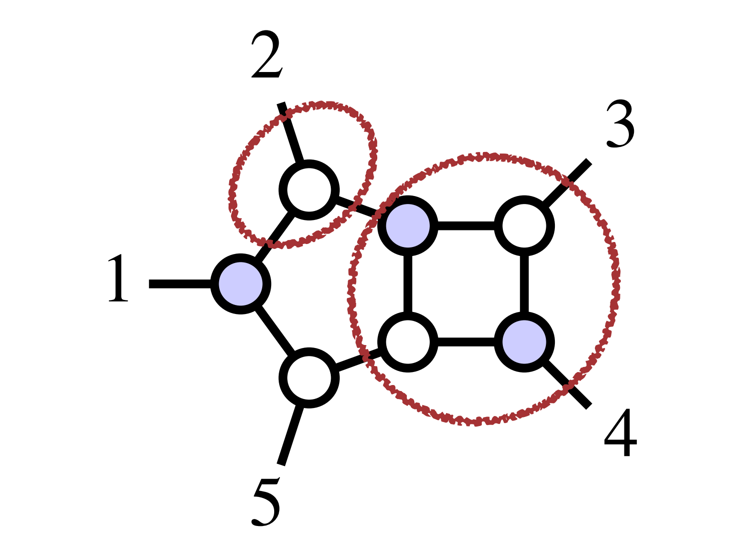
Simple examples

- Four point: only one factorization channel



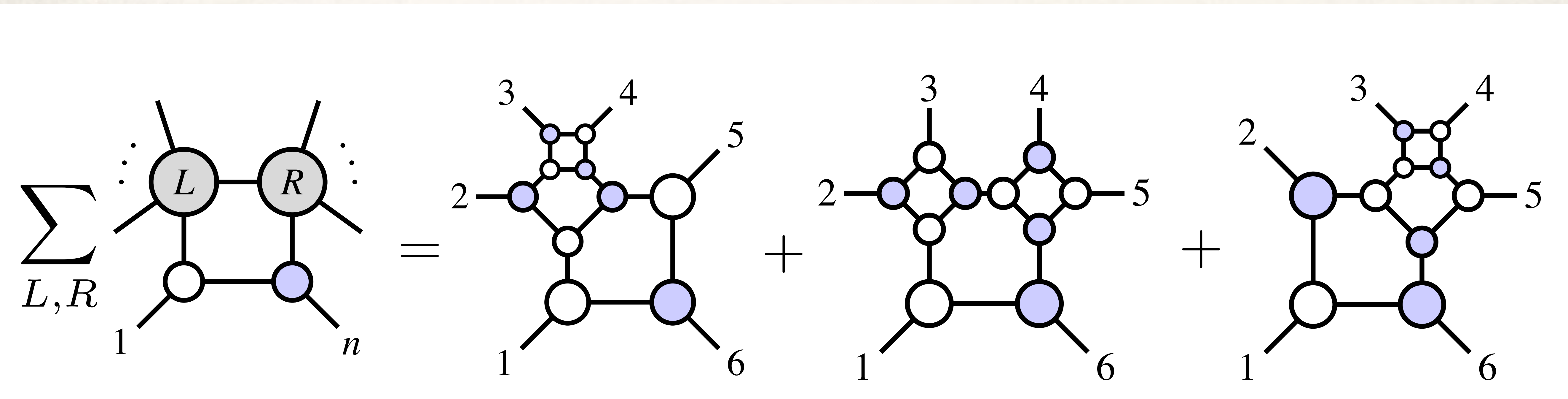
- Five point amplitude

Bridge 5,1 on 3pt
and 4pt amplitudes



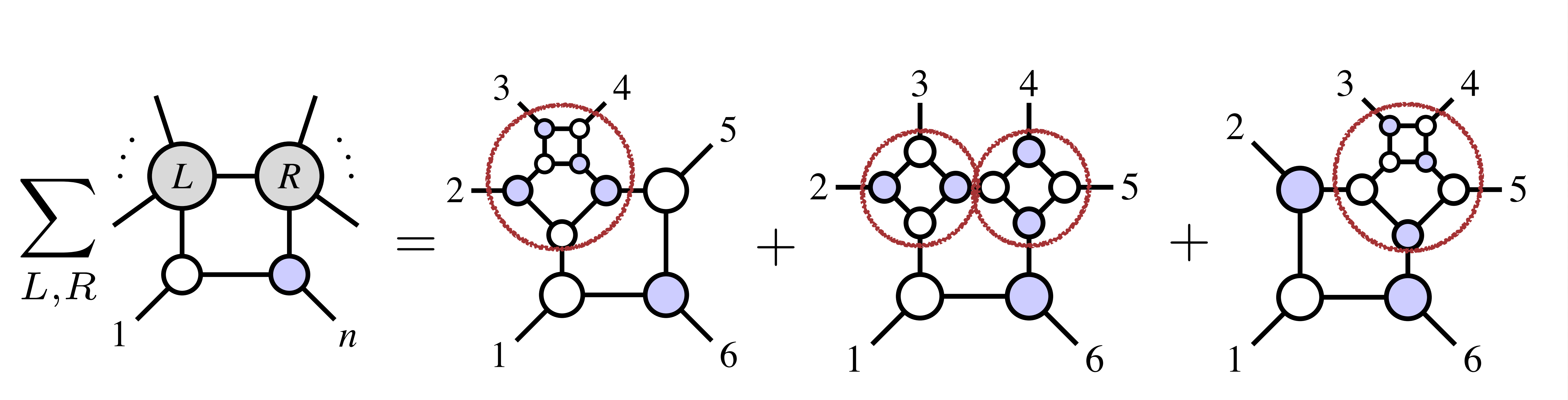
Six point example

- Three diagrams



Six point example

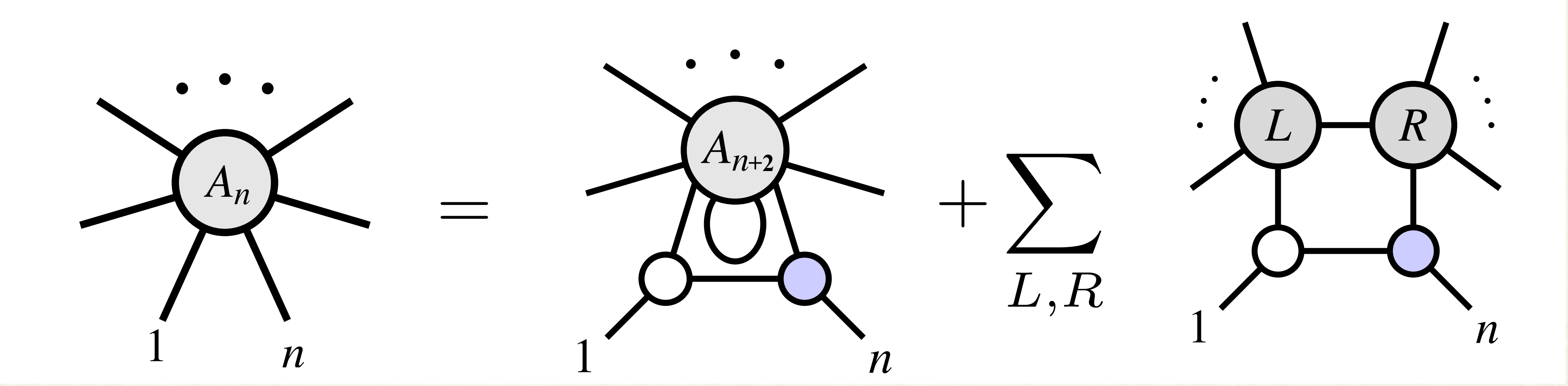
- Three diagrams



Loop recursion relations

(Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, JT 2010)

- Recursion relations for ℓ-loop integrand (limited use)



Loop recursion relations

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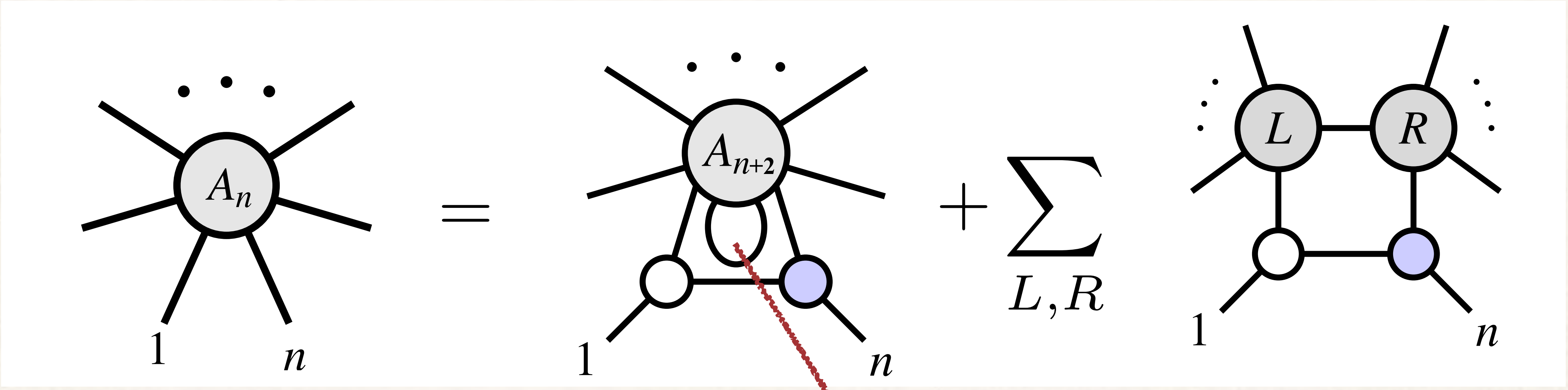


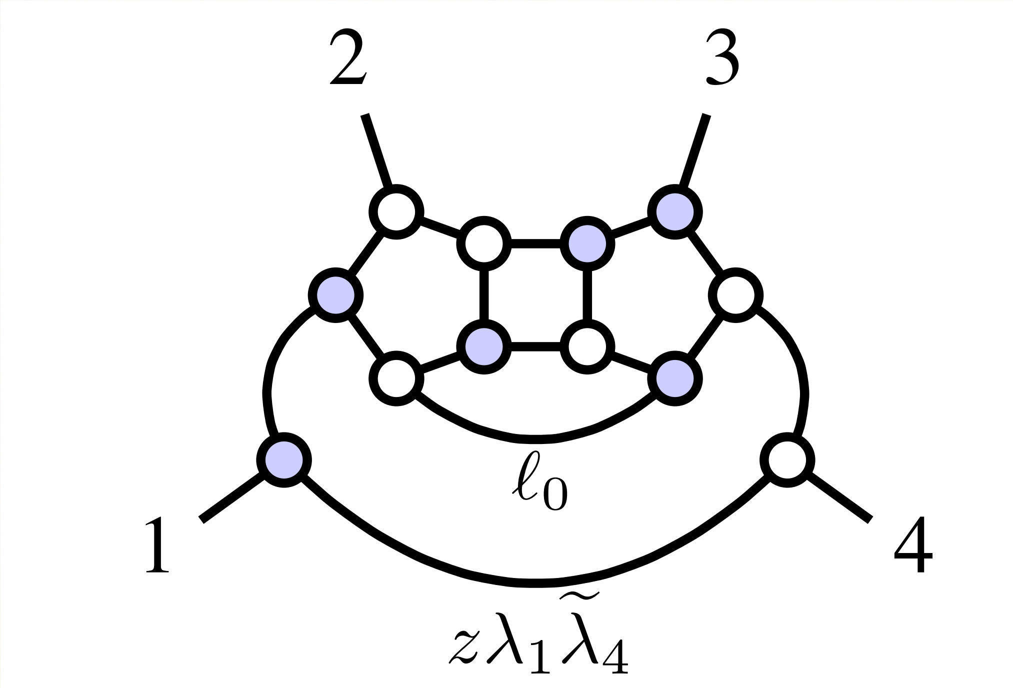
- Loop orders: $(\ell-1)$ ℓ_1,ℓ_2 $\ell_1+\ell_2=\ell$

- New loop momentum $\ell^{(L)} = \ell_0^{(L)} + z\lambda_1\widetilde{\lambda}_n$

$$(\ell_0^{(L)})^2 = 0$$

Four point one loop amplitude

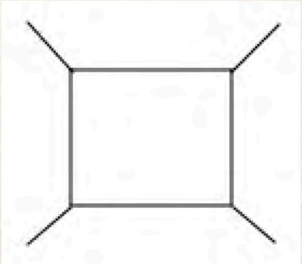
- It is given by one diagram



$$\ell = \ell_0 + z\lambda_1\widetilde{\lambda}_4$$

- 4 complex parameters -> impose reality condition

- 5-loop on-shell diagram = 1-loop off-shell box

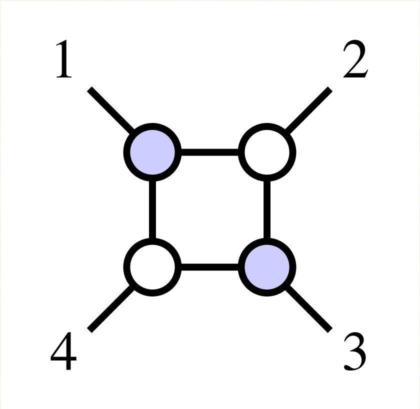


Dimensionality of diagrams

- Tree-level recursion: diagrams with $P = 4L$ contribute

  rational functions of external kinematics

  no delta functions, no free parameters



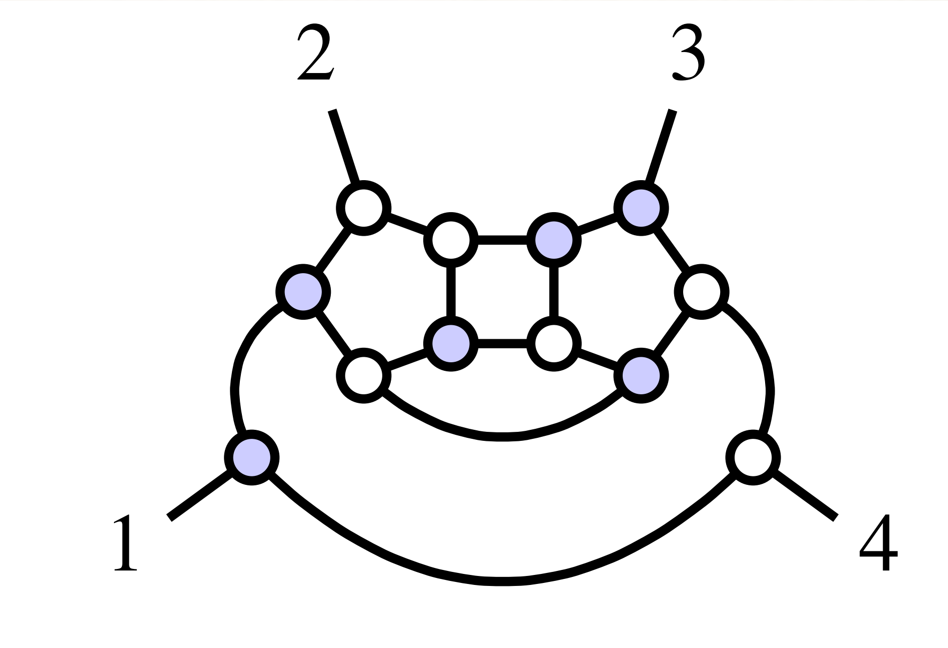
- These are also leading singularities of loop amplitudes

- Loop level: free parameters left

  components of loop momenta

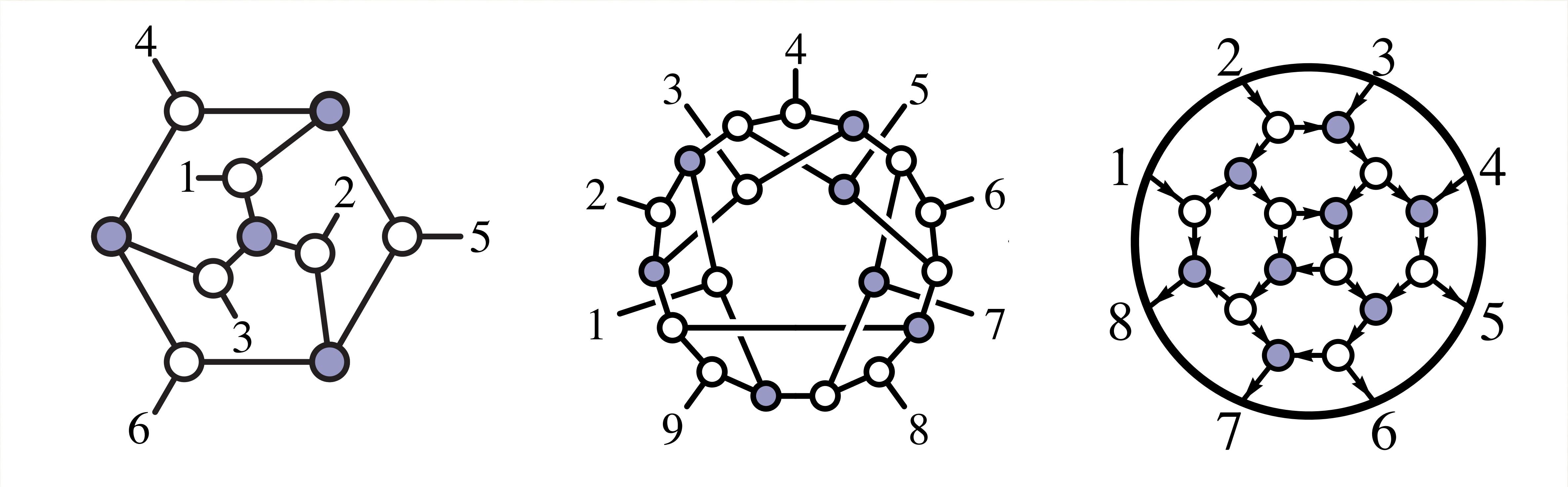
  $\text{free} = 4L - P$

  $P = 16$
  $L = 5$
  $\text{free} = 4$



On-shell diagrams in other theories

- On-shell diagrams are well defined in any QFT



- Gauge invariant on-shell functions, product of amplitudes
- Open question: how to reconstruct amplitudes from them?

Thank you for attention!