Scattering Amplitudes

LECTURE 3

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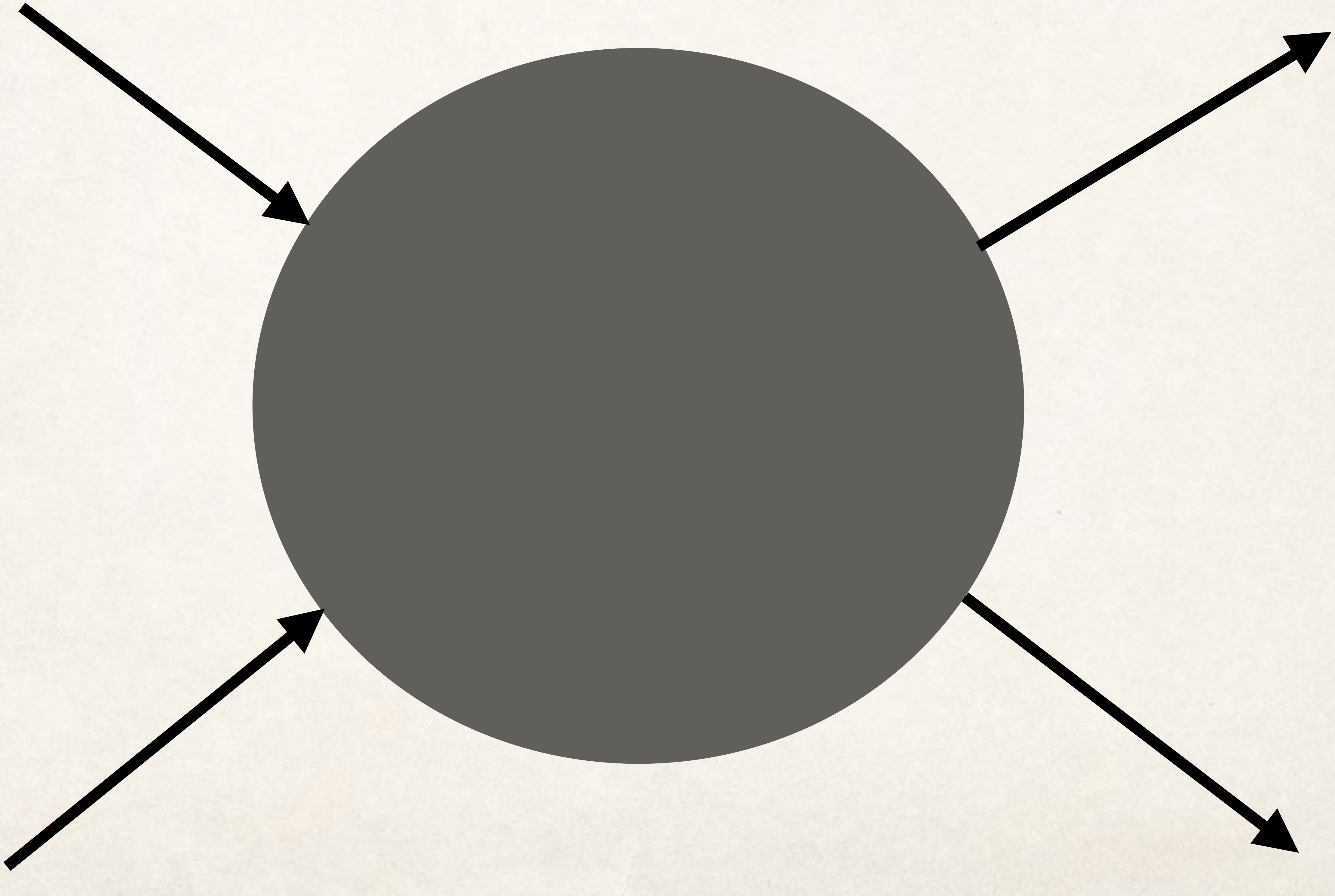
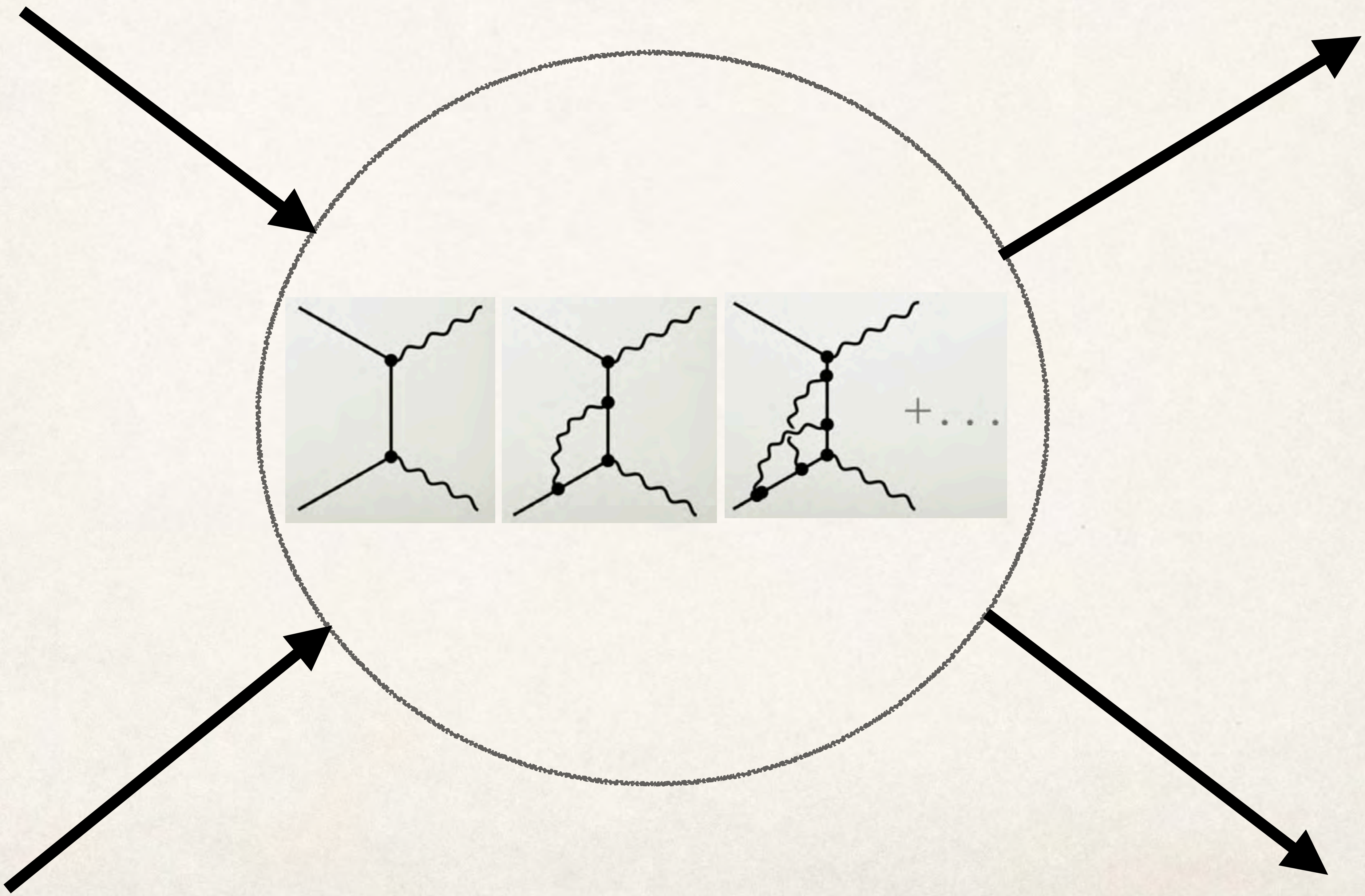
ICTP Summer School, June 2017

Review of Lectures 1-2

What does the blob represent?



Standard picture:
Feynman diagrams



Feynman diagrams

- Yang-Mills Lagrangian

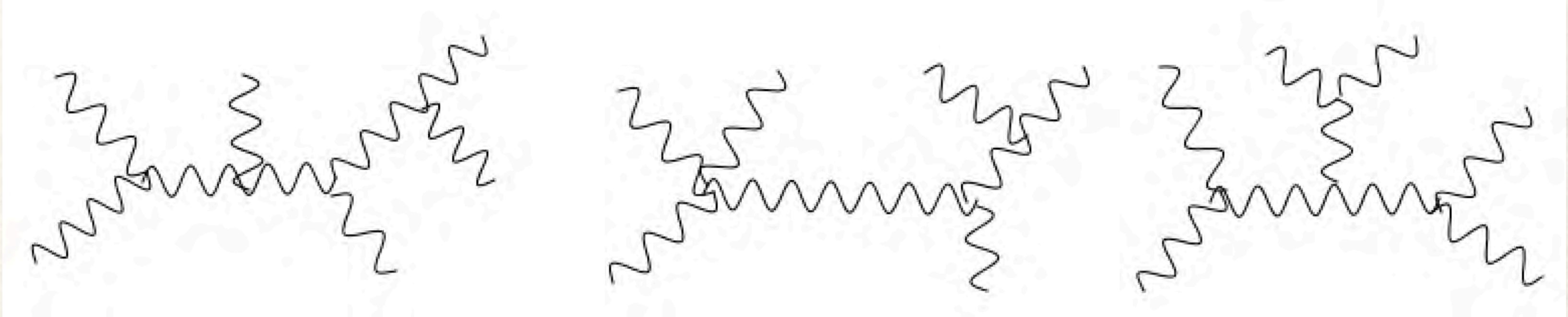
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \sim (\partial A)^2 + A^2\partial A + A^4$$

$$\sim f^{abc} g_{\mu\nu} p_\alpha \qquad\qquad \sim f^{abe} f^{cde} g_{\mu\nu} g_{\alpha\beta}$$

- Draw diagrams

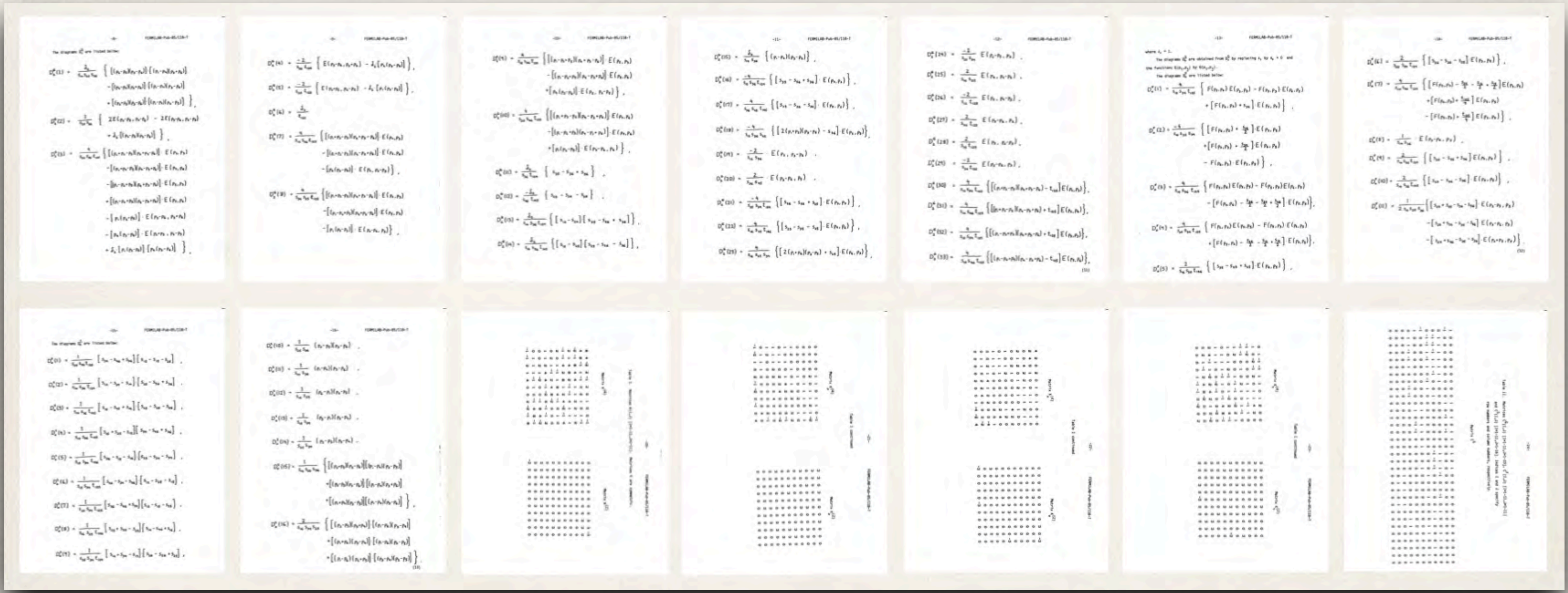
  Feynman rules

  Sum everything



Parke-Taylor formula

- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations



Parke-Taylor formula


Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

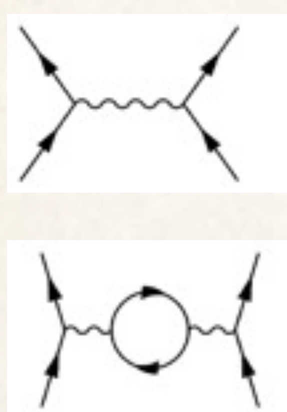
- Surprisingly simple expression for the final answer:

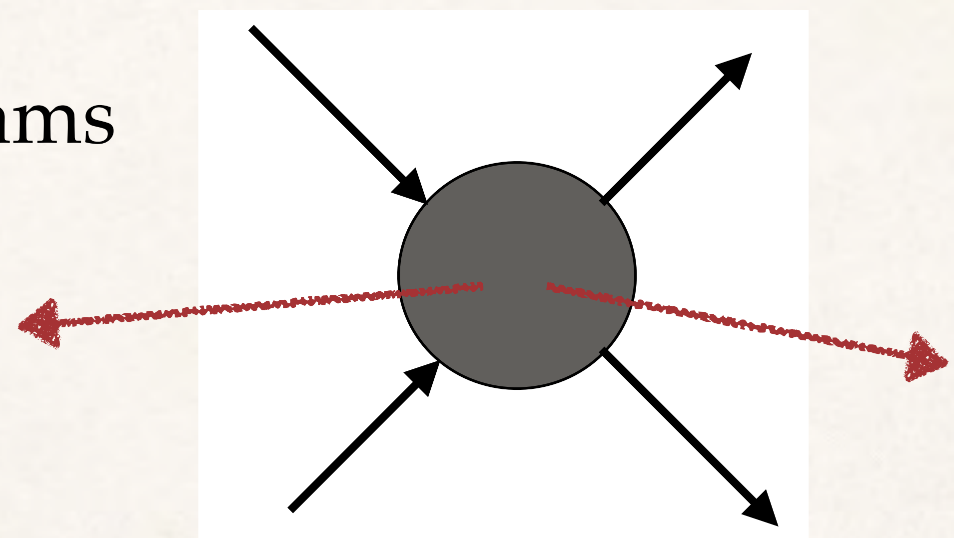
$$\mathcal{M}_6 = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}$$

Amplitude: unique object

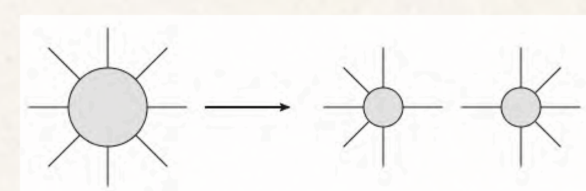
What is the scattering amplitude?

Feynman diagrams



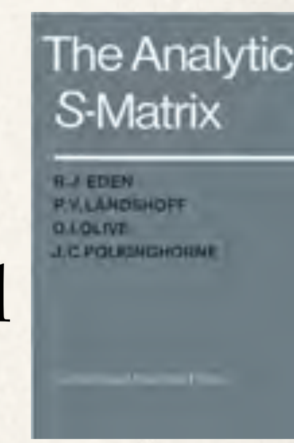


Unique object fixed
by physical properties



Was not successful

(1960s)



Modern methods use both:

- Calculate the amplitude directly
- Use perturbation theory

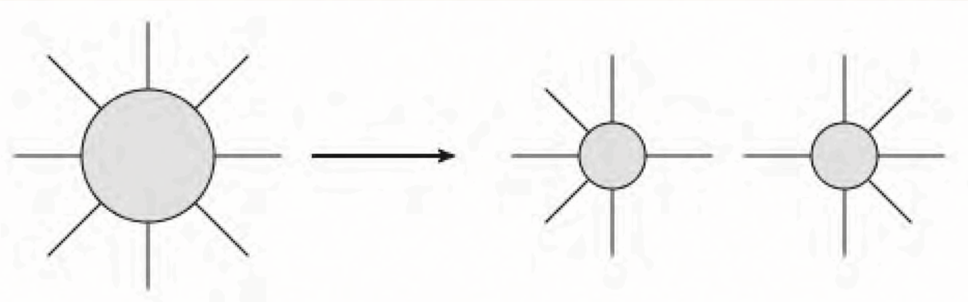
Locality and tree-level unitarity

- Only poles: Feynman propagators

Locality

$$\frac{1}{P^2} \quad \text{where} \quad P = \sum_{k \in \mathcal{P}} p_k$$

- On the pole



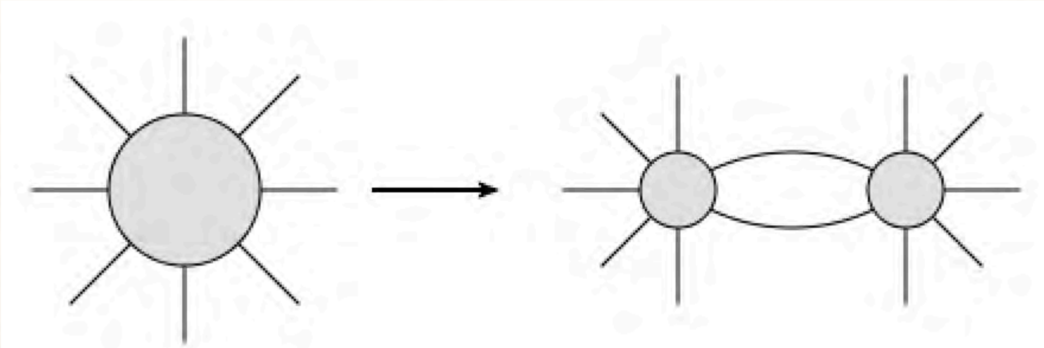
Unitarity

Feynman diagrams recombine on both sides into amplitudes

$$\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

Loop unitarity

- Analogue of tree-level unitarity at one-loop



$$\mathcal{M}^{1-loop} \xrightarrow[\ell^2=(\ell+Q)^2=0]{} \mathcal{M}_L^{tree} \frac{1}{\ell^2(\ell+Q)^2} \mathcal{M}_R^{tree}$$

Unitarity cut

- In general $\quad \text{Cut} \leftrightarrow \ell^2 = 0$

New viewpoint

- Rigidity of the final answer after we provide an input
- Feynman diagrams: input = Lagrangian
- New methods: locality, unitarity and gauge invariance
- Amplitude is a unique gauge invariant function which factorizes properly on all factorization channels

Unitarity methods



(Bern, Dixon, Kosower)

- Expansion of the amplitude

$$\mathcal{M}^{\ell-loop} = \sum_j a_j \int d\mathcal{I}_j$$

Cuts give product of trees

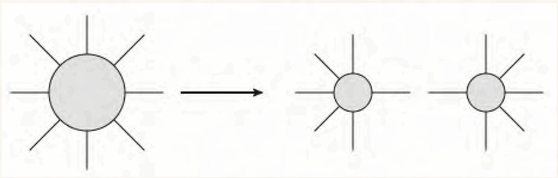
Linear combinations of coefficients a_j

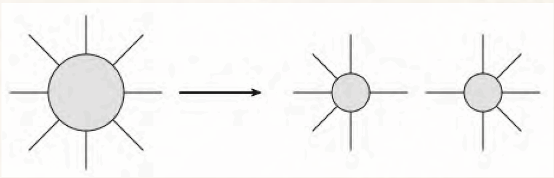
- Very successful method for loop amplitudes in different theories

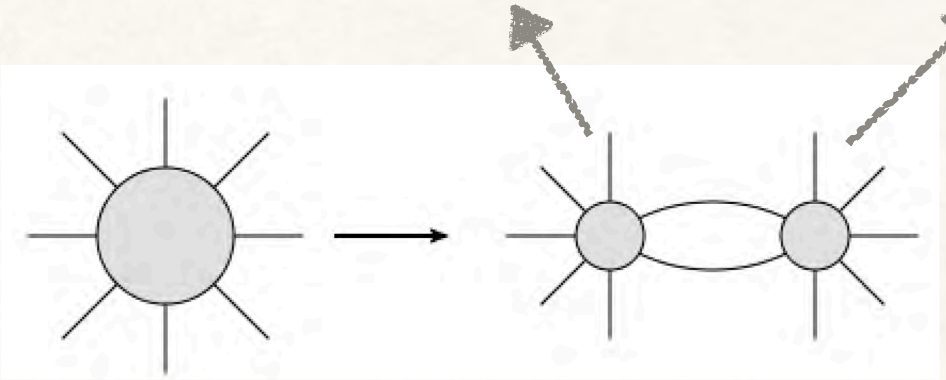
- Practical problems:
  - Find basis of integrals
  - Solve (long) system of equations

One-loop unitarity

- Higher cuts

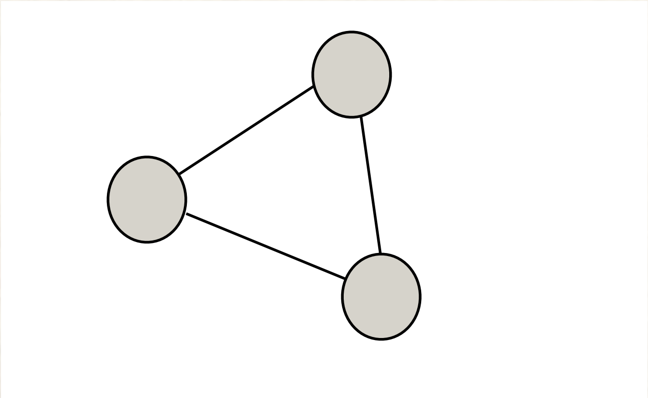






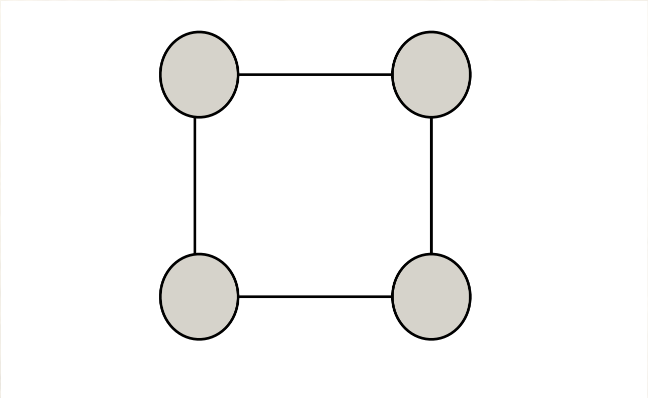
Triple cut

$$\ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = 0$$



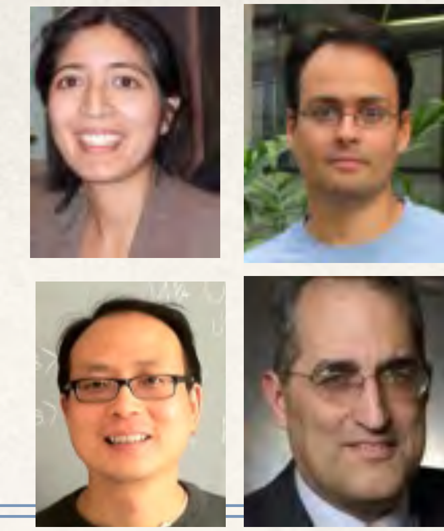
Quadruple cut

$$\ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = (\ell + Q_3)^2 = 0$$



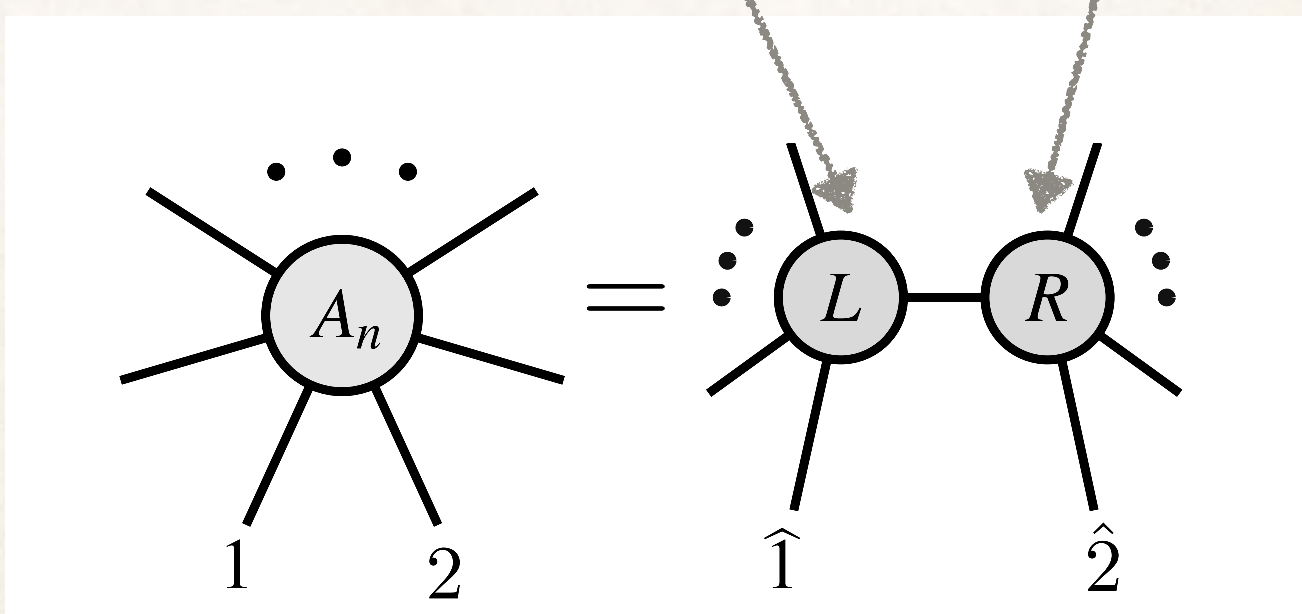
BCFW recursion relations

(Britto, Cachazo, Feng, Witten, 2005)



$$A_n = -\sum_j A_L(z_j)\frac{1}{P_j^2}A_R(z_j)$$

$$z_j = \frac{P_j^2}{2\langle 1|P_j|2]}$$



Chosen such that internal line is on-shell

Sum over all distributions of legs keeping 1,2 on different sides

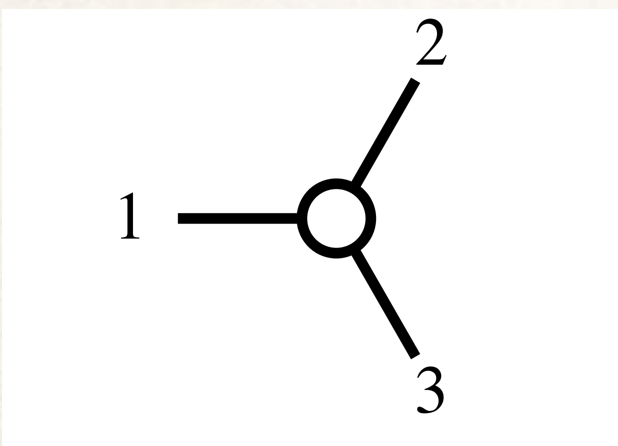
New starting point

- Both methods are very efficient
- Based on conservative ideas of applying general principles to uniquely fix the answer
- Main goal of this effort (at least for me): completely new picture for Quantum Field Theory
- No locality, unitarity — we need new starting point

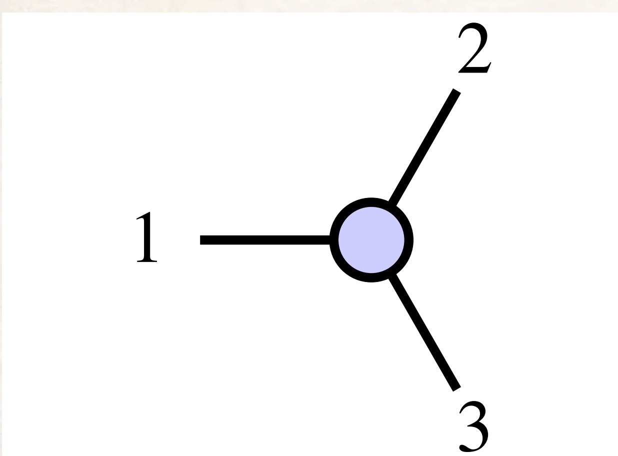
What is next?

Three point kinematics

- Two options



$$\lambda_1 \sim \lambda_2 \sim \lambda_3$$



$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

<u>Spinor helicity variables</u>

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \widetilde{\lambda}_{\dot{a}}$$

$$\langle 12 \rangle = \epsilon_{ab} \lambda_{1a} \lambda_{2b}$$

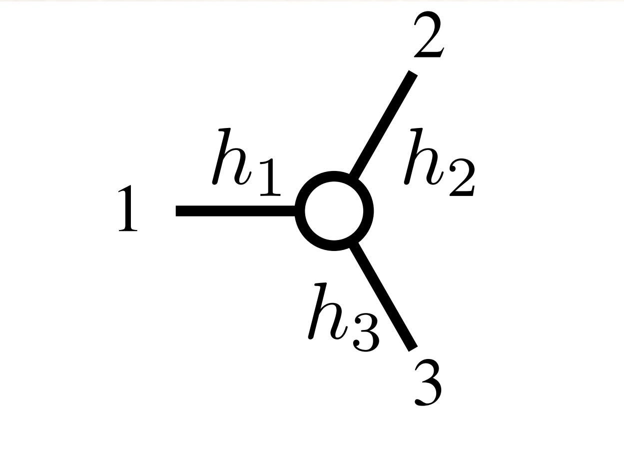
$$[12] = \epsilon_{\dot{a}\dot{b}} \lambda_{1\dot{a}} \lambda_{2\dot{b}}$$

Two solutions for 3pt kinematics

$$p_1^2 = p_2^2 = p_3^2 = (p_1 + p_2 + p_3) = 0$$

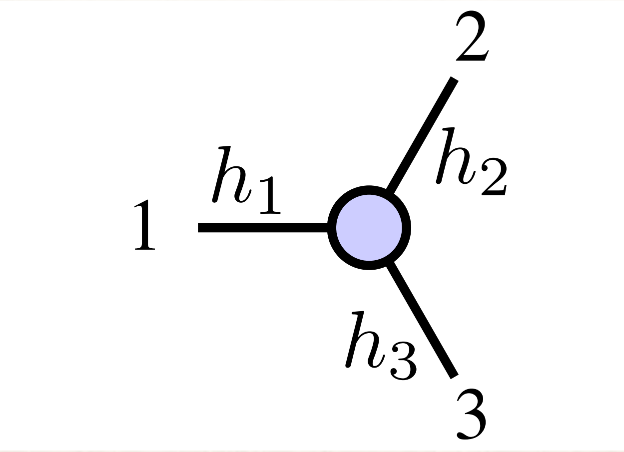
Three point amplitudes

- Two solutions for amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

$$h_1 + h_2 + h_3 \geq 0$$



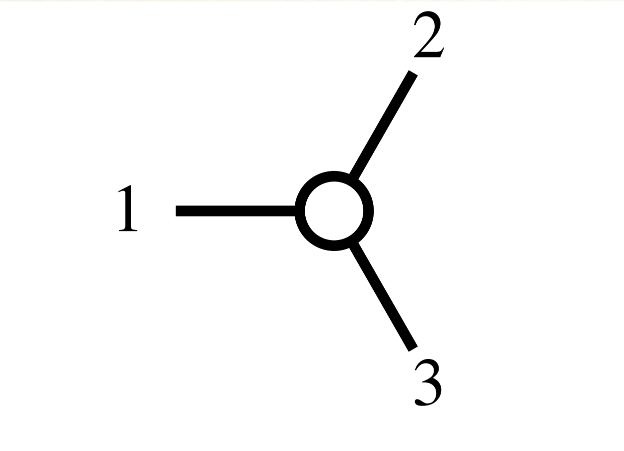
$$A_3 = \langle 12\rangle^{-h_1-h_2+h_3}\langle 23\rangle^{+h_1-h_2-h_3}\langle 31\rangle^{-h_1+h_2-h_3}$$

$$h_1 + h_2 + h_3 \leq 0$$

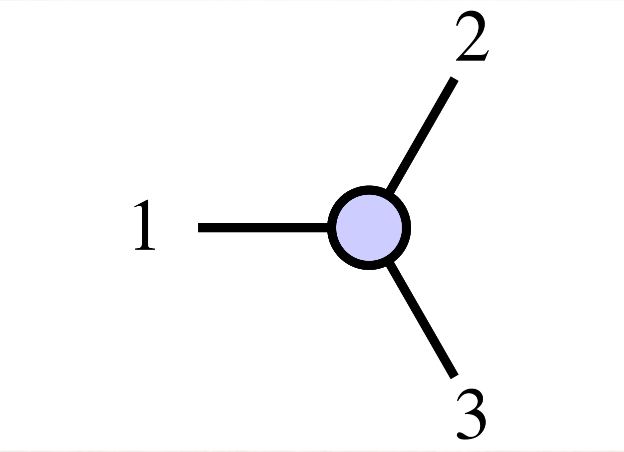
Supersymmetry: amplitudes of super-fields
(all component fields included)

Three point amplitudes

- In N=4 SYM: no need to specify helicities



$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2+[12]\widetilde{\eta}_3)}{[12][23][31]}$$



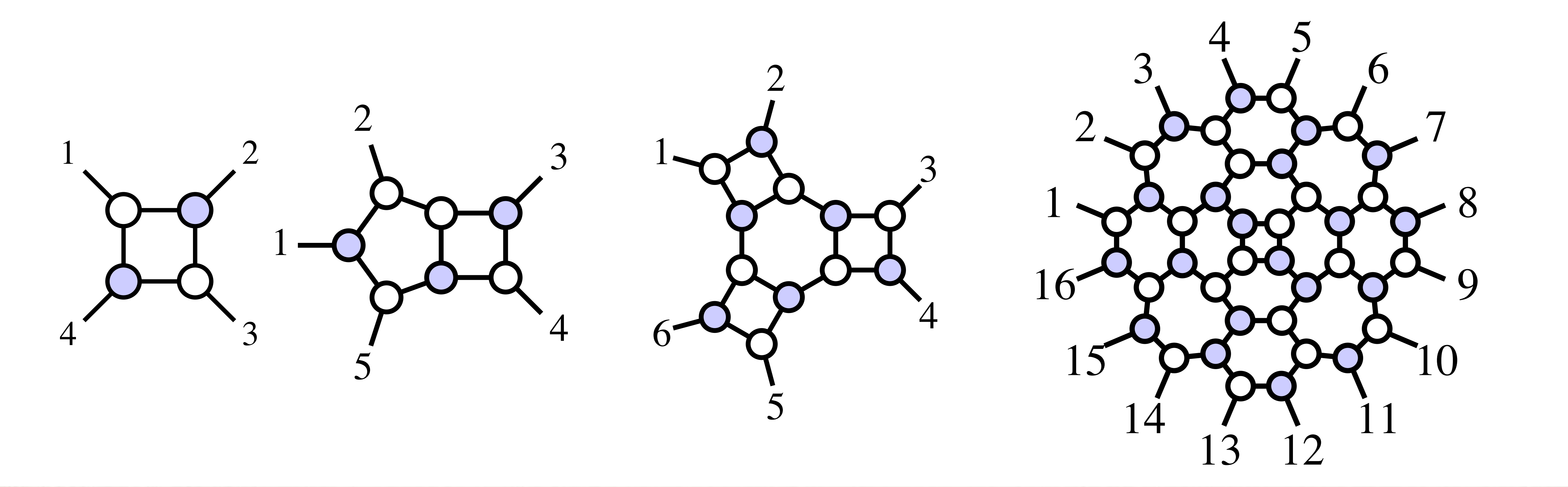
$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Easy book-keeping

Fully fixed in any QFT up to coupling

On-shell diagrams

- Draw arbitrary graph with three point vertices

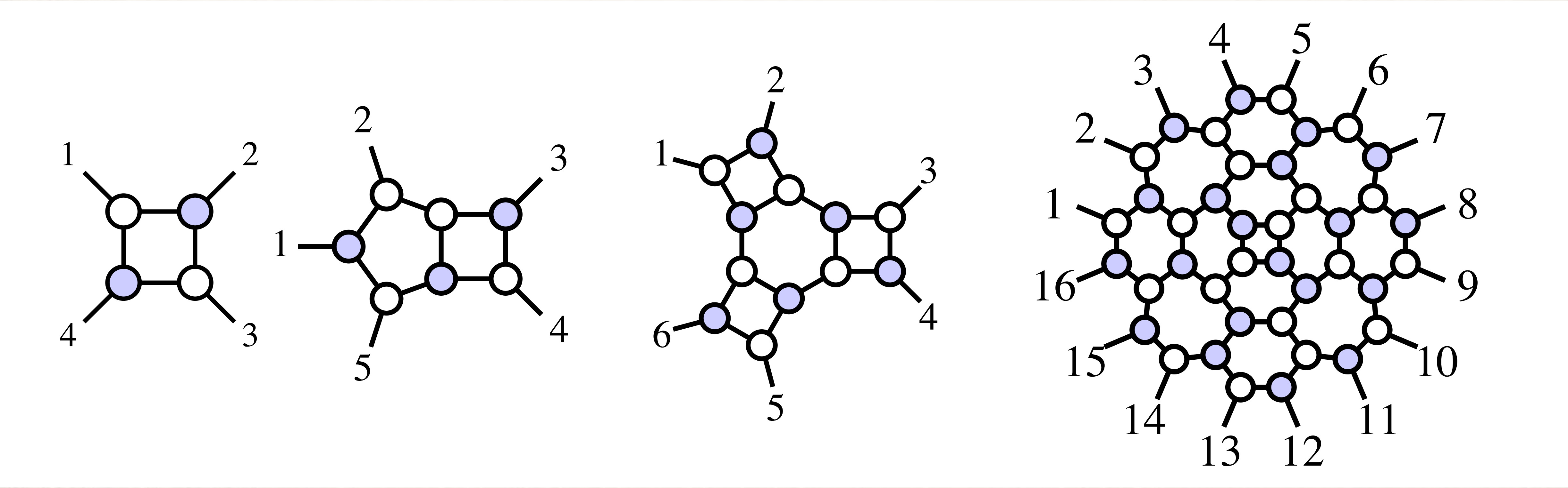


- Products of 3pt amplitudes: gauge invariant functions

- Well defined in any Quantum Field Theory

On-shell diagrams

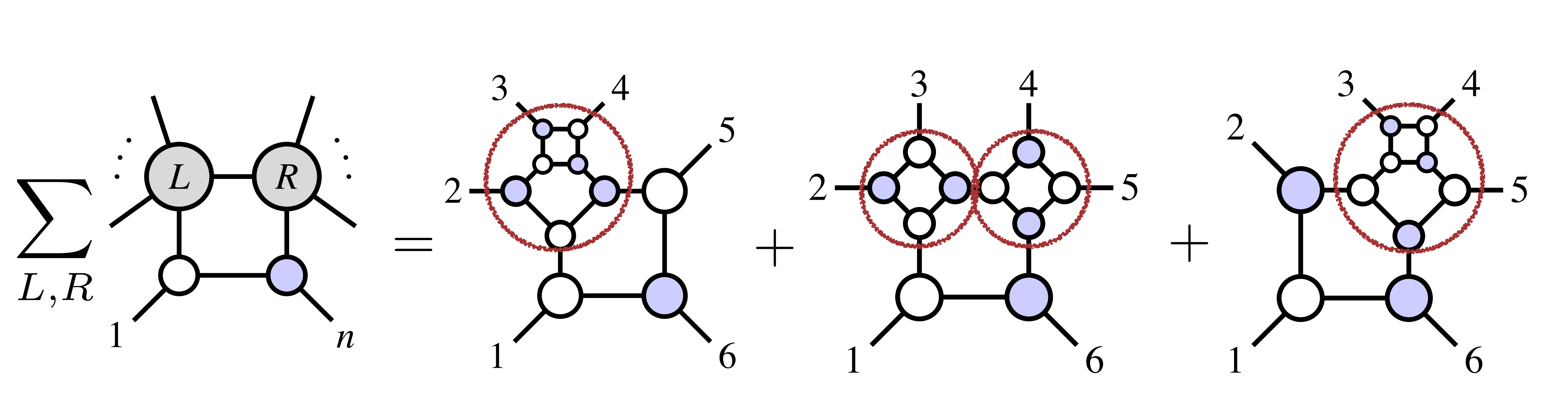
- Draw arbitrary graph with three point vertices



Question: Can we build amplitude from on-shell diagrams?

Recursion relations

- Six point example



- Implementation of known method in this language

On-shell diagrams

- On-shell diagrams: natural gauge invariant objects
- Based on the complete rigidity of 3pt amplitudes
- Recursion relations in this language, hopefully in the future also at loops in more generality

On-shell diagrams

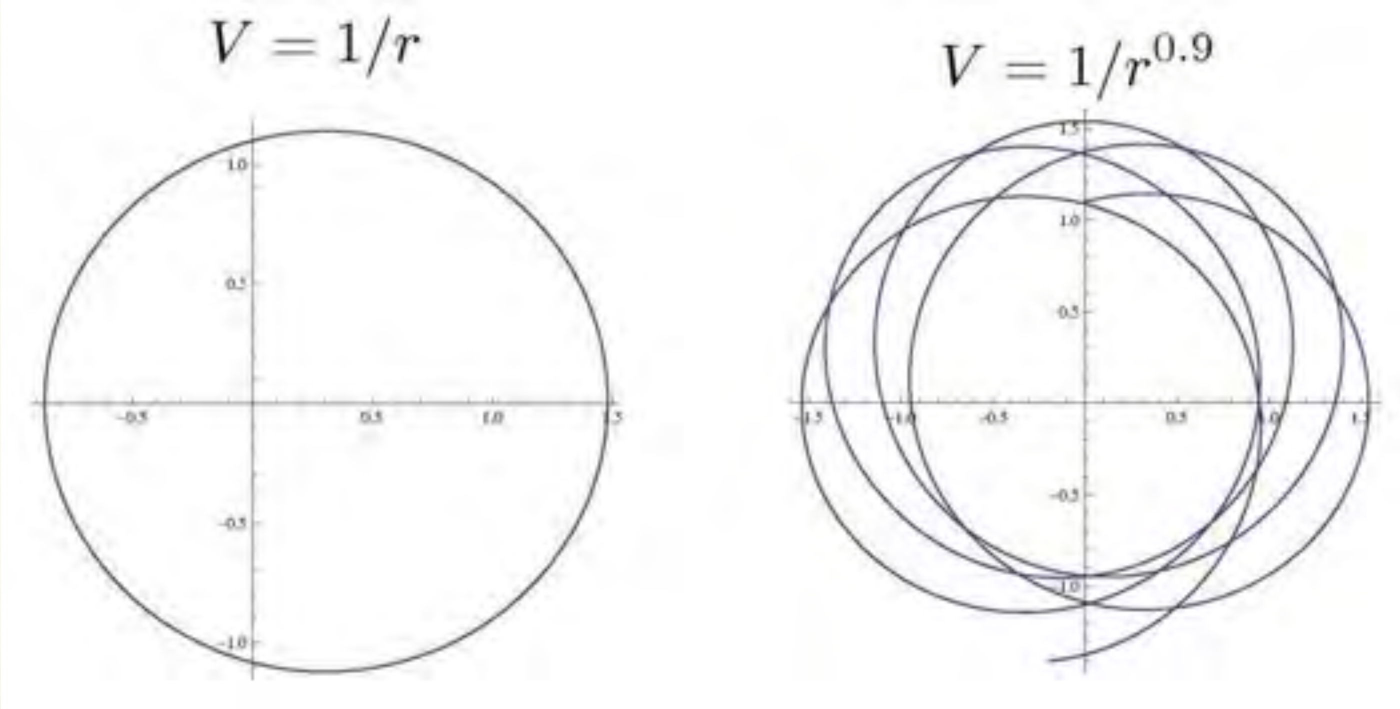
- Input: 3pt amplitude = fixed by Lorentz group and helicities of particles
- Still the same physics origin as Feynman diagrams, but implemented in much better language
- However, very surprisingly they are also starting point to a completely new story which brings us to the world of geometry

Hydrogen atom of gauge theories

Toy models

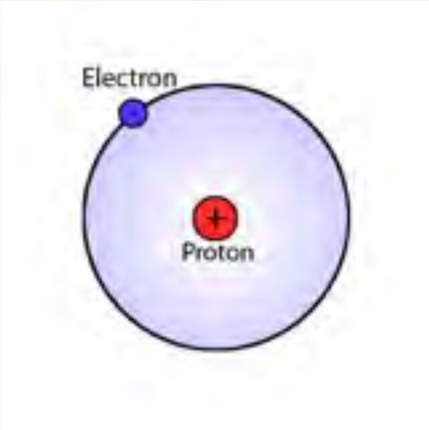
- Hard to make progress on difficult questions in full generality: time-proven method - choose toy model
- Long history of "integrable models": exactly solvable
- **Kepler problem:**
  - orbits do not precess
  - Runge-Lenz vector

$$\vec{A} = \frac{1}{2}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \mu \frac{\lambda}{4\pi} \frac{\vec{x}}{|x|}$$



Toy models

- **Hydrogen atom:** $$H = \frac{1}{2m}p^2 - \frac{k}{r}$$



    - Hidden symmetry: Runge-Lenz-Pauli vector
    - Allows to find spectrum

- Toy model for QFT: **planar N=4 SYM theory**

(Brink, Schwarz, Scherk) (1984)

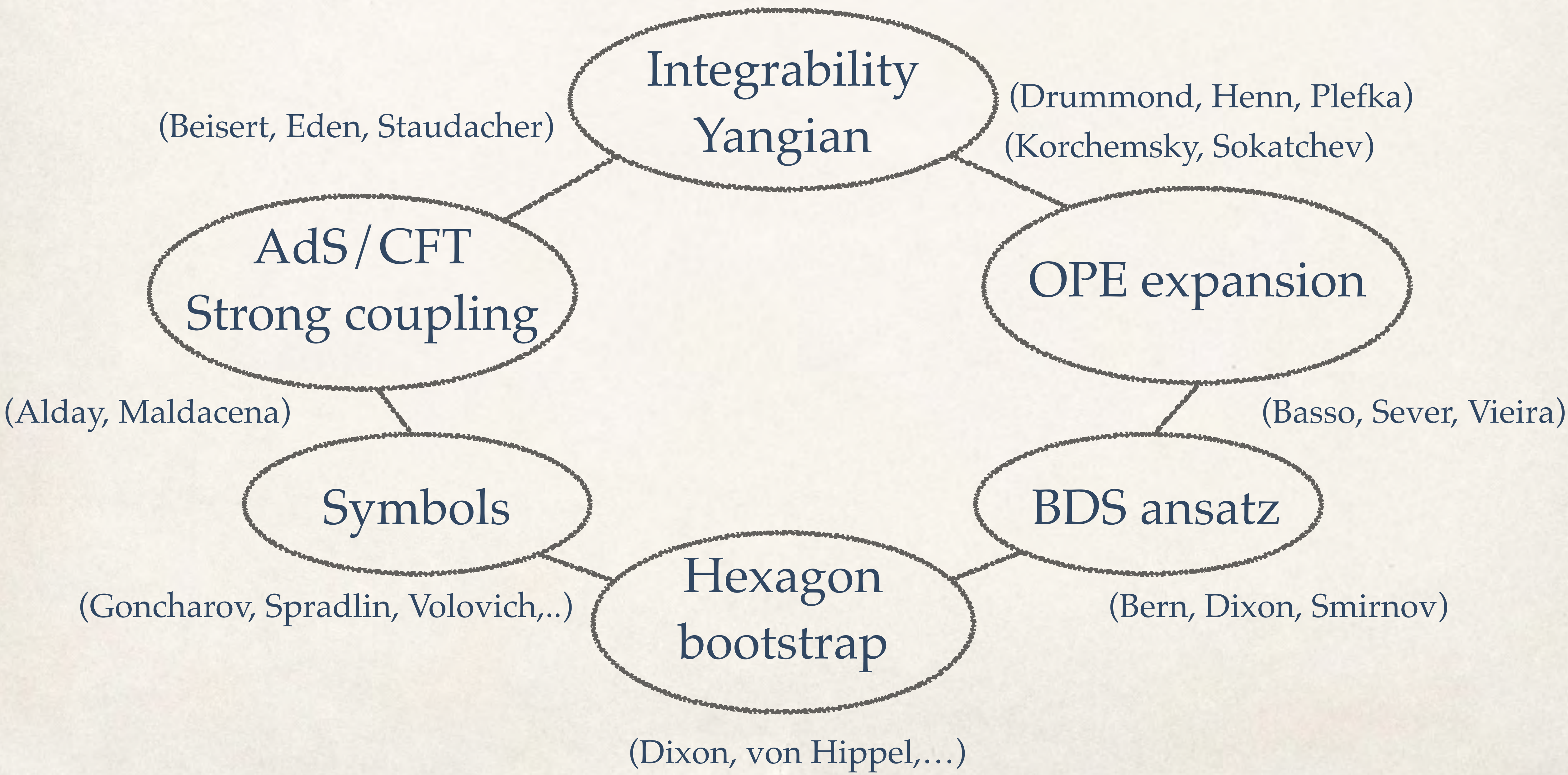
    - Theory of quarks and gluons, similar to QCD but no confinement
    - Hidden symmetry: Yangian - connection to 2d integrable models

(Drummond, Henn, Plefka, Korchemsky, Sokatchev) (2007)

    - Great theory to test new ideas in QFT

Hydrogen atom of gauge theories

- Useful playground for many theoretical ideas



Amplitudes in N=4 SYM

- N=4 superfield

$$\Phi = G_+ + \tilde{\eta}_A \Gamma_A + \frac{1}{2}\tilde{\eta}^A\tilde{\eta}^B S_{AB} + \frac{1}{6}\epsilon_{ABCD}\tilde{\eta}^A\tilde{\eta}^B\tilde{\eta}^C\overline{\Gamma}^D + \frac{1}{24}\epsilon_{ABCD}\tilde{\eta}^A\tilde{\eta}^B\tilde{\eta}^C\tilde{\eta}^D G_-$$

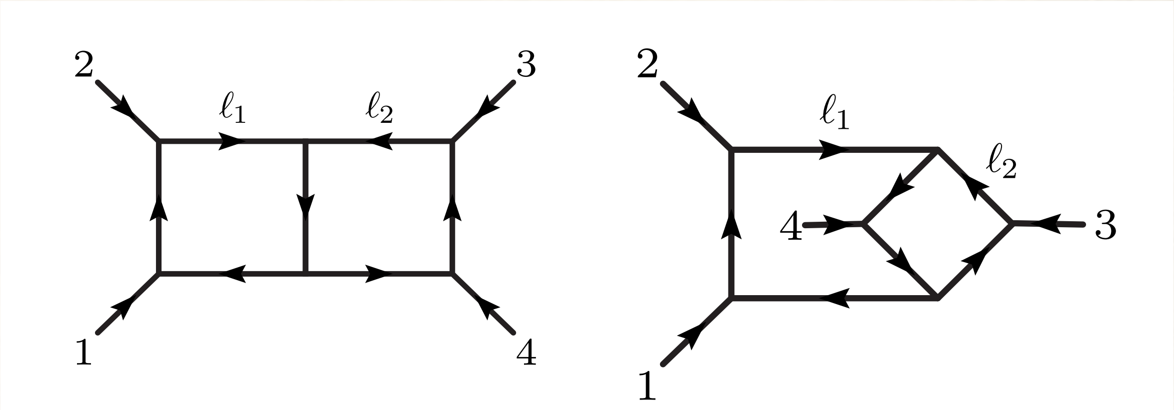
- Superamplitudes: $\mathcal{A}_n = \sum_{k=2}^{n-2} \mathcal{A}_{n,k}$

Component amplitudes with power $\tilde{\eta}^{4k}$

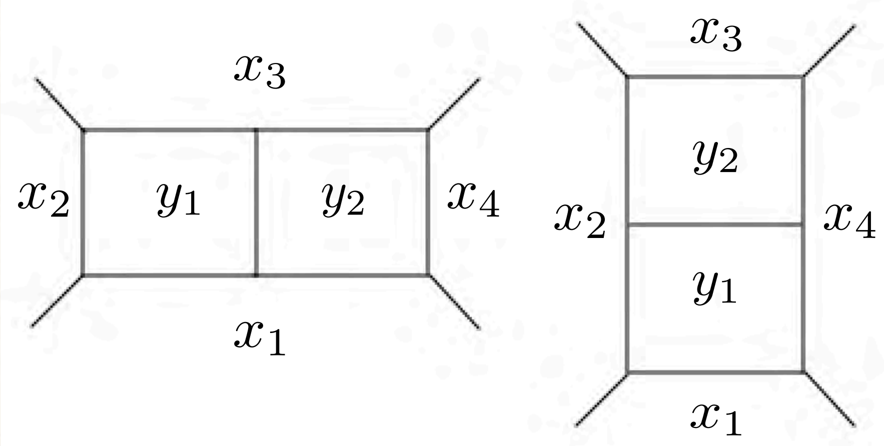
- Planarity: limit $N \to \infty$ - simplification

Dual variables

- Generally, each diagram has its own variables
  - No global loop momenta
  - Each diagram: its own labels



- Planar limit: dual variables



$$k_1 = (x_1 - x_2) \quad k_2 = (x_2 - x_3)$$

$$\ell_1 = (x_3 - y_1) \quad \ell_2 = (y_2 - x_3)$$

etc

Global variables

Integrand

- Using these variables: define a single function

$$\mathcal{M} = \int d^4y_1 \ldots d^4y_L \, \mathcal{I}(x_i, y_j)$$

Integrand

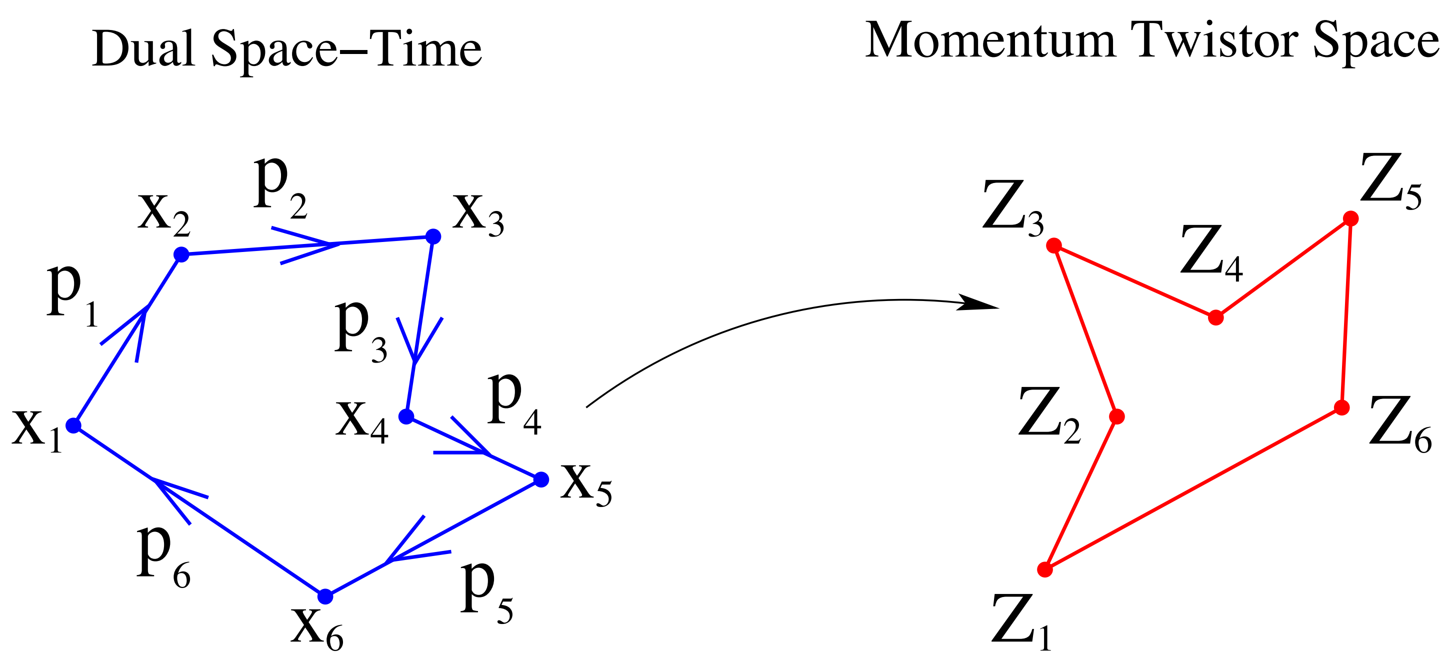
- Ideal object to study: rational function, no divergencies
- Hidden dual conformal symmetry in these variables
- (There is a hidden symmetry in QCD at tree-level)

Momentum twistors

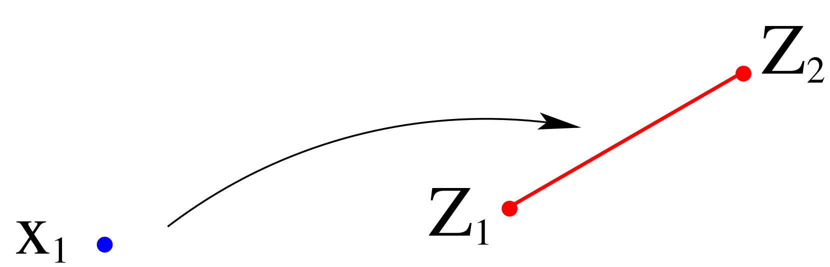
(Hodges 2009)

- New variables: points in $\mathbb{P}^3$

$$Z = \begin{pmatrix} \lambda_a \\ x_{a\dot{a}}\tilde{\lambda}_{\dot{a}} \end{pmatrix}$$



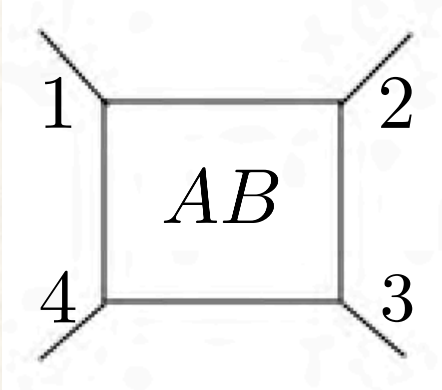
Cyclic ordering crucial



$$p_j = x_{j+1} - x_j$$

Momentum twistors

- Dual conformal: SL(4) on momentum twistors
- Dual conformal invariants: $\langle 1234\rangle = \epsilon_{abcd} Z_1^a Z_2^b Z_3^c Z_4^d$
  $\langle 1234\rangle = \langle 12\rangle\langle 23\rangle\langle 34\rangle[23]$
- Loop momenta: $\ell \leftrightarrow Z_A Z_B$



$$\frac{d^4\ell\, st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

Back to on-shell diagrams

Historic coincidence

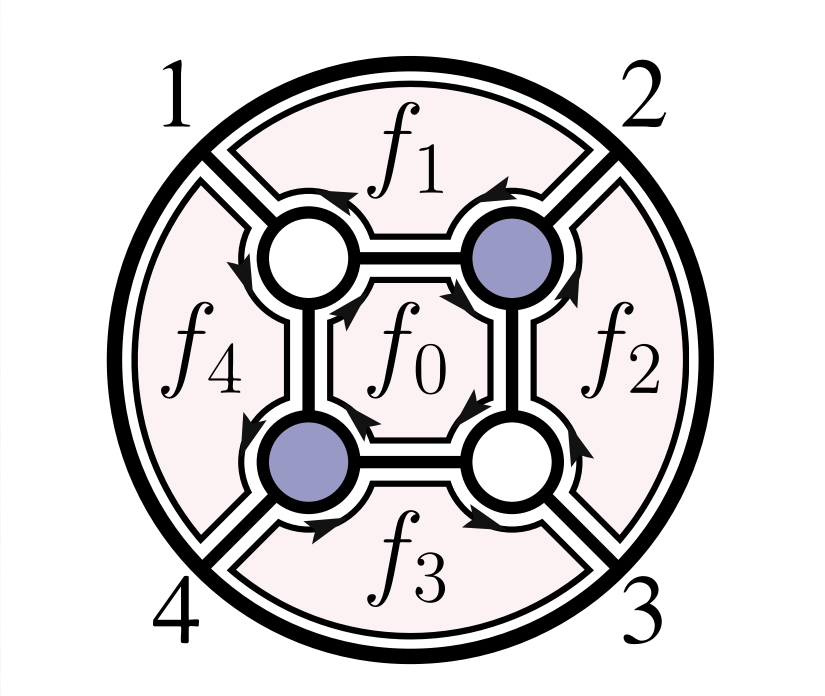
- Same diagrams appeared in mathematics around 2005
- Very different motivation:

$$k \overset{n}{\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}} \qquad \begin{vmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{vmatrix} \geq 0$$

- Goal: find algorithm for writing real matrices with positive minors (mod GL(k)): positive Grassmannian

Plabic graphs

- Draw a graph with two types of three point vertices
- Associate variables with the faces of diagram

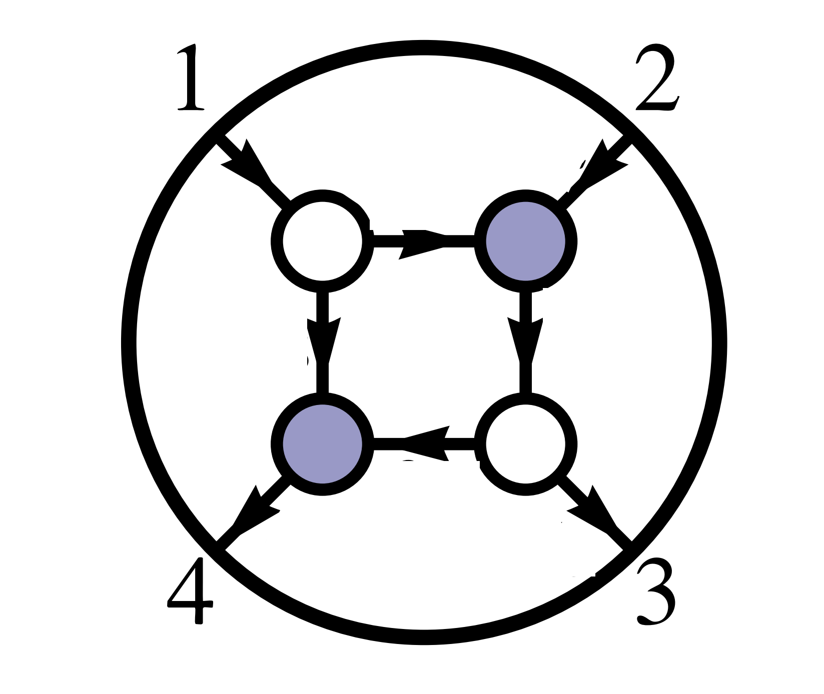


with the property

$$\prod_j f_j = -1$$

Perfect orientation

- Arrows on all edges



Perfect orientation

White vertex: one in, two out

Black vertex: two in, one out

- Not unique, always exists at least one
- Two (k) incoming, two (n-k) outgoing

Entries of matrix

- Define elements of $(k \times n)$ matrix

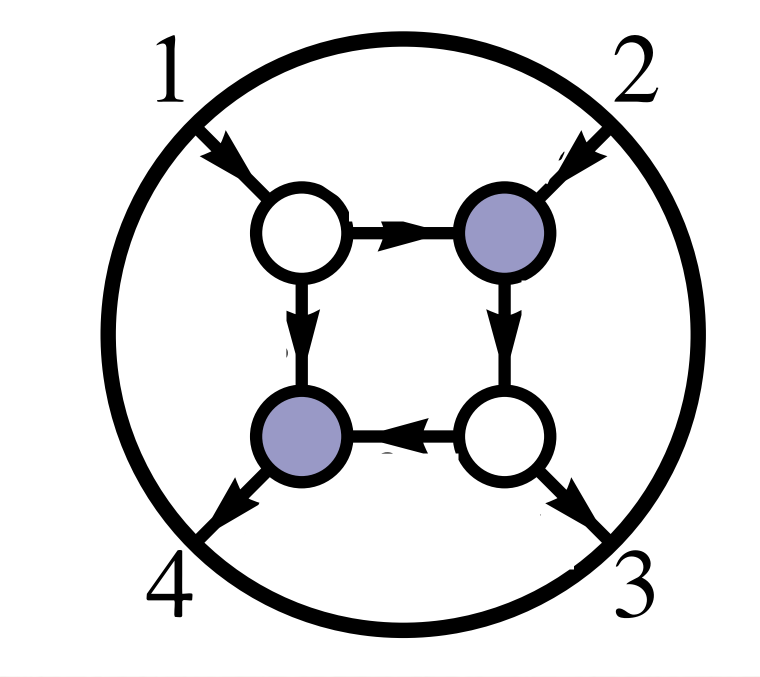
$$c_{ab} = -\sum_{\Gamma} \prod_{j} (-f_j)$$

product of all face variables to the right of the path

incoming

if b incoming
$c_{aa} = 1$
$c_{ab} = 0$

sum over all allowed paths



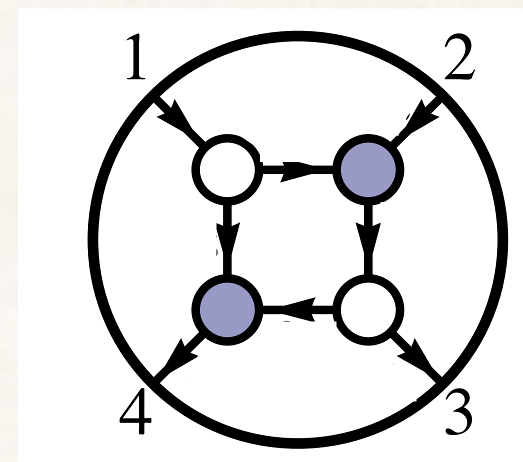
- Example: $c_{11} = c_{22} = 1 \quad c_{12} = c_{21} = 0$

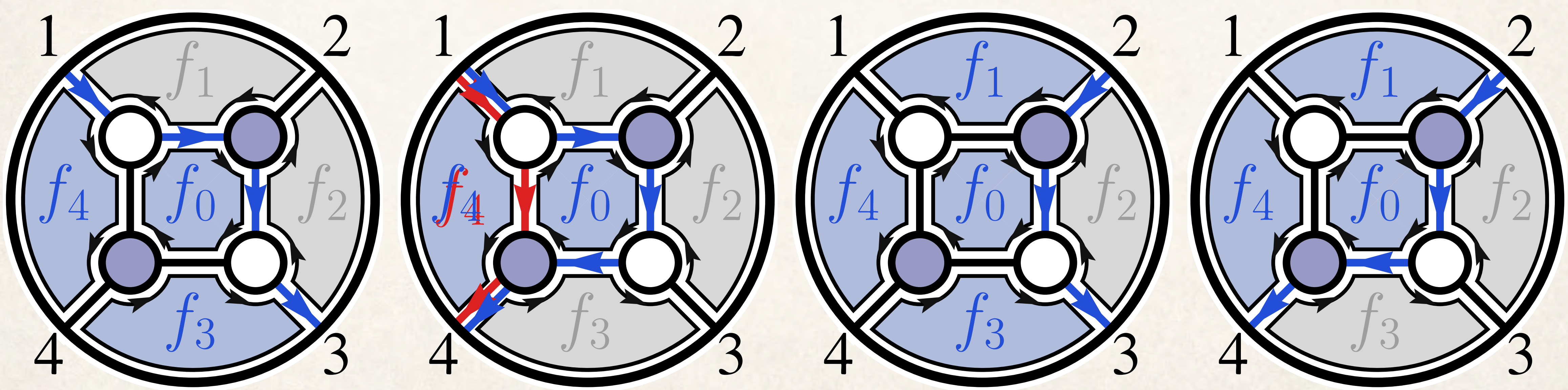
$c_{13} = *,\ c_{14} = *,\ c_{23} = *,\ c_{24} = *$

Entries of matrix

Apply on our example

$$c_{ab} = -\sum_{\Gamma}\prod_{j}(-f_j)$$





$-c_{13} = -f_0\, f_3\, f_4$ $\quad -c_{14} = f_0\, f_4 - f_4$ $\quad -c_{23} = f_0\, f_1\, f_3\, f_4$ $\quad -c_{24} = f_0\, f_1\, f_4$

Entries of matrix

- The matrix is

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

f_2 eliminated

- There always exists choice of signs for f_i such that

$$C \in G_+(k,n)$$

- For our case:

$$m_{12} = 1 \quad m_{23} = -f_0 f_3 f_4$$
$$m_{13} = -f_0 f_1 f_3 f_4 \quad m_{24} = -f_4(1-f_0)$$
$$m_{14} = -f_0 f_1 f_4 \quad m_{34} = f_0 f_1 f_3 f_4^2$$

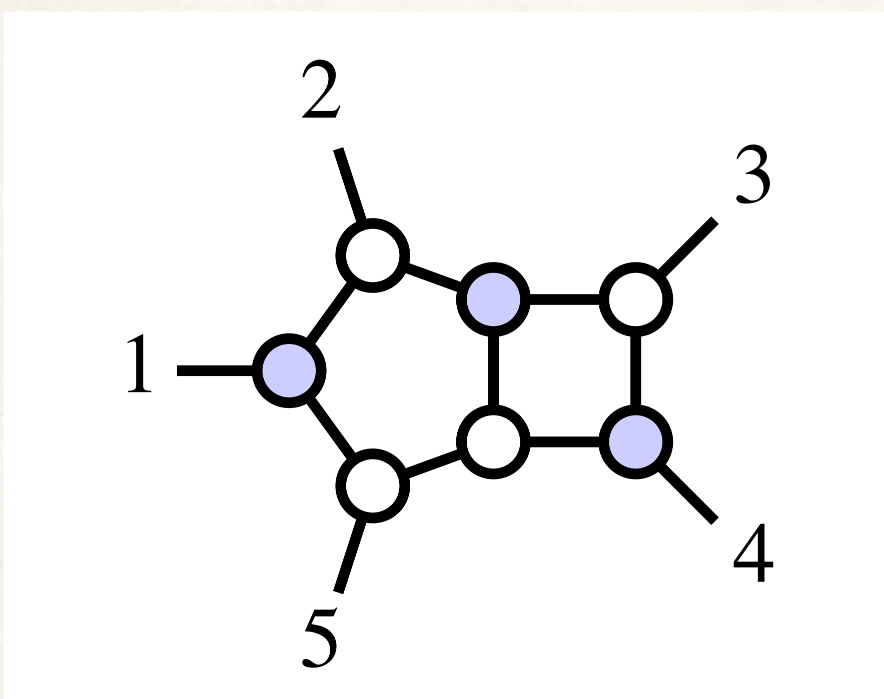
→

$$f_0 < 0$$
$$f_1 < 0$$
$$f_3 > 0$$
$$f_4 < 0$$

All minors positive

Positive Grassmannian from on-shell diagram

- On-shell diagram: method how to generate $C \in G_+(k,n)$



$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

$$\begin{vmatrix} * & * \\ * & * \end{vmatrix} \geq 0$$

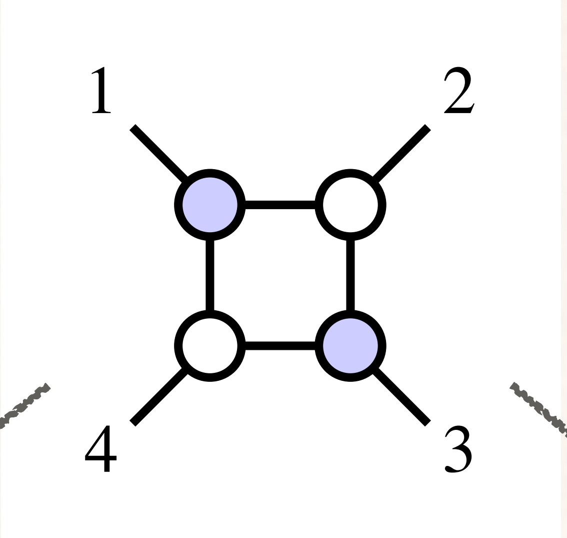
- All such matrices generated using on-shell diagrams

It is very interesting that the same objects appear in physics and mathematics.

But is it useful for something?

Physics from Grassmannian

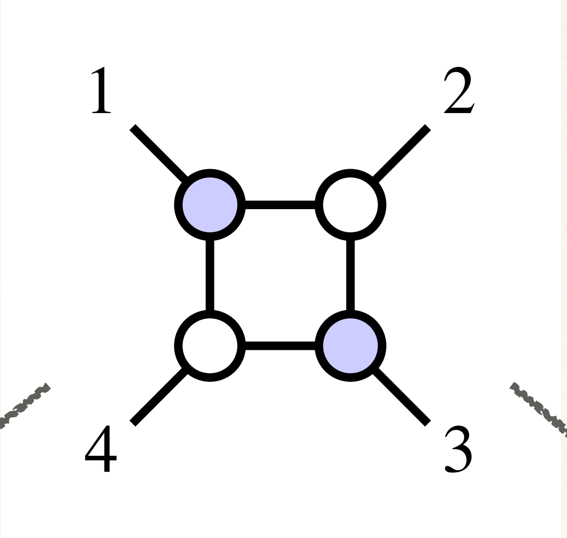
Connection



$$R = \mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\mathcal{M}_3^{tree}\mathcal{M}_4^{tree}$$

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

Connection



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$$R = \frac{df_0}{f_0}\frac{df_1}{f_1}\frac{df_2}{f_2}\frac{df_3}{f_3}\delta(C \cdot Z)$$

Momentum conservation

$$\delta(C \cdot Z) = \delta(C \cdot \widetilde{\lambda})\delta(C^{\perp} \cdot \lambda)$$

- Simple motivation: linearize momentum conservation

$$\delta(P) = \delta\left(\sum_a \lambda_a \widetilde{\lambda}_a\right)$$

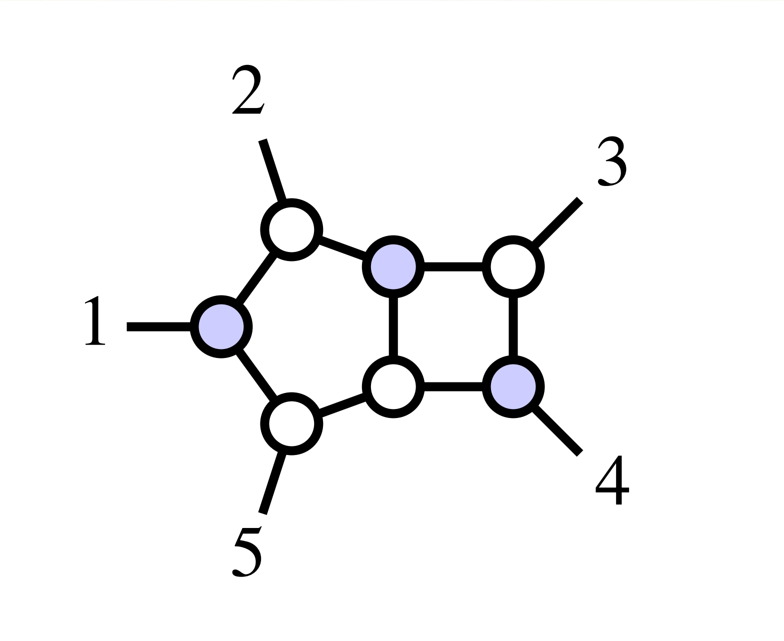
- We want to write it as two linear factors

$$\delta\left(C_{ab}\widetilde{\lambda}_b\right)\,\delta\left(D_{ab}\lambda_b\right)$$

and get the condition: $D_{ab} = C^{\perp}_{ab}$

Dual picture for on-shell diagrams

For arbitrary on-shell diagram



- Label face variables
- Find perfect orientation
- Construct the Grassmannian matrix
- Write a logarithmic form

$$R = \frac{df_0}{f_0}\frac{df_1}{f_1}\frac{df_2}{f_2}\dots\frac{df_d}{f_d}\delta(C\cdot Z) \quad \leftrightarrow \quad \mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\dots\mathcal{M}_m^{tree}$$

Definition of the theory

- Why is this for N=4 SYM? What about other theories?
- Diagrams and connection to Grassmannian is general
- Specific for theory: differential form

planar N=4 SYM:

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\dots\frac{d\alpha_n}{\alpha_n}\delta(C\cdot Z)$$

Definition of the theory

- Why is this for N=4 SYM? What about other theories?
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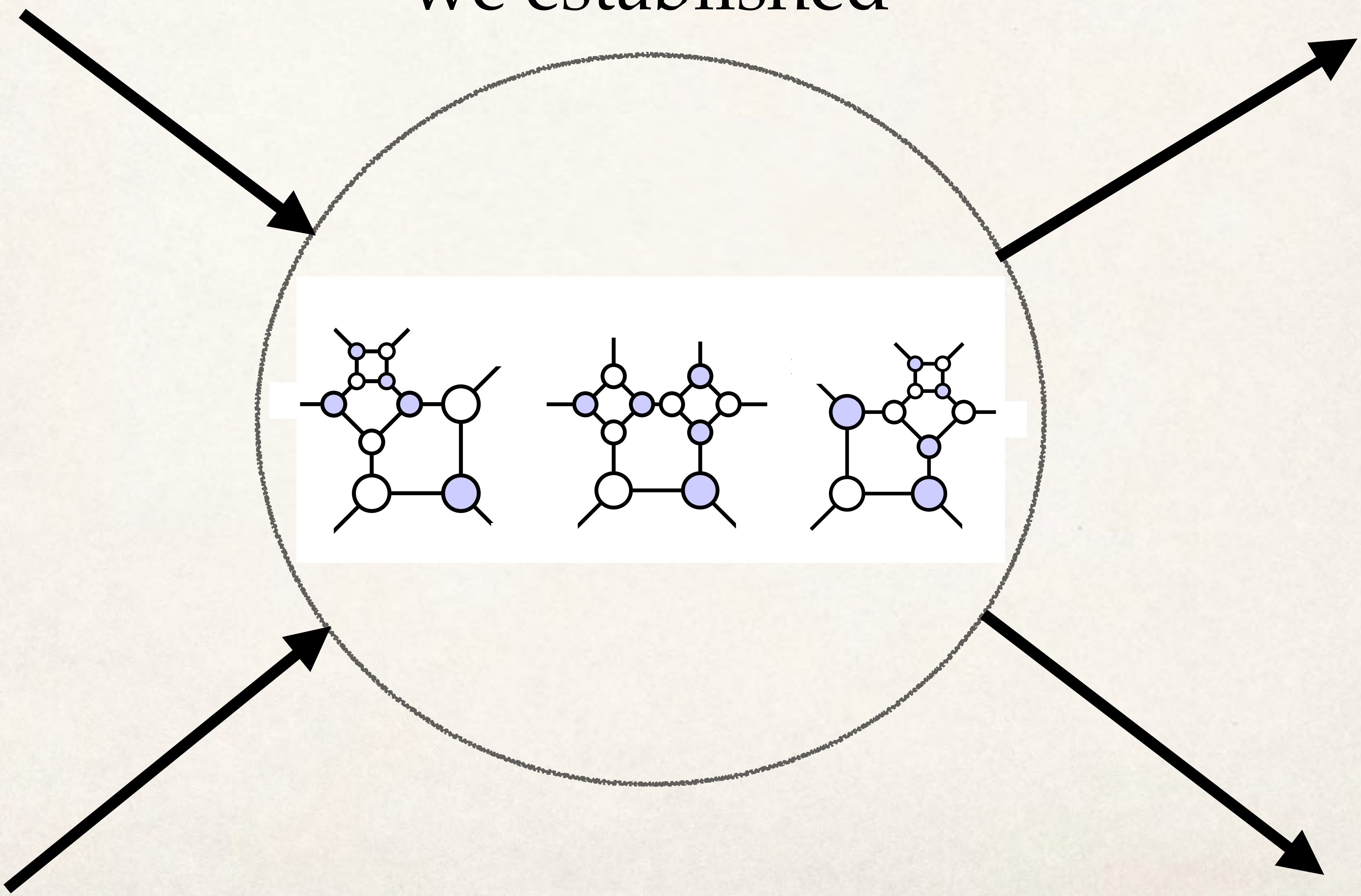
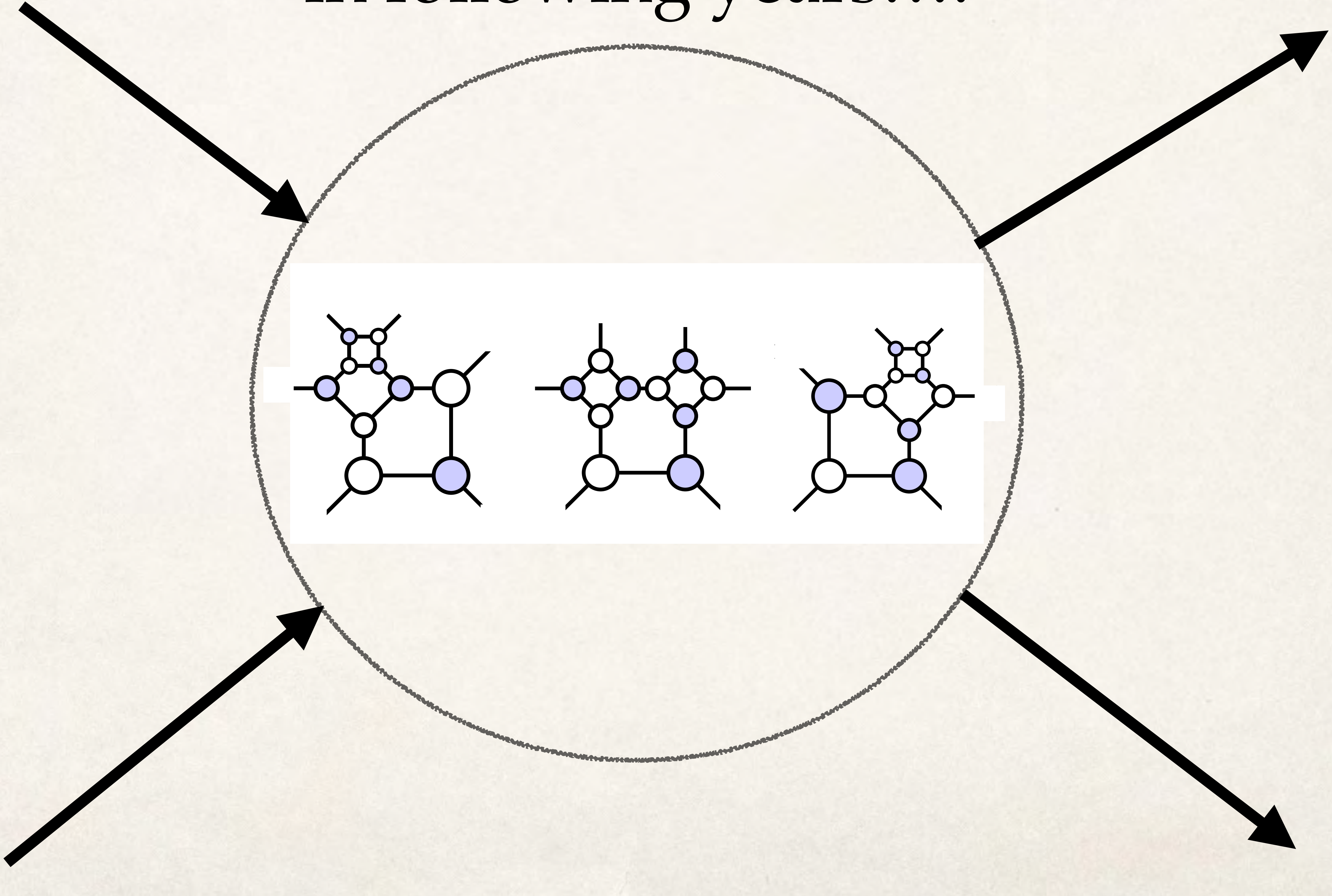
General QFT: $$\Omega = F(\alpha)\,\delta(C\cdot Z)$$

- In a sense $F(\alpha)$ defines a theory (as Lagrangian does)

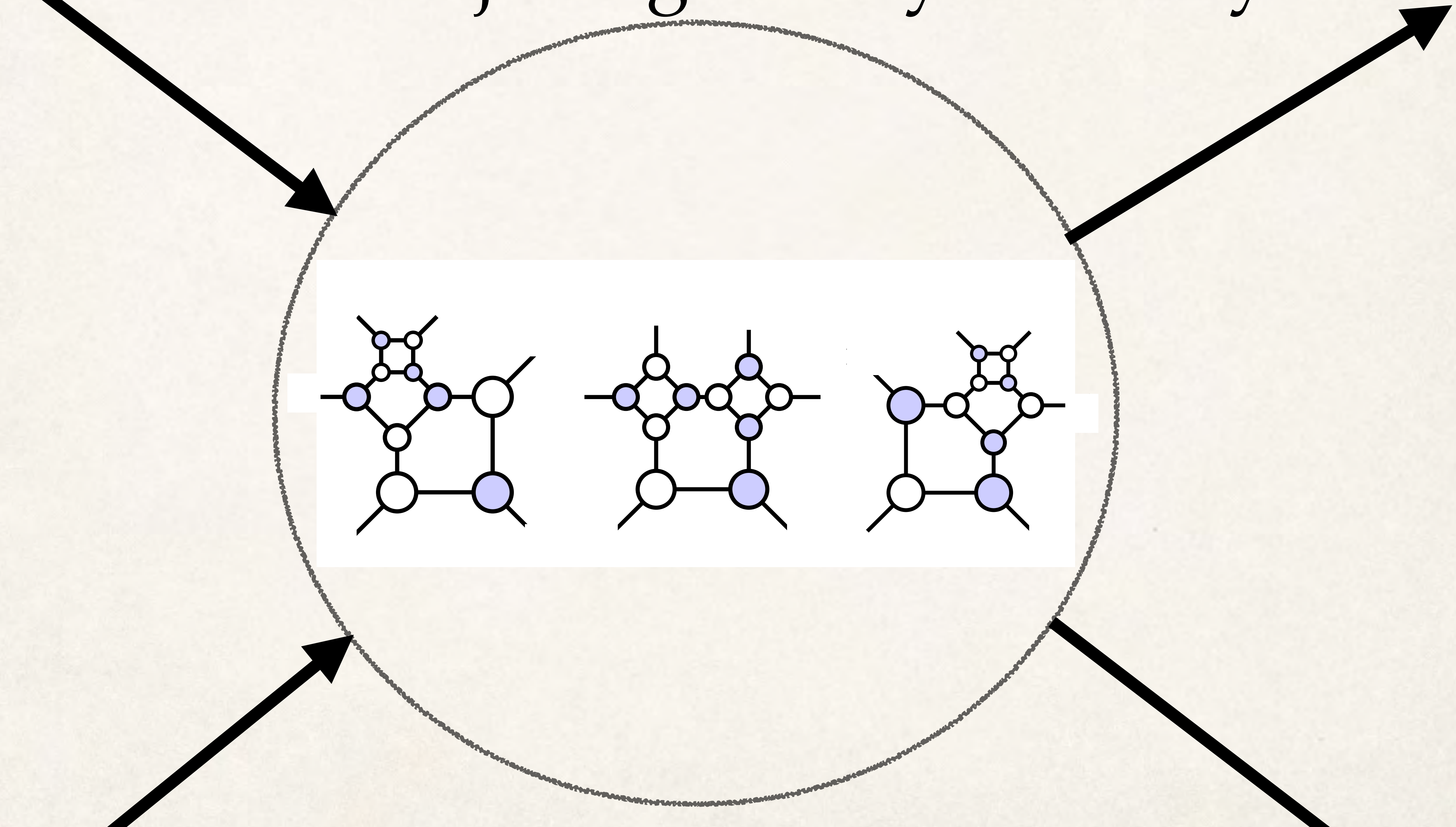
At least for planar N=4 SYM we established



Hopefully for other theories
in following years….



Even for planar N=4 SYM not completely satisfactory: sum of objects given by unitarity



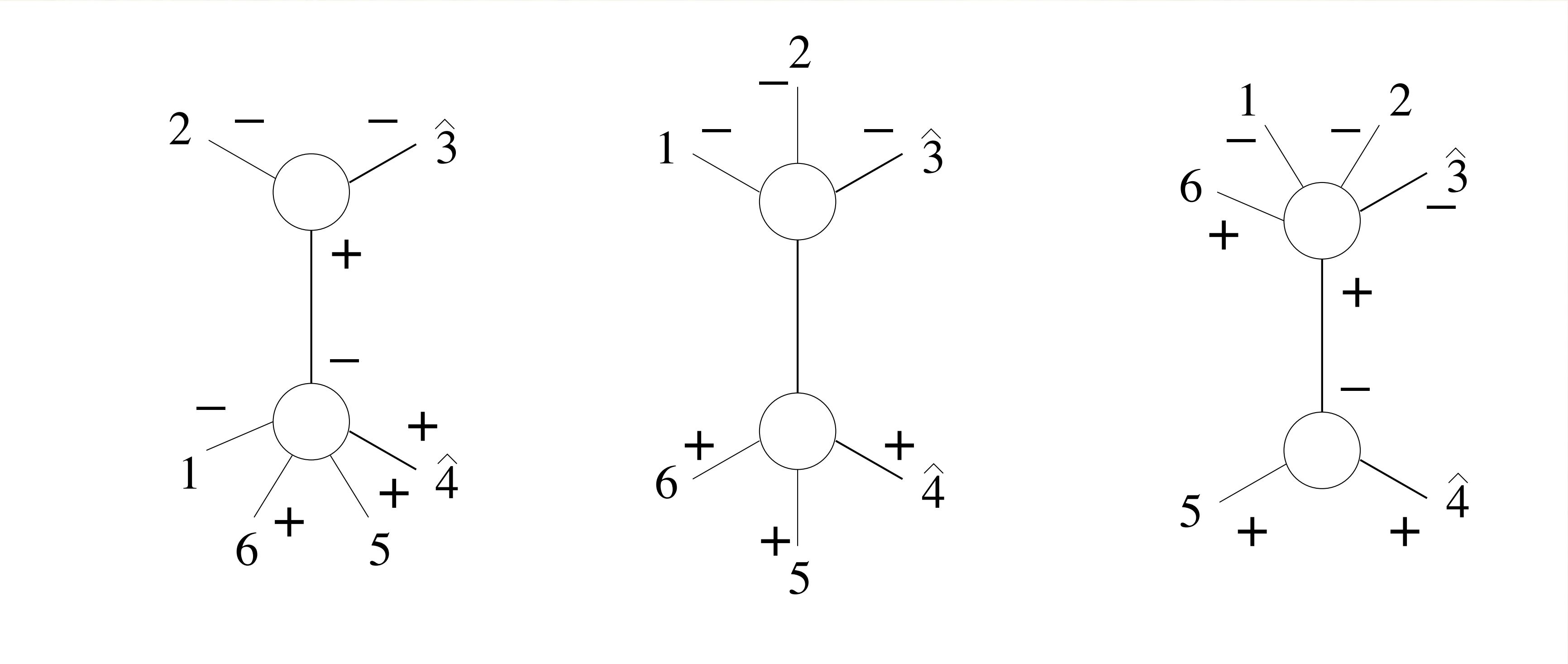
Search for a single object

Prelude

Volume of polyhedron

(Hodges 2009)

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space



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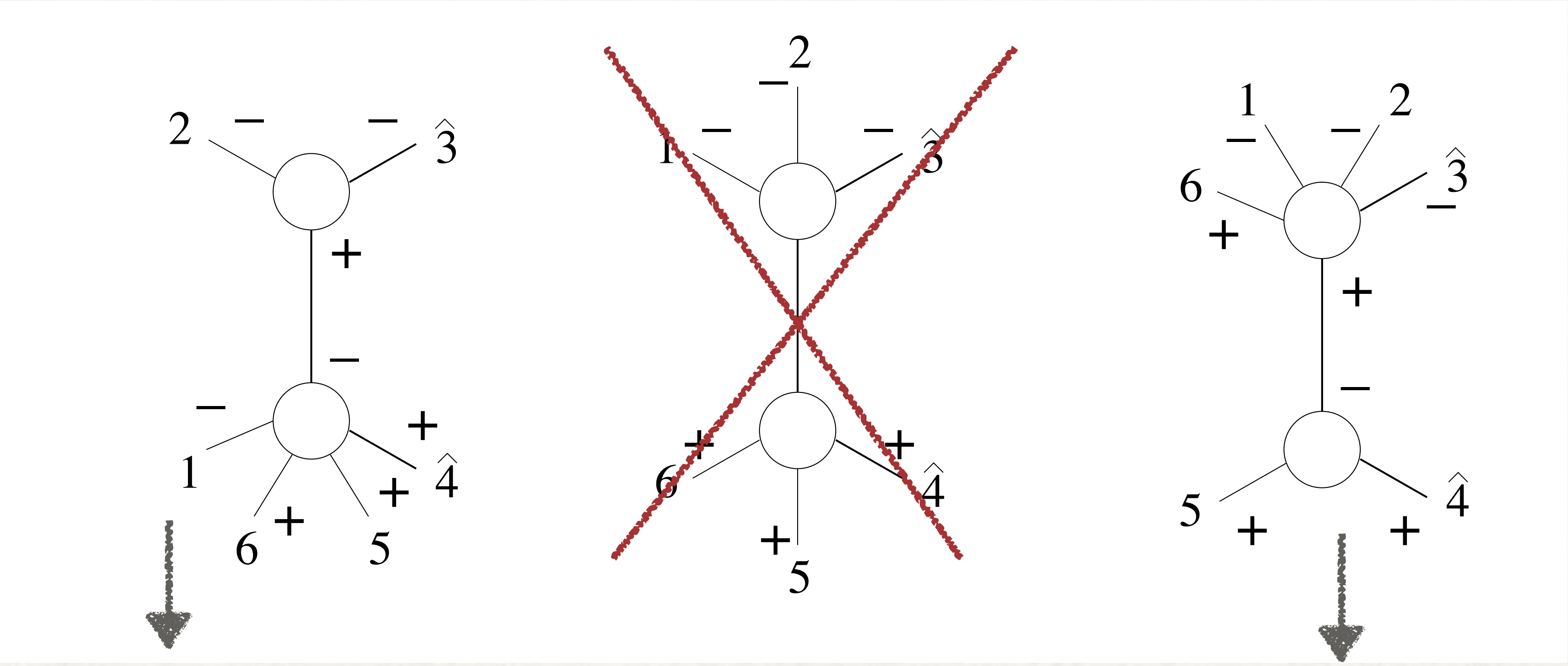

$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

Volume of polyhedron

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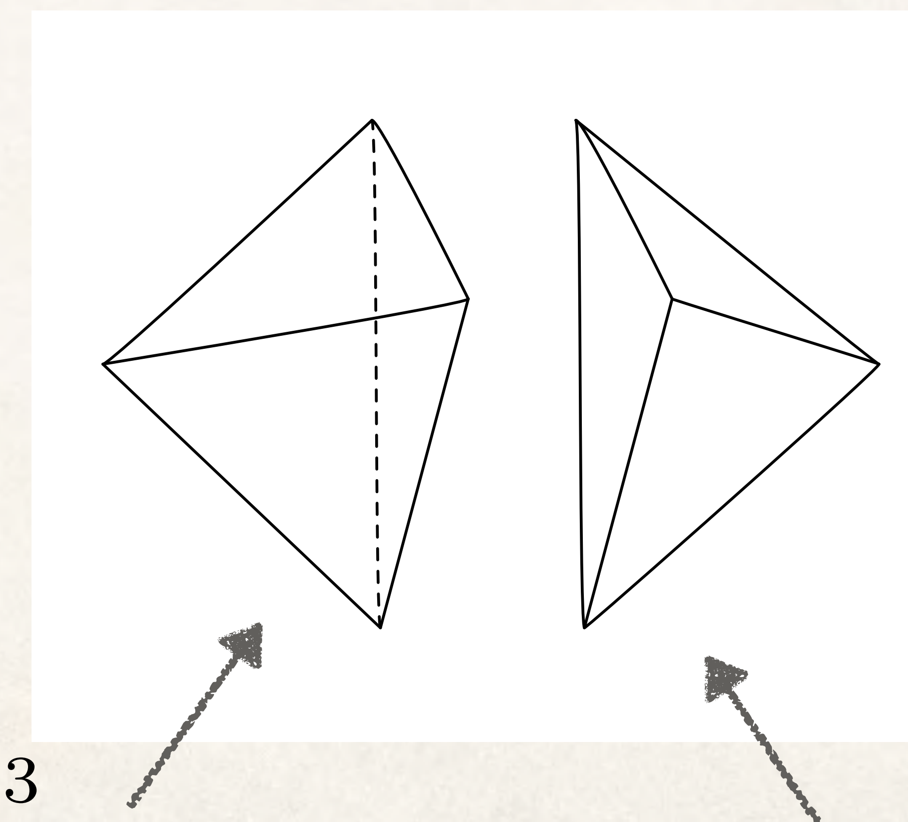
Volume of tetrahedron in momentum twistor space!



Each face labeled by $\langle abcd \rangle$

$$\frac{\langle 1345 \rangle^3}{\langle 1234 \rangle \langle 1245 \rangle \langle 2345 \rangle \langle 1235 \rangle} \qquad \frac{\langle 1356 \rangle^3}{\langle 1235 \rangle \langle 1256 \rangle \langle 2356 \rangle \langle 1236 \rangle}$$

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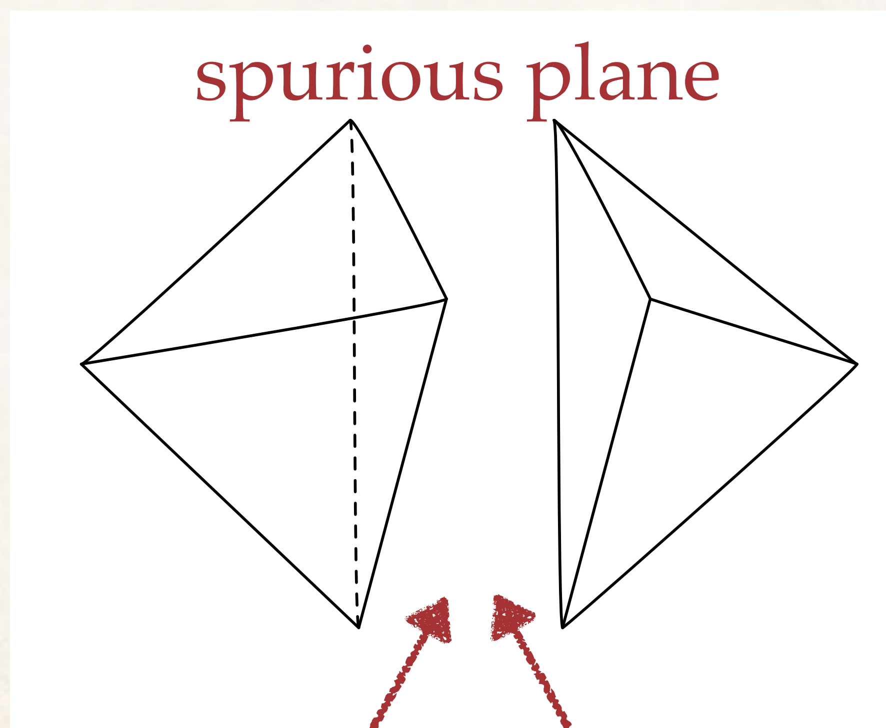

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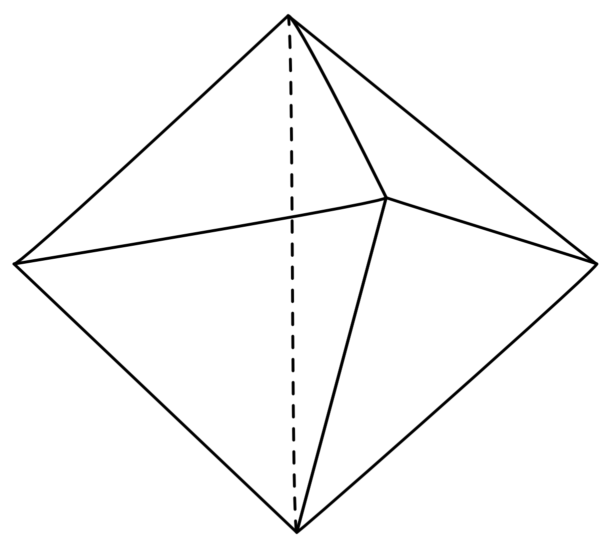
Volume of polyhedron

(Hodges 2009)

- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Amplitude is a volume of polyhedron



Each face labeled by $\langle abcd\rangle$

$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

"Conjecture"

Amplitudes are volumes
of *some regions* in *some space*

“Conjecture”

Amplitudes are volumes
of *some regions* in *some space*

Must be related to
positive Grassmannian

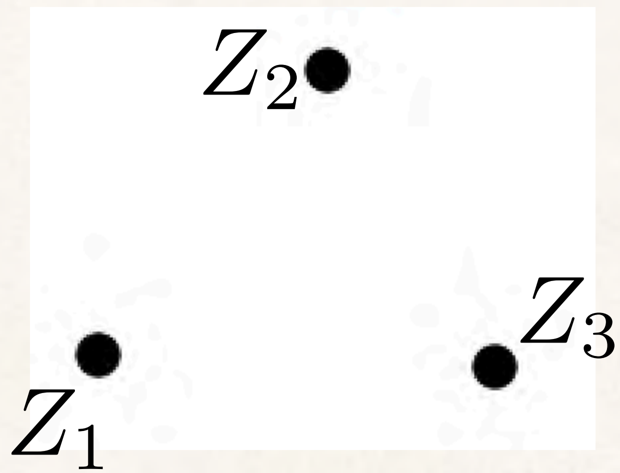
Strategy

- Simple intuitive geometric ideas
- Use suitable mathematical language to describe them
- Generalize to more complicated (non-intuitive) cases

Inside of the triangle

Inside of the triangle

- Let us consider three points in a projective plane

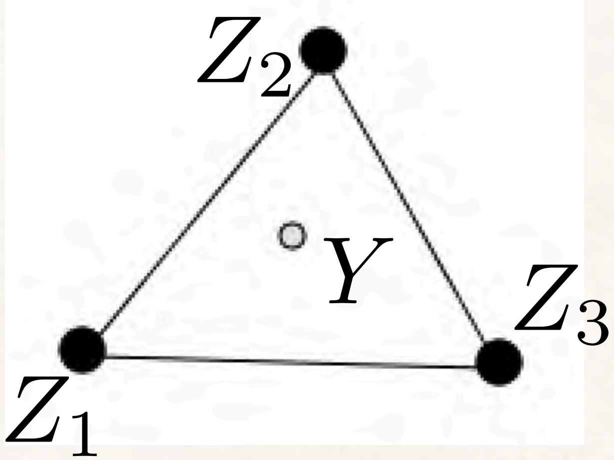


$$Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad Z_j \sim tZ_j$$

We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

Inside of the triangle

- Point inside the triangle



$$Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad Z_j \sim tZ_j$$

We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

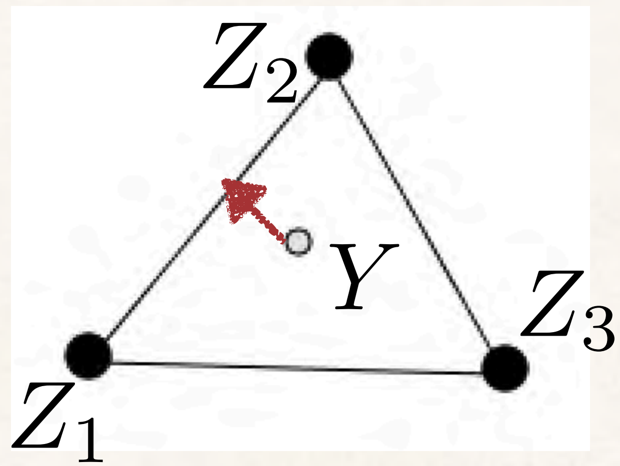
- Point inside the triangle

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \qquad c_1, c_2, c_3 > 0$$

Projective: one of c_j can be fixed to 1

Inside of the triangle

- Point inside the triangle



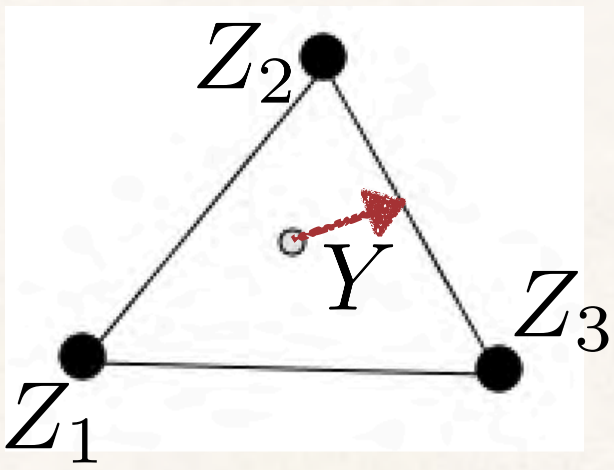
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_3 = 0$$

Inside of the triangle

- Point inside the triangle



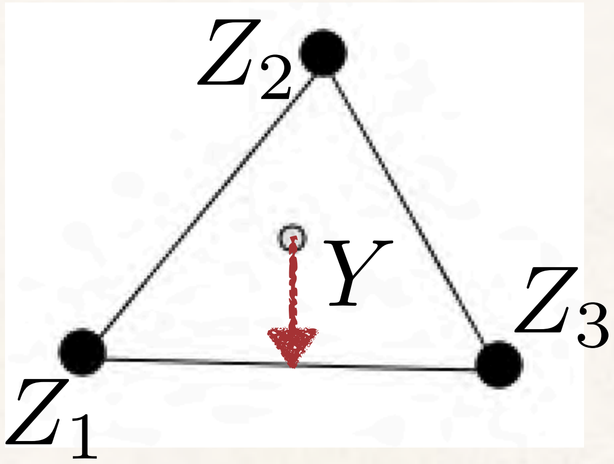
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_1 = 0$$

Inside of the triangle

- Point inside the triangle



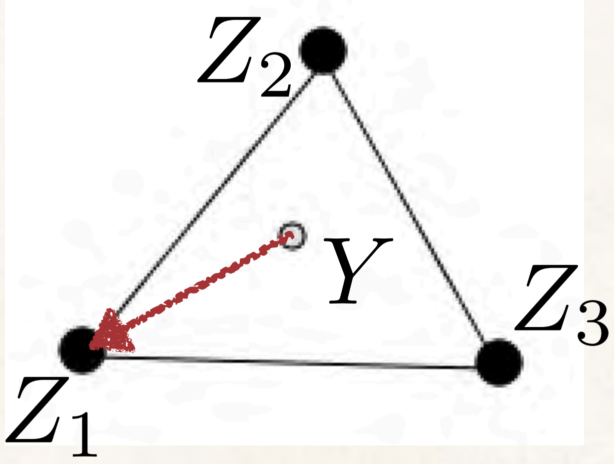
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = 0$$

Inside of the triangle

- Point inside the triangle



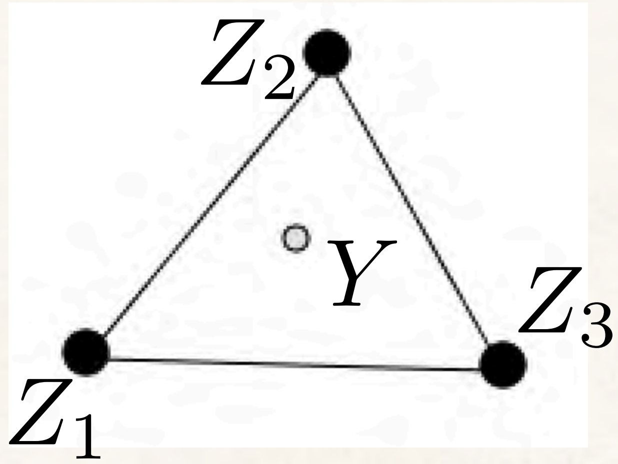
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = c_3 = 0$$

Logarithmic form

- Point inside the triangle



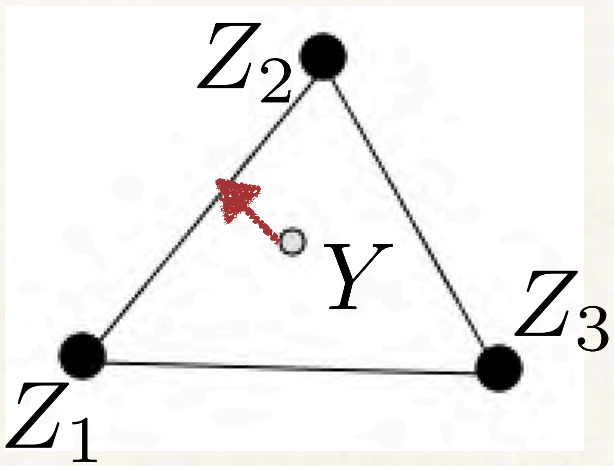
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_3 = 0$$

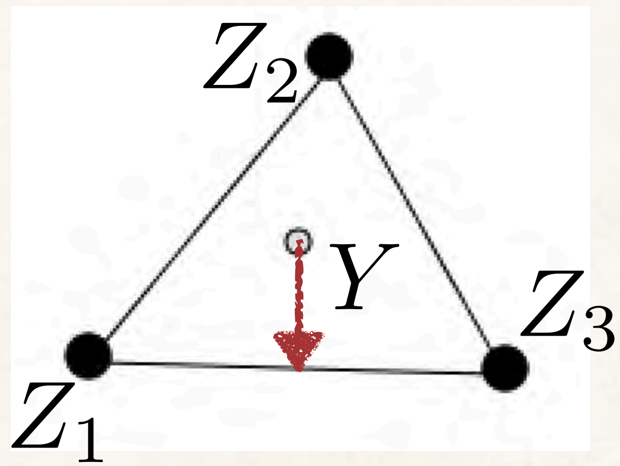
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \rightarrow \frac{dc_2}{c_2}$$

Logarithmic form

- Point inside the triangle



$$c_2 = 0$$

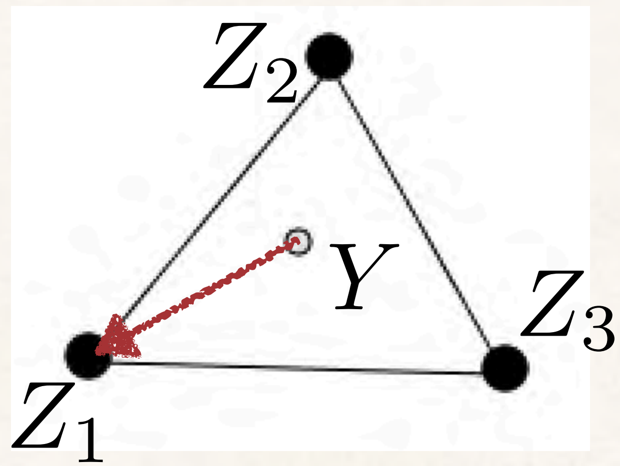
$$Y = Z_1 + \cancel{c_2 Z_2} + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \to \frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_2 = c_3 = 0$$

$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \rightarrow \frac{dc_3}{c_3} \rightarrow 1$$

- Other boundaries can correspond to $c_2, c_3 \rightarrow \infty$

Logarithmic form

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

$$\langle X_1 X_2 X_3 \rangle = \epsilon_{abc} X_1^a X_2^b X_3^c$$

$$d^2Y = dc_2\, dc_3\, Z_2 Z_3$$

- Solve for c_2, c_3 from $Y = Z_1 + c_2 Z_2 + c_3 Z_3$

↓

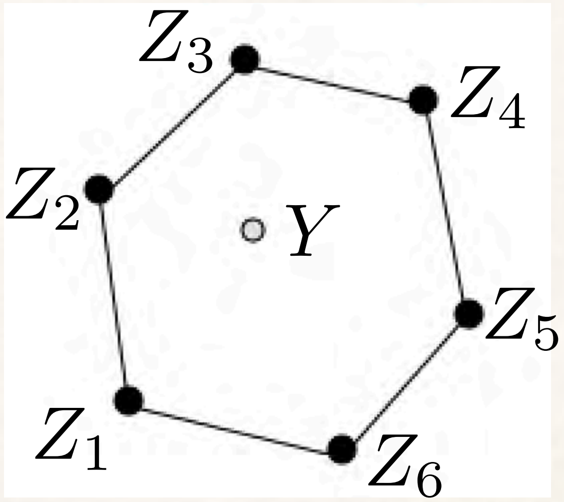
$$\Omega = \frac{\langle Y\, d^2Y \rangle \langle 123 \rangle^2}{\langle Y12 \rangle \langle Y23 \rangle \langle Y31 \rangle}$$

Projective in all variables

Polygon

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \dots c_n Z_n$$

$$c_j > 0$$

interior of the polygon

- Convex polygon: condition on points Z_i

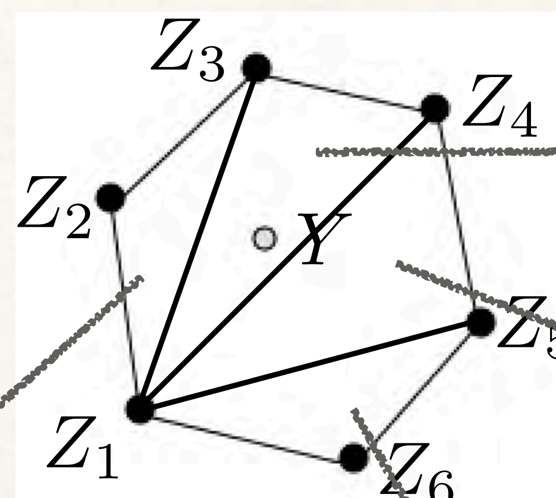
$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \dots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

All main minors positive

$$\begin{vmatrix} * & * & * \\ * & * & * \\ * & * & * \end{vmatrix} > 0$$

Logarithmic form

- Easiest way how to write the form is to triangulate



$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{dc_3}{c_3}\frac{dc_4}{c_4} \quad c_3, c_4 \geq 0$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{dc_4}{c_4}\frac{dc_5}{c_5} \quad c_4, c_5 \geq 0$$

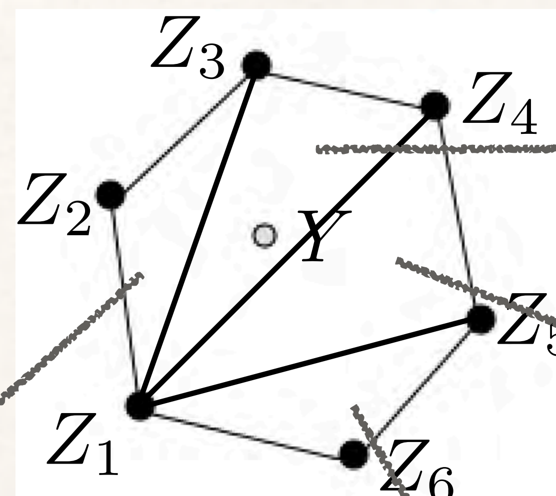
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \quad c_2, c_3 \geq 0$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{dc_5}{c_5}\frac{dc_6}{c_6} \quad c_5, c_6 \geq 0$$

How to sum them?

Logarithmic form

- Easiest way how to write the form is to triangulate



$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle}$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle}$$

$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

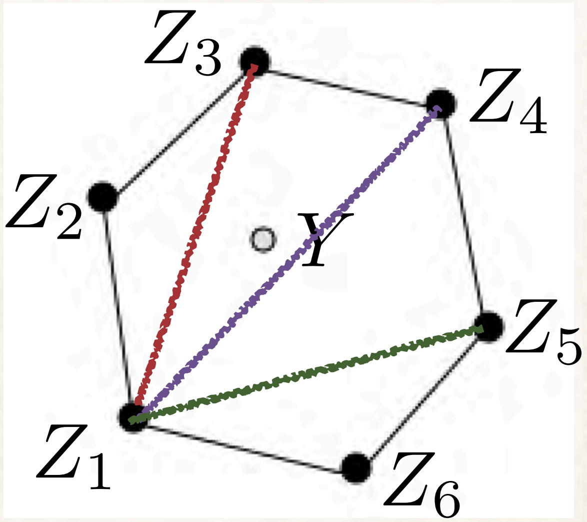
Write in projective form

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$



Spurious poles

Cancel in the sum

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\,d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\,d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\,d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\,d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

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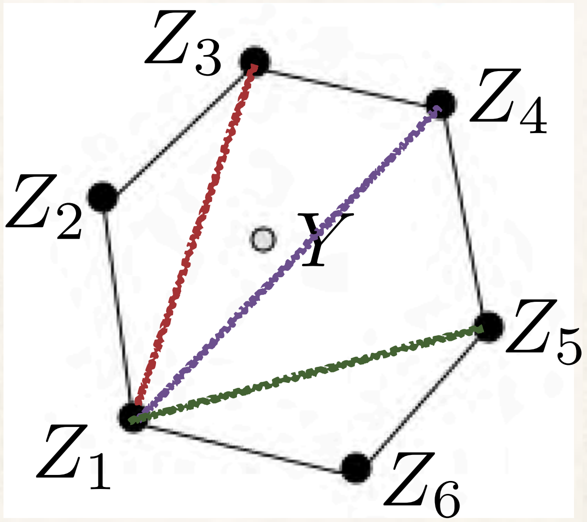


$$\Omega = \frac{\langle Y\,d^2Y\rangle\,\mathcal{N}(Y,Z_j)}{\langle Y12\rangle\langle Y23\rangle\langle Y34\rangle\langle Y45\rangle\langle Y56\rangle\langle Y61\rangle}$$

From Y to supersymmetry

- Let us take the form for the triangle

$$\Omega = \frac{\langle Y\, d^2 Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

- Rewrite external Z:

$$Z_j = \begin{pmatrix} z_j^{(1)} \\ z_j^{(2)} \\ (\phi \cdot \eta_j) \end{pmatrix}$$

$z_j \in \mathbb{P}^2$ bosonic

η_j^A ferimionic

ϕ^A auxiliary

$A = 1, 2$

- Also define

$$Y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From Y to supersymmetry

- We plug them into the form for triangle:

$$\frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \to \frac{(\langle 12\rangle(\phi\cdot\eta_3) + \langle 23\rangle(\phi\cdot\eta_1) + \langle 31\rangle(\phi\cdot\eta_2))^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

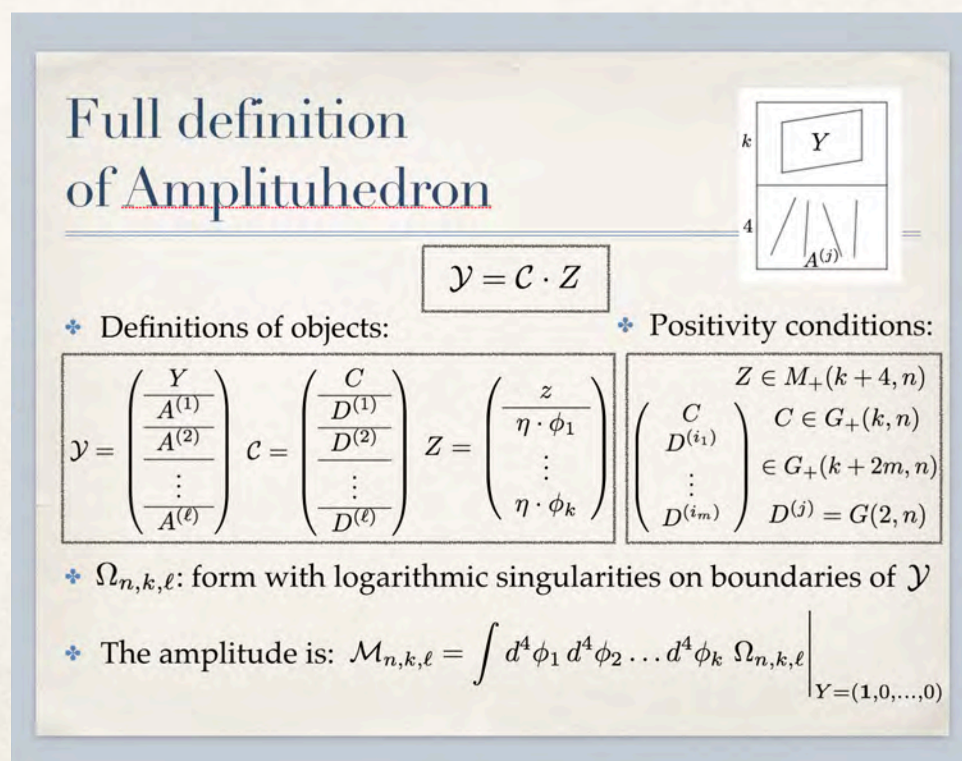
- Final step: integrate over ϕ:

$$\int d^2\phi \int \Omega\, \delta(Y - Y_0) = \boxed{\frac{(\langle 12\rangle\eta_3 + \langle 23\rangle\eta_1 + \langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

Amplituhedron

Definition

space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$



Full definition of Amplituhedron

$$\mathcal{Y} = \mathcal{C} \cdot Z$$

- Definitions of objects:

$$\mathcal{Y} = \begin{pmatrix} Y \\ \hline A^{(1)} \\ \hline A^{(2)} \\ \hline \vdots \\ \hline A^{(\ell)} \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} C \\ \hline D^{(1)} \\ \hline D^{(2)} \\ \hline \vdots \\ \hline D^{(\ell)} \end{pmatrix} \quad Z = \begin{pmatrix} z \\ \hline \eta \cdot \phi_1 \\ \vdots \\ \eta \cdot \phi_k \end{pmatrix}$$

- Positivity conditions:

$$Z \in M_+(k+4,n) \quad C \in G_+(k,n) \quad \begin{pmatrix} C \\ D^{(i_1)} \\ \vdots \\ D^{(i_m)} \end{pmatrix} \in G_+(k+2m,n) \quad D^{(j)} = G(2,n)$$

- $\Omega_{n,k,\ell}$: form with logarithmic singularities on boundaries of $\mathcal{Y}$
- The amplitude is: $\mathcal{M}_{n,k,\ell} = \int d^4\phi_1\, d^4\phi_2 \ldots d^4\phi_k\, \Omega_{n,k,\ell} \Big|_{Y=(1,0,\ldots,0)}$

Triangulation

space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$

↓

triangulate in terms of "simplices"

$\Omega_0 \sim \frac{dx}{x}$ for each

→ Set of regions:

- cover the whole space
- each region specified by $f_j \in (0, \infty)$

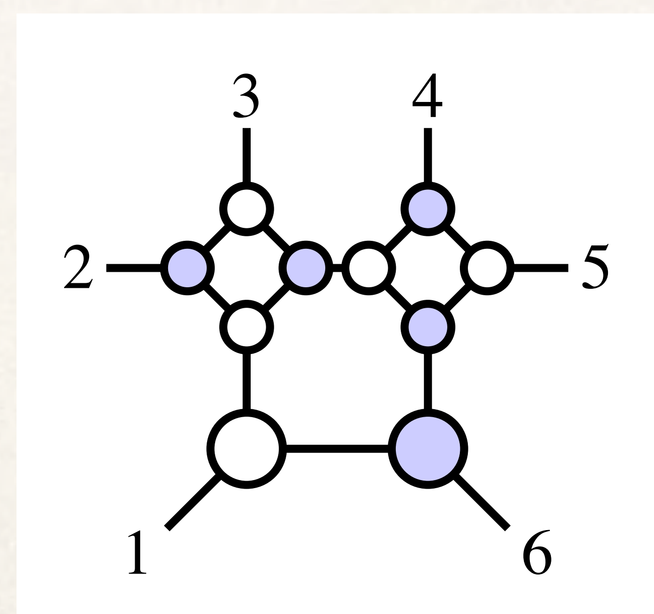
→ sum them

Triangulation

space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$

triangulate in terms of "simplices"

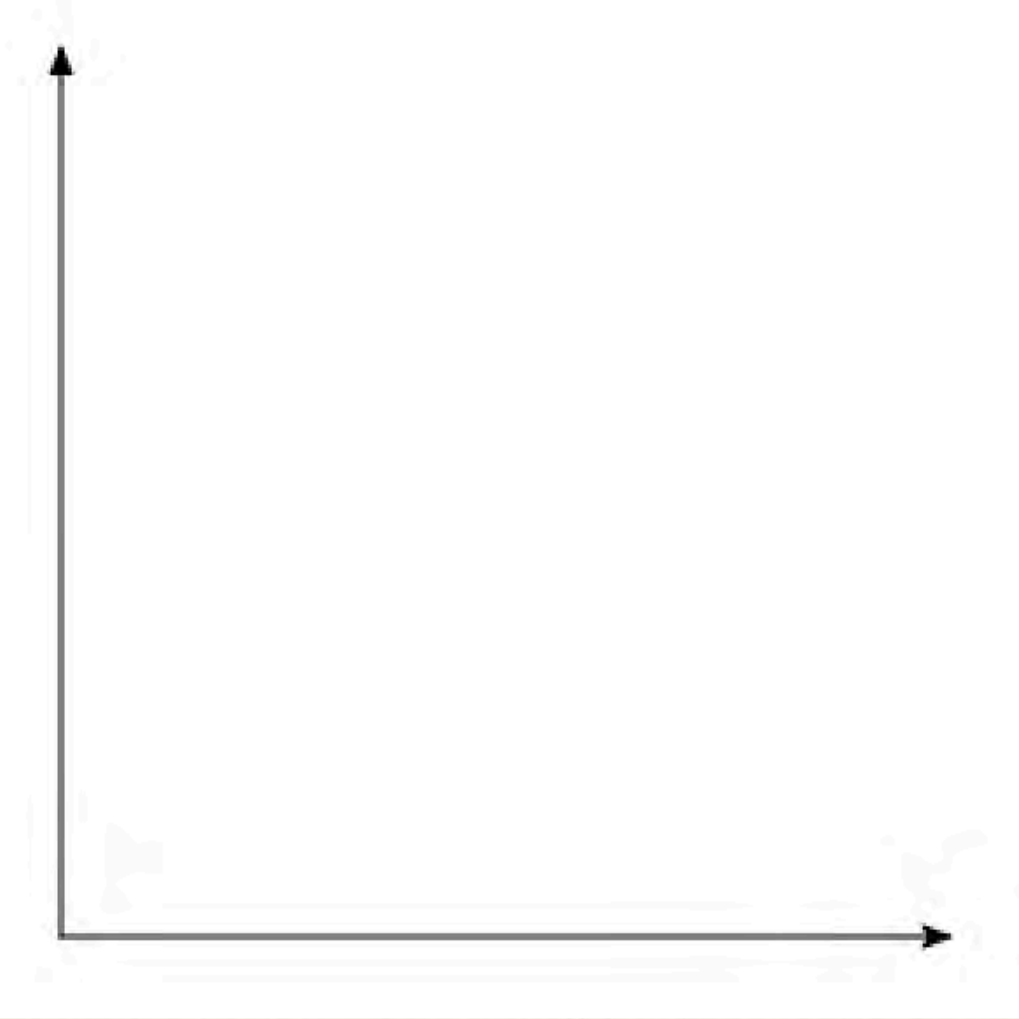
$$\Omega_0 \sim \frac{dx}{x}$$ for each →



→ sum them

High school problem $gg \to gg$

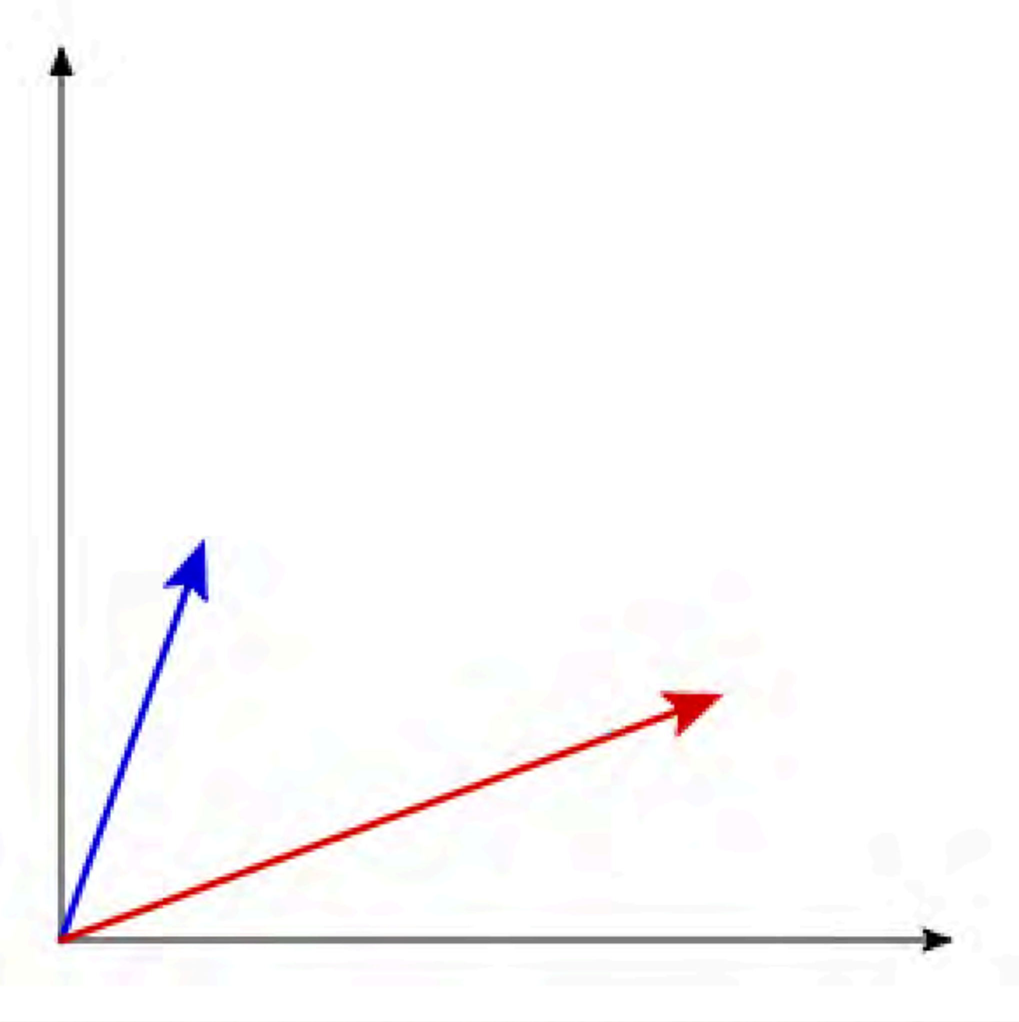
- Positive quadrant



High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$



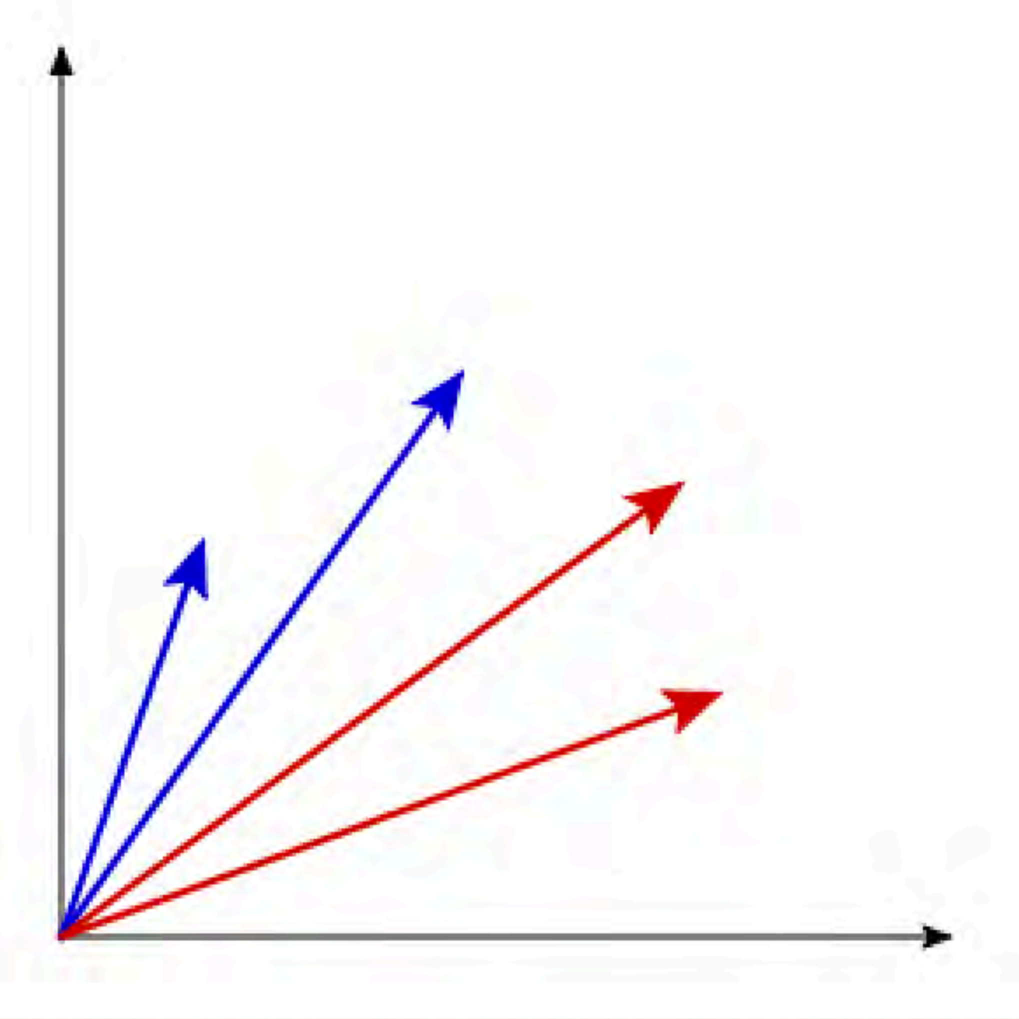
$$\text{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}$$

High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$$[\mathrm{Vol}\,(1)]^2 = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2}$$

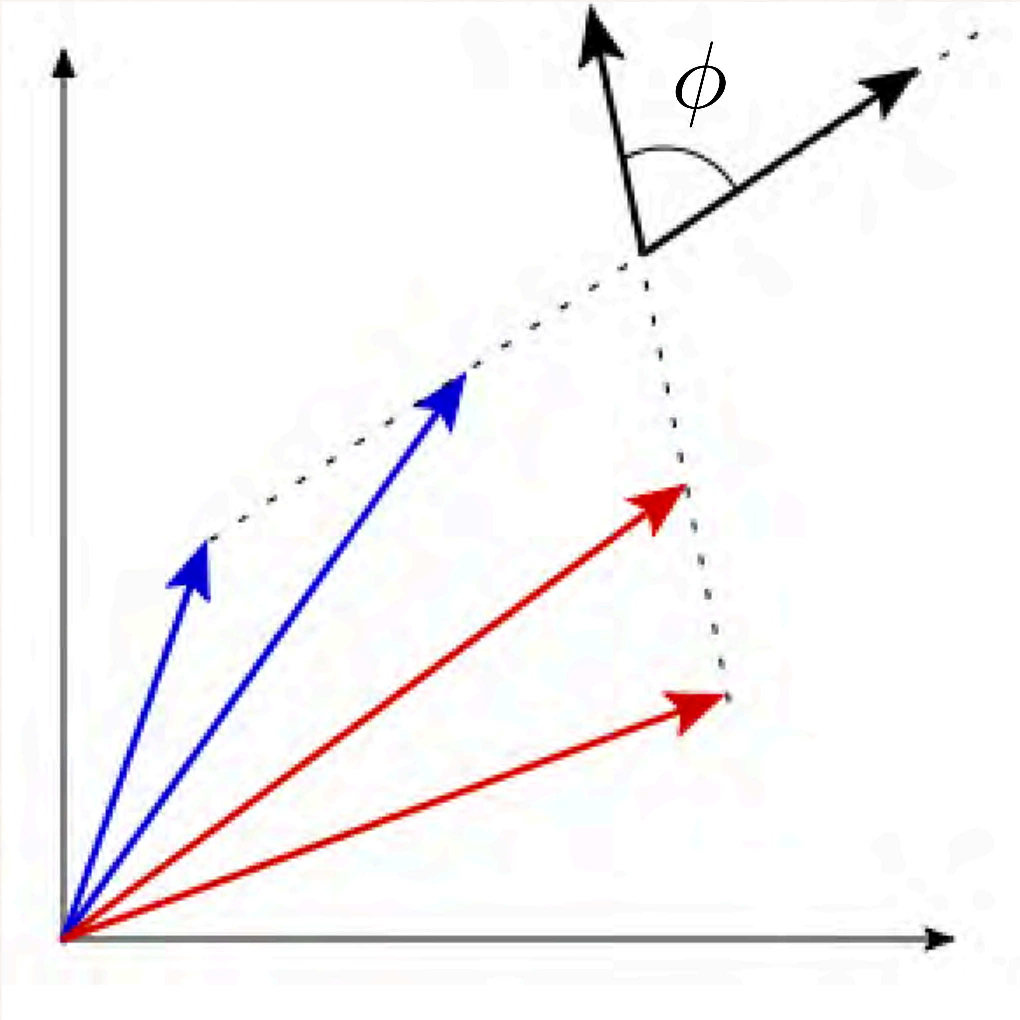
High school problem $gg \to gg$

- Positive quadrant

- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



- Impose: $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1) \leq 0$ $\quad \phi > 90^o$

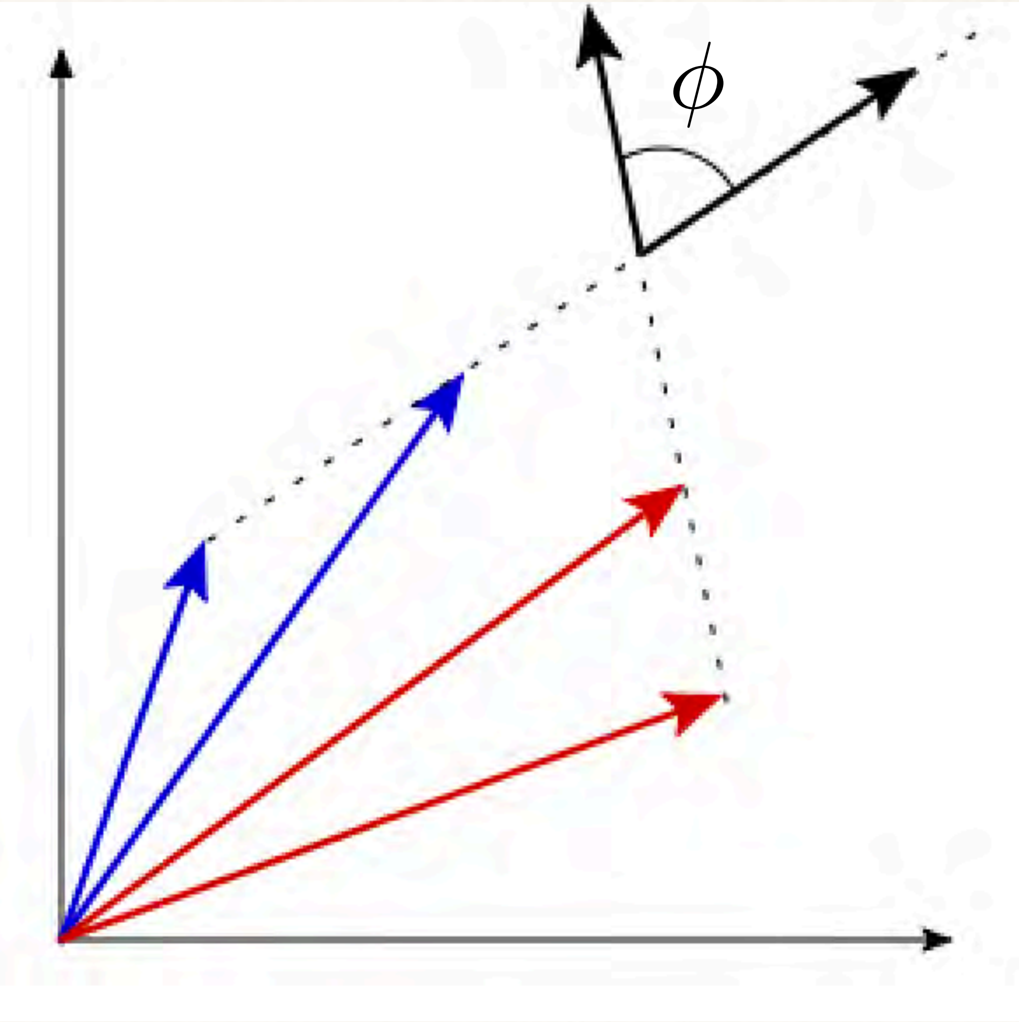
Subset of configurations allowed: triangulate

High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$$\mathrm{Vol}\,(2) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2}\left[\frac{\vec{a}_1 \cdot \vec{b}_2 + \vec{a}_2 \cdot \vec{b}_1}{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1)}\right]$$

High school problem $gg \to gg$

- Positive quadrant
- Vectors

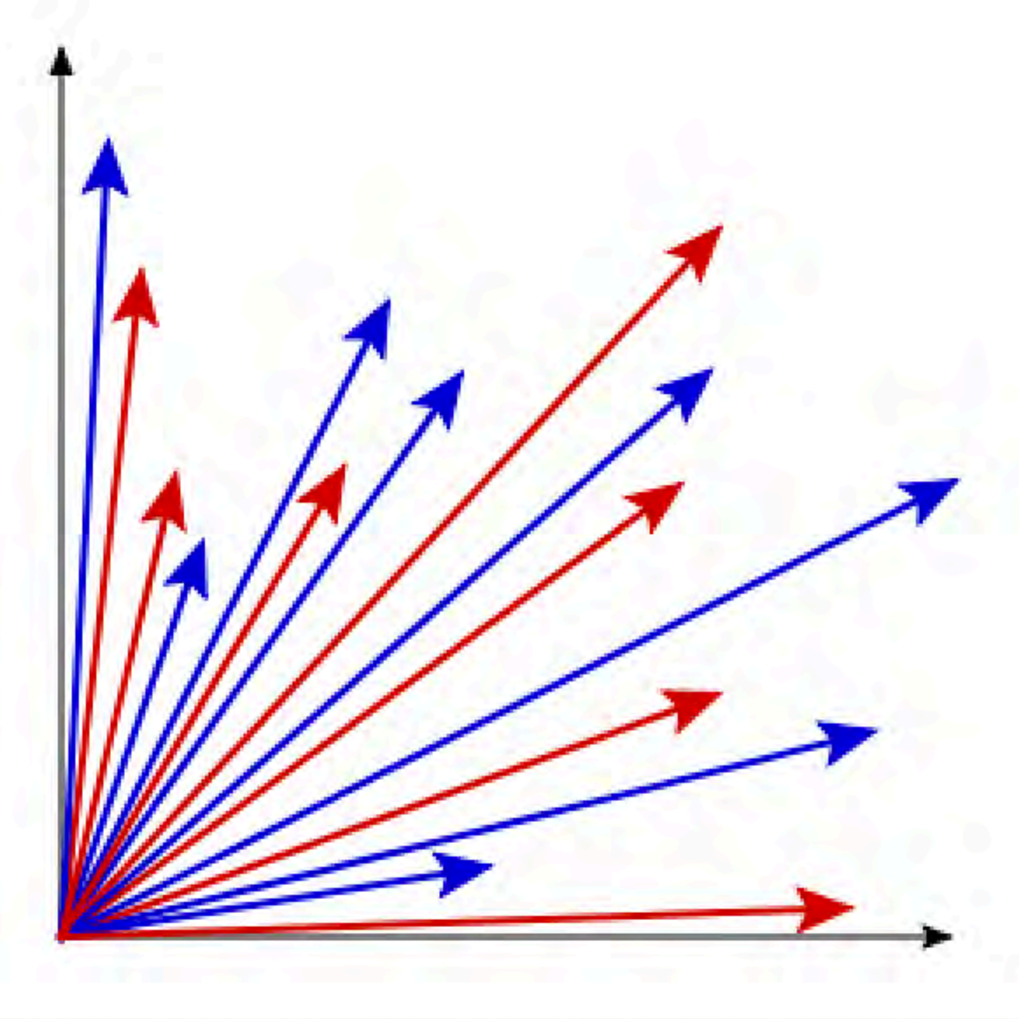
$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_\ell \quad \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_\ell$

- Conditions

$$(\vec{a}_i - \vec{a}_j) \cdot (\vec{b}_i - \vec{b}_j) \leq 0$$

for all pairs i, j

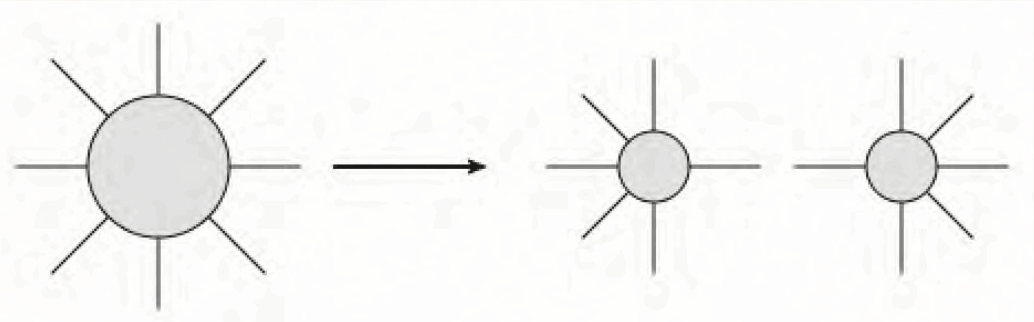
Let me know if you solve it!

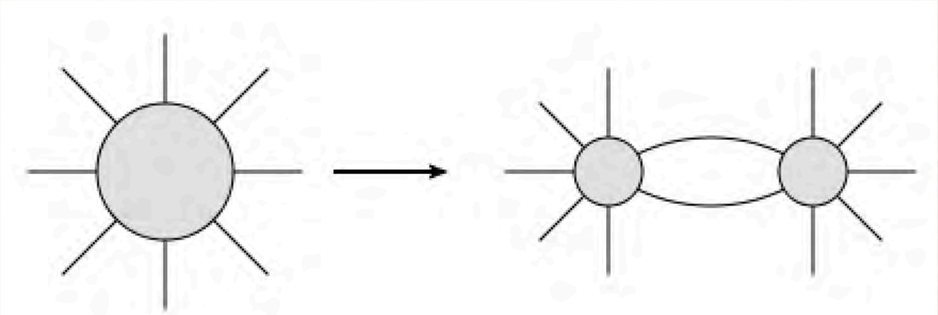


$\text{Vol}\,(\ell) = \ldots\ldots$

Why true?

- No QFT proof because it is not QFT but geometry
- It is correct: the result satisfies locality and unitarity

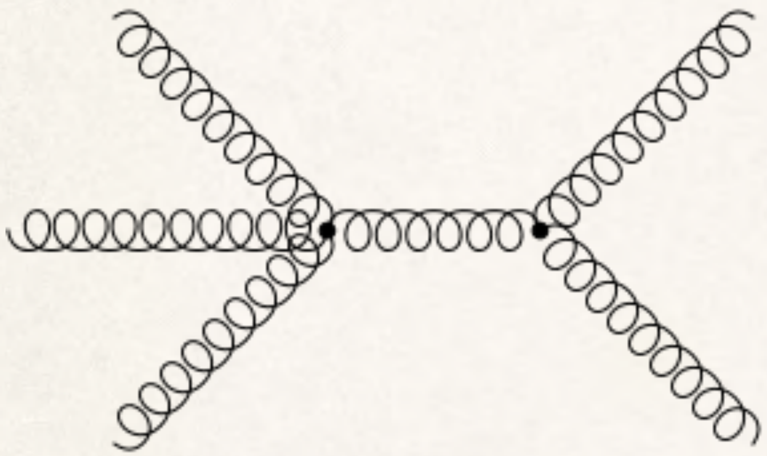


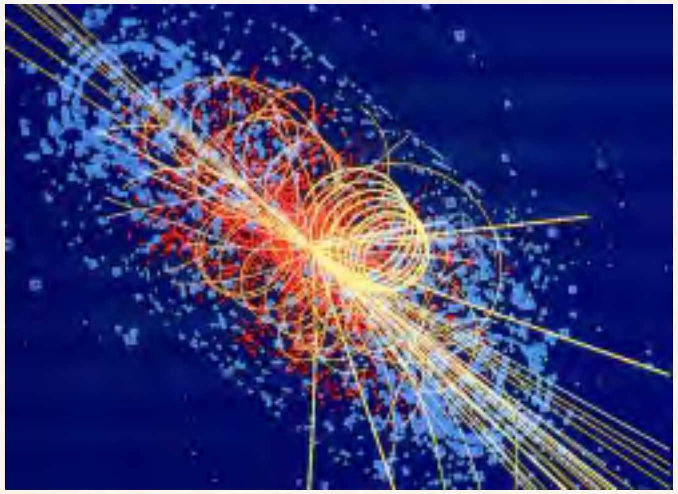


- Totally different approach: same answer
- Many open questions: triangulations, mathematical structure…..

Physics vs geometry

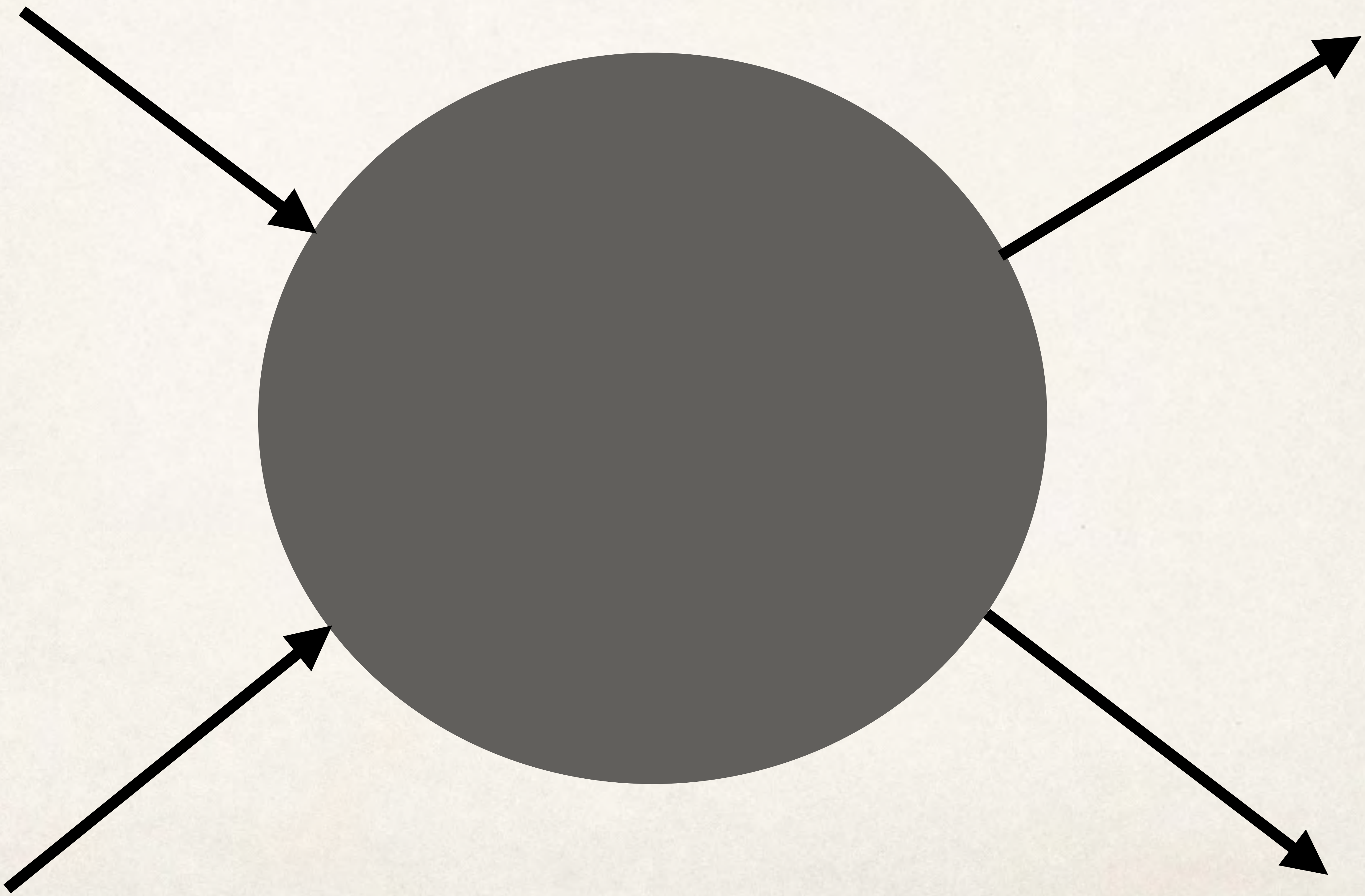
- Dynamical particle interactions in 4-dimensions



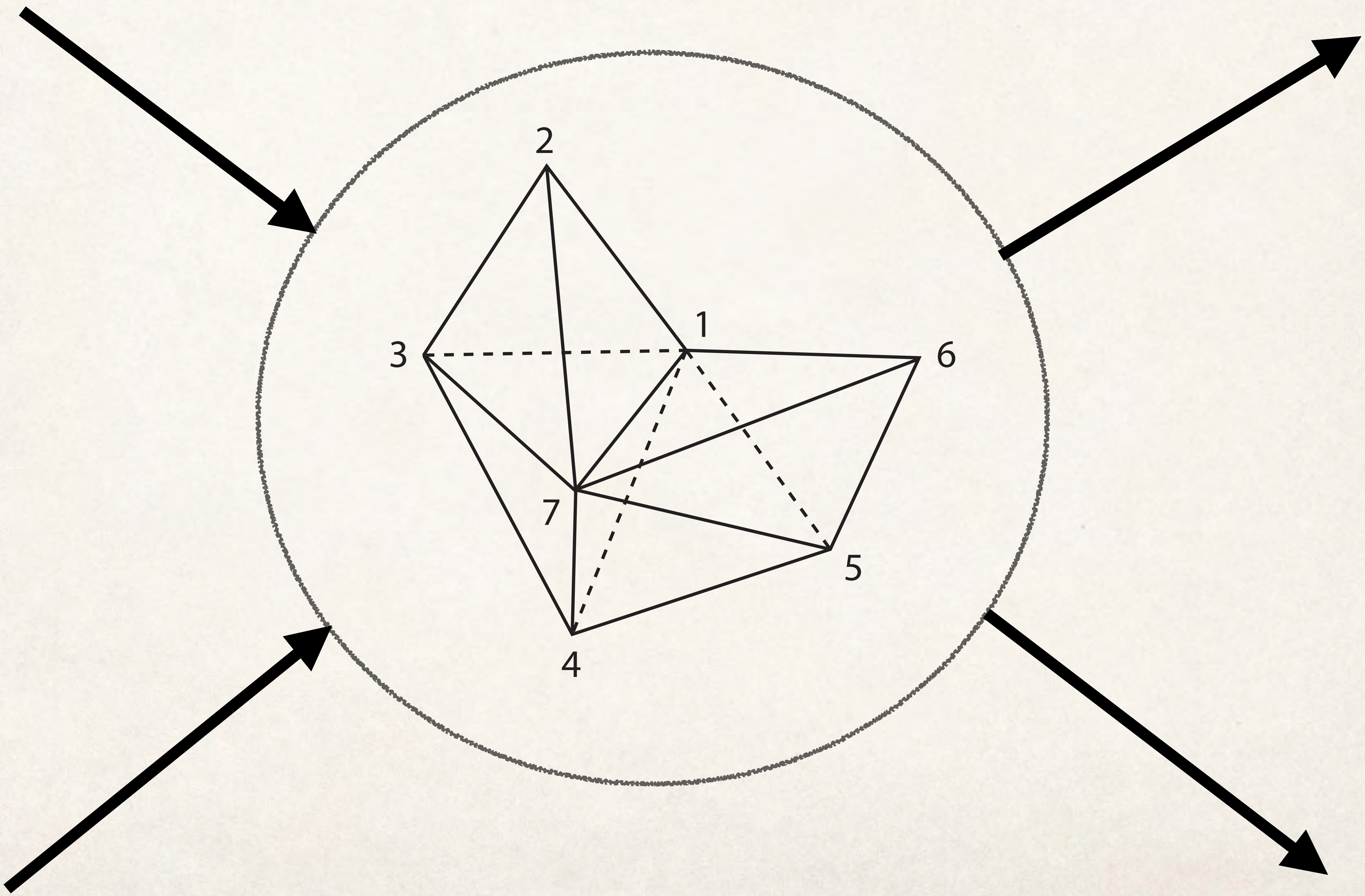


- Static geometry in high dimensional space

What is scattering amplitude?



What is scattering amplitude?



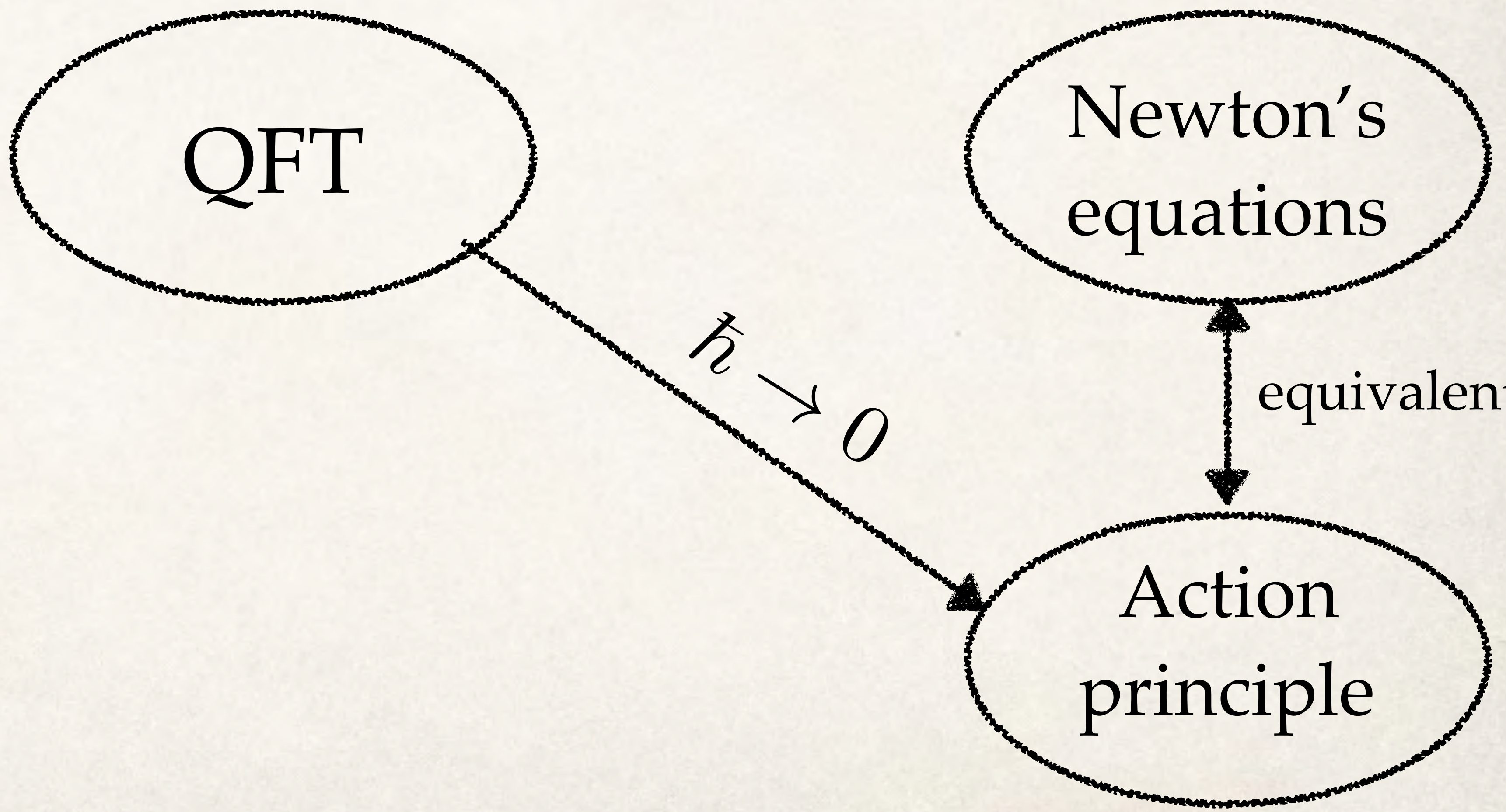
Step 1.1.1.

- It is very early to say if/how this can generalize
- Some encouraging news but more work needed
- New formulation of QFT?
  - Integrals
  - Masses
  - RG flow
  - Correlation functions
  - Beyond perturbation theory

Establish as an efficient computational tool

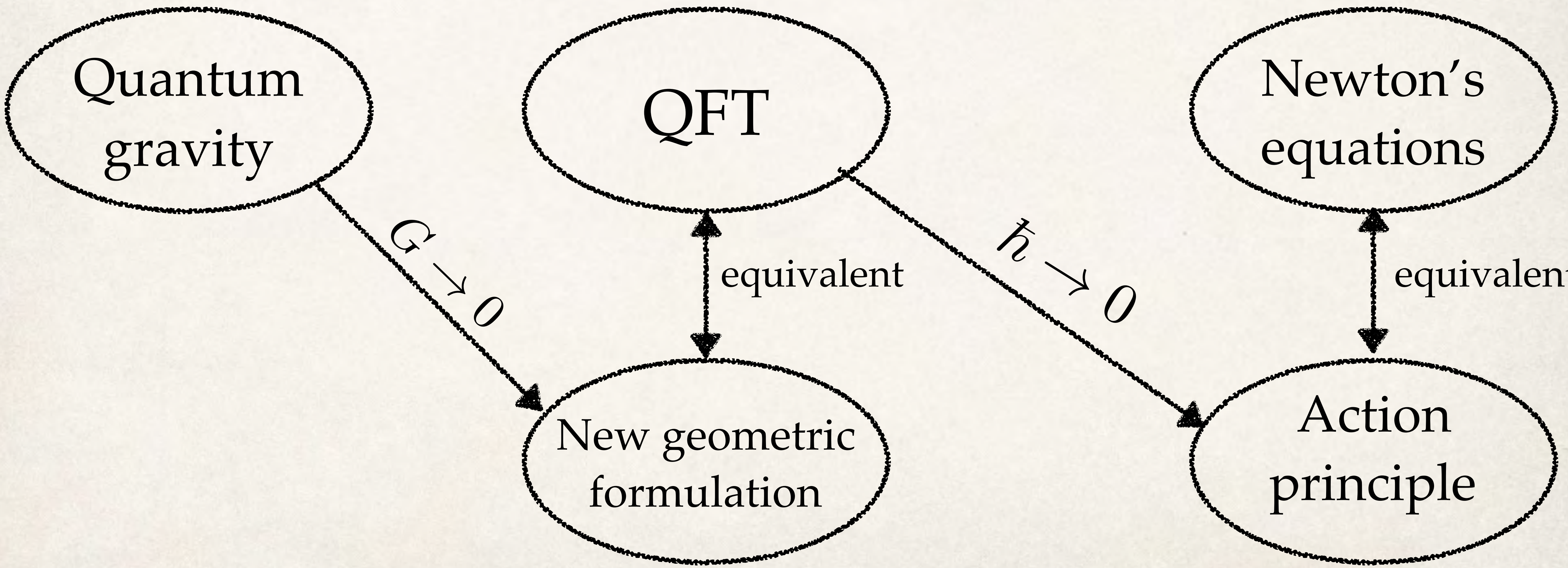
Fantasy

- Beyond understanding QFT better there is one more motivation

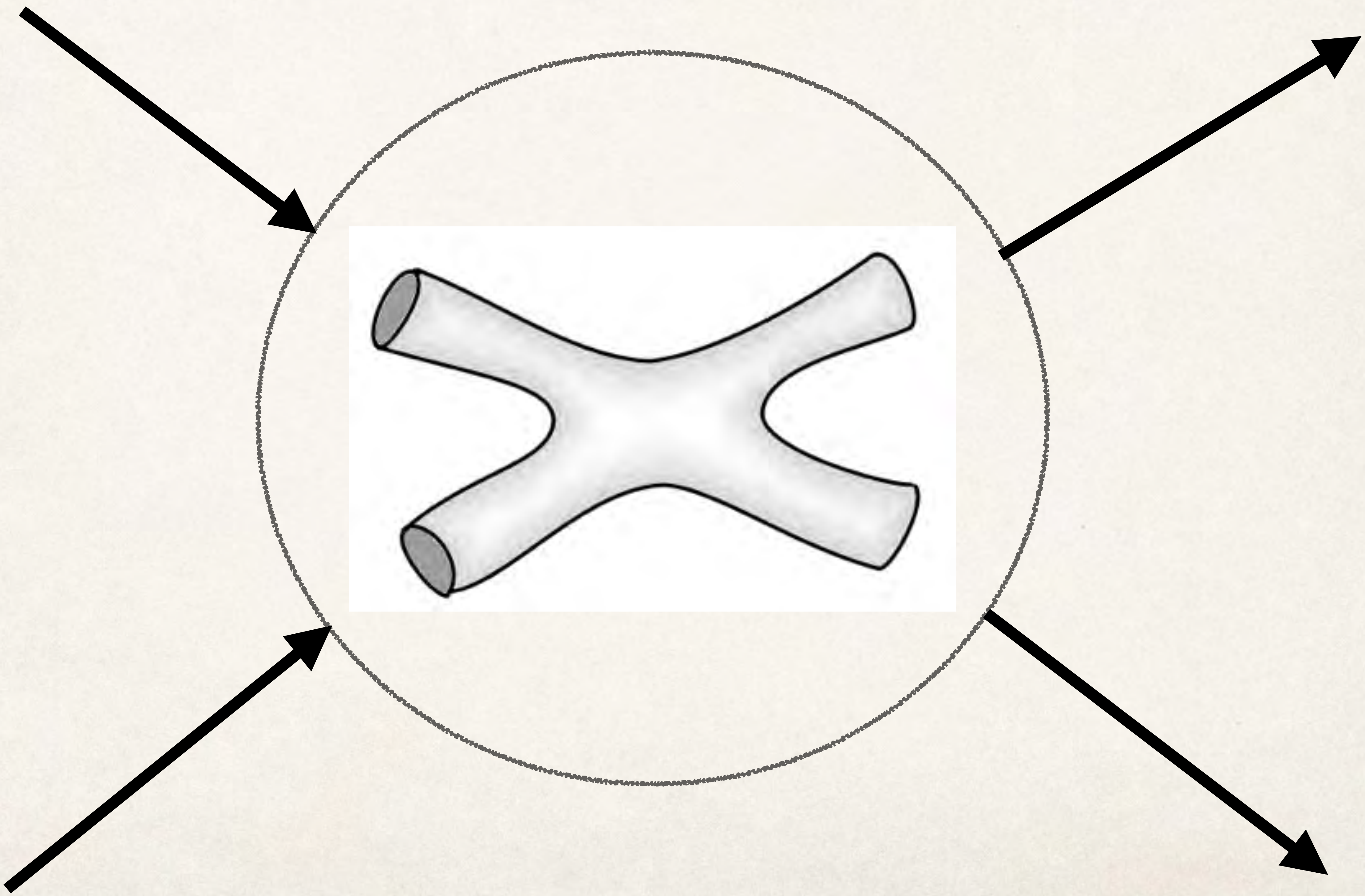


Fantasy

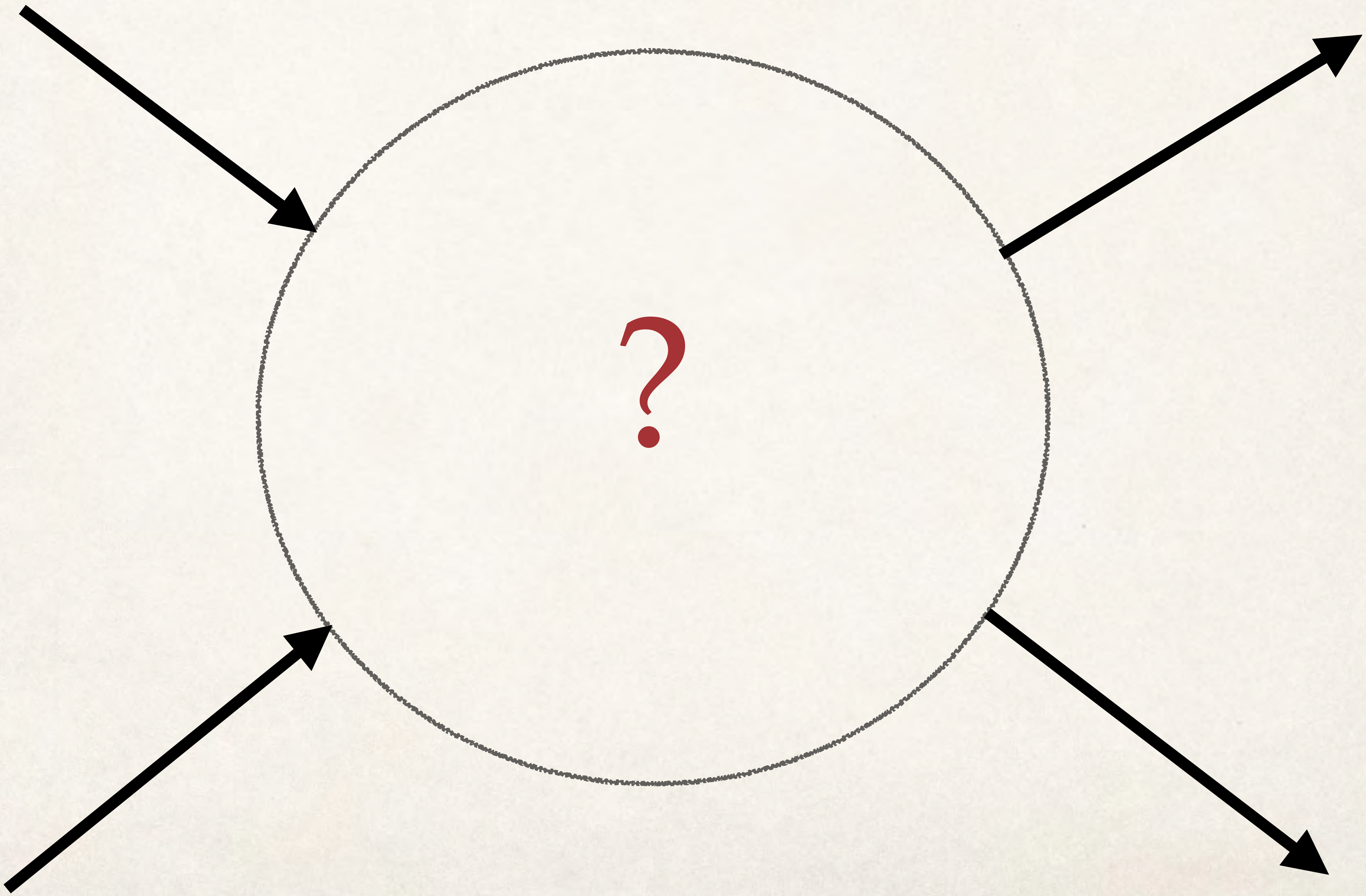
- Beyond understanding QFT better there is one more motivation



We have a theory of quantum gravity: string theory



New geometric picture for string theory?

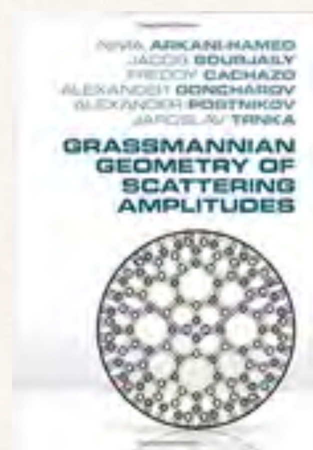


Amplitudes as a new field

- This is one of the directions in fast developing field
- More: scattering equations, BCJ duality, string amplitudes, supergravity finiteness, hexagon bootstrap, cluster polylogarithms, worldsheet models, integration techniques, LHC calculations,…..
- Zeroth order problems open, many chances for young people to make big discoveries!

Resources

Books and reviews



https://arxiv.org/abs/1308.1697

https://arxiv.org/abs/1310.5353

https://arxiv.org/abs/1610.05318

Conferences

Summer school in July in Edinburgh, you can still apply!

https://higgs.ph.ed.ac.uk/workshops/amplitudes-2017-summer-school

Thank you for your attention