Lectures on Perturbative QCD

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Nuclear Physics
(N, π, ...)

$\not{CP}$ & flavor

BSM search
& Precision SM
(Collider physics)

QCD is everywhere

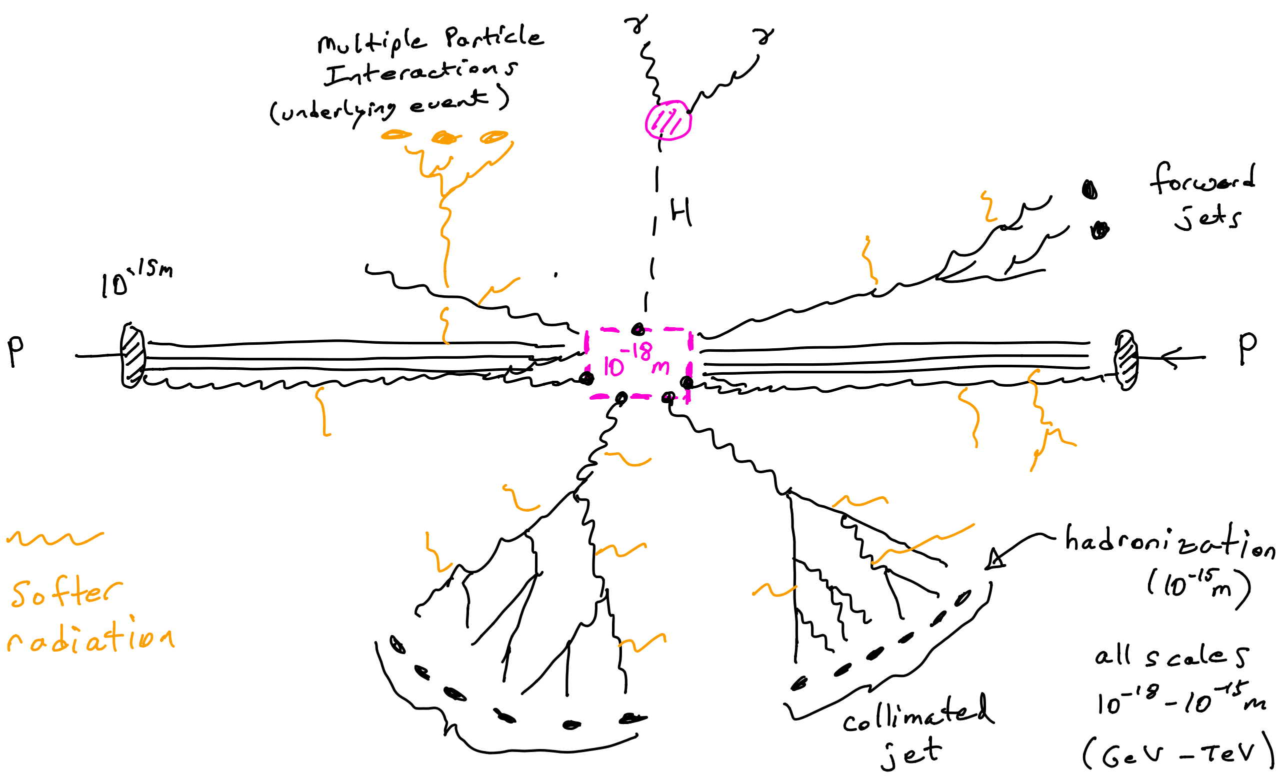
high Temperature
Heavy Ions

ν - Physics
(int. w Nuclei)

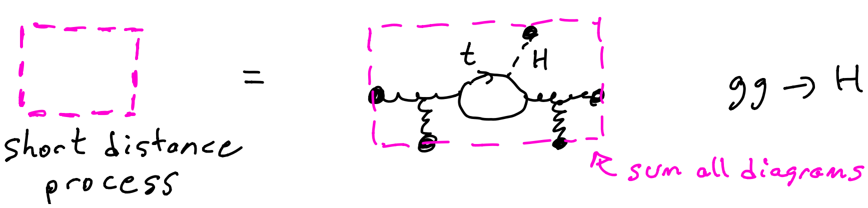
QFT
(prototypical non-abelian gauge thy)

...

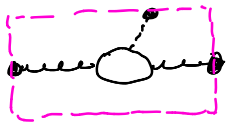
LHC collision of two protons

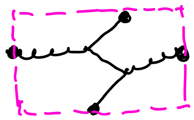


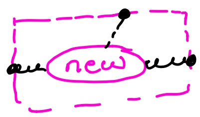
all scales
$10^{-18} - 10^{-15}$m
(GeV – TeV)
are probed



$gg \to H + gg$ $\quad (pp \to H + 2\,\text{jets})$

or  $gg \to H$ ($pp \to H + 0$ jets)

or  $gg \to q\bar{q}$ ($pp \to 2$ jets) $> 10^5$ more likely

or 

Concepts to Explore:

- Perturbative & Non-Perturbative interactions
  collide protons not q or g, pQCD applies at high E = short dist.
  but observe hadrons in jets
- Duality: quarks & gluons vs. hadrons
- Soft & Collinear Singularities in pQCD
- Jets & Shower of Radiation
- Factorization (very successful!)

QCD SU(3) gauge theory # colors = $N_c = 3$

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \sum_{i=u,d,s,c,b,t} \bar{\psi}^\alpha_i (i\not{D} - m_i)^{\alpha\beta} \psi^\beta_i$$

ψ^β_i: quark field, $\beta = 1,2,3$ fundamental rep.

$$+ \mathcal{L}_{\text{gauge fix}}$$

$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g f^{ABC} A^B_\mu A^C_\nu$$

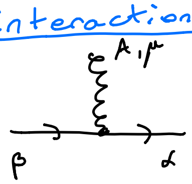
A^C_ν: gluon field

$A = 1, \dots, 8$ adjoint rep. $\qquad [T^A, T^B] = i f^{ABC} T^C$

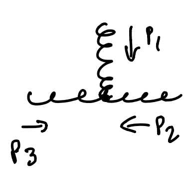
$$iD_\mu = i\partial_\mu - g A^A_\mu T^A$$

interactions:

universal coupling



$= -ig T^A_{\alpha\beta} \gamma^\mu$



$= -g f^{ABC} \left[g^{\mu_1\mu_2} (p_1 - p_2)^{\mu_3} + (1 \to 2 \to 3 \to 1) + (1 \to 3 \to 2 \to 1) \right]$

I. QCD SUMMARY

The $SU(N_c)$ QCD Lagrangian without gauge fixing

$$\mathcal{L} = \bar{\psi}(i\not{D}-m)\psi - \frac{1}{4}G^A_{\mu\nu}G^{\mu\nu A}\,, \qquad G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g f^{ABC} A^B_\mu A^C_\nu \tag{1}$$
$$D_\mu = \partial_\mu + igA^A_\mu T^A\,, \qquad [D_\mu, D_\nu] = igG^A_{\mu\nu}T^A\,.$$

The equations of motion and Bianchi

$$(i\not{D}-m)\psi = 0\,, \qquad \partial^\mu G^A_{\mu\nu} = gf^{ABC}A^{B\mu}G^C_{\mu\nu} + g\bar{\psi}\gamma_\nu T^A\psi\,, \qquad \epsilon^{\mu\nu\lambda\sigma}(D_\nu G_{\lambda\sigma})^A = 0. \tag{2}$$

Color identites

$$[T^A, T^B] = if^{ABC}T^C\,, \qquad \mathrm{Tr}\big[T^AT^B\big] = T_F\delta^{AB}\,, \qquad \bar{T}^A = -T^{A*} = -(T^A)^T\,,$$
$$T^AT^A = C_F\mathbf{1}\,, \qquad f^{ACD}f^{BCD} = C_A\delta^{AB}\,, \quad f^{ABC}T^BT^C = \frac{i}{2}C_AT^A\,,$$
$$T^AT^BT^A = \Big(C_F - \frac{C_A}{2}\Big)T^B\,, \quad d^{ABC}d^{ABC} = \frac{40}{3}\,, \qquad d^{ABC}d^{A'BC} = \frac{5}{3}\delta^{AA'}\,, \tag{3}$$

where $C_F = (N_c^2-1)/(2N_c)$, $C_A = N_c$, $T_F = 1/2$, and $C_F - C_A/2 = -1/(2N_c)$. The color reduction formula and Fierz formula are

$$T^AT^B = \frac{\delta^{AB}}{2N_c}\mathbf{1} + \frac{1}{2}d^{ABC}T^C + \frac{i}{2}f^{ABC}T^C\,, \qquad (T^A)_{ij}(T^A)_{k\ell} = \frac{1}{2}\,\delta_{i\ell}\delta_{kj} - \frac{1}{2N_c}\,\delta_{ij}\delta_{k\ell}\,. \tag{4}$$

Feynman gauge rules, fermion, gluon, ghost propagators, and Fermion-gluon vertex

$$\frac{i(\not{p}+m)}{p^2-m^2+i0}\,, \qquad \frac{-ig^{\mu\nu}\delta^{AB}}{k^2+i0}\,, \qquad \frac{i}{k^2+i0}\,, \qquad -ig\gamma^\mu T^A\,. \tag{5}$$

Triple gluon and Ghost Feynman rules in covariant gauge for $\{A^A_\mu(k), A^B_\nu(p), A^C_\rho(q)\}$ all with incoming momenta, and $\bar{c}^A(p)A^B_\mu c^C$ with outgoing momenta p:

$$-gf^{ABC}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big]\,, \qquad gf^{ABC}p^\mu\,. \tag{6}$$

Triple gluon Feynman rule in bkgnd Field covariant gauge $\mathcal{L}_{gf} = -(D^A_\mu Q^A_\mu)^2/(2\xi)$ for $\{A^A_\mu(k), Q^B_\nu(p), Q^C_\rho(q)\}$ with A^A_μ a bkgnd field:

$$-gf^{ABC}\Big[g^{\mu\nu}\Big(k-p-\frac{q}{\xi}\Big)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}\Big(q-k+\frac{p}{\xi}\Big)^\nu\Big]\,. \tag{7}$$

Lorentz gauge:

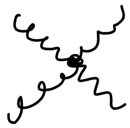
$$\mathcal{L} = -\frac{(\partial_\mu A^\mu)^2}{2\xi}\,, \qquad D^{\mu\nu}(k) = \frac{-i}{k^2+i0}\Big(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\Big)\,, \tag{8}$$

where Landau gauge is $\xi \to 0$. Coulomb gauge:

$$\vec{\nabla}\cdot\vec{A} = 0\,, \qquad D^{\mu\nu}(k) = \frac{-i}{k^2+i0}\bigg(g^{\mu\nu} - \frac{[g^{\nu0}k^0k^\mu + g^{\mu0}k^0k^\nu - k^\mu k^\nu]}{\vec{k}^2}\bigg)\,,$$
$$D^{00}(k) = \frac{i}{\vec{k}^2-i0}\,, \qquad D^{ij}(k) = \frac{i}{k^2+i0}\Big(\delta^{ij} - \frac{k^ik^j}{\vec{k}^2}\Big). \tag{9}$$

Running coupling with $\beta_0 = 11C_A/3 - 4T_Fn_f/3 = 11 - 2n_f/3$:

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1+\frac{\beta_0}{2\pi}\alpha_s(\mu_0)\ln\frac{\mu}{\mu_0}} = \frac{2\pi}{\beta_0\ln\frac{\mu}{\Lambda_{\rm QCD}}}\,, \qquad \frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_s(\mu_0)} + \frac{\beta_0}{2\pi}\ln\frac{\mu}{\mu_0}\,. \tag{10}$$

 $= \cdots$

$T^A T^A = C_F \mathbb{1}$ $\quad C_F = 4/3$ "quark color charge"

$f^{ACD} f^{BCD} = C_A \delta^{AB}$ $\quad C_A = 3$ "adj. color charge"

covariant gauge (Fadeev-Popov) ← needed to invert $\mathcal{L}$ to obtain gluon propagator

$$\mathcal{L}_{\text{gauge fix}} = -\frac{1}{2\xi}(\partial_\mu A^{\mu A})^2 + \bar{c}^A(-\partial_\mu D^\mu_{AB})\, c^B$$

← ghosts, anti-commuting Lorentz scalars

path integral integrates over gauge inv. configurations (2 gluon polarizations), turned into $\mathcal{L}_{\text{gauge fix}}$ by Fadeev-Popov procedure

- we'll use Feynman gauge $\xi = 1$,  $= \dfrac{-i g^{\mu\nu} \delta^{AB}}{p^2 + i0}$

Running Coupling

In QCD resolution scale μ of a process is very important

$$\alpha_s = \frac{g^2}{4\pi} = \alpha_s(\mu)$$

parameters in QFT defined by $\overline{MS}$ renormalization scheme here

scheme parameter μ $\qquad \alpha_s^{\text{bare}} = Z_\alpha\, \mu^{2\epsilon} \alpha_s(\mu)$ ← $1/\epsilon$ poles

(4 indep. ways to compute β)

n_f fermions

$$\mu \frac{d}{d\mu} \alpha_s^{(n_f)}(\mu) = -\frac{\beta_0^{(n_f)}}{2\pi}\left[\alpha_s^{(n_f)}(\mu)\right]^2 + \cdots$$

β - function $\qquad \beta_0^{(n_f)} = \frac{11}{3} C_A - \frac{2}{3} n_f$

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\beta_0}{2\pi}\alpha_s(\mu_0) \ln\frac{\mu}{\mu_0}} = \frac{2\pi}{\beta_0 \ln\left(\frac{\mu}{\Lambda_{QCD}}\right)},$$

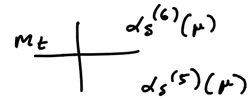
$\Lambda_{QCD} \sim 250\,\text{MeV}$ dimensional transmutation

- processes with physical scale s will involve $\alpha_s(\mu) \ln\frac{\mu}{s}$ terms, so we pick $\mu \simeq s$ to avoid large logs

$\Rightarrow$ more than one scale s_i $\Rightarrow$ more than one relevant $\alpha_s(\mu_i)$

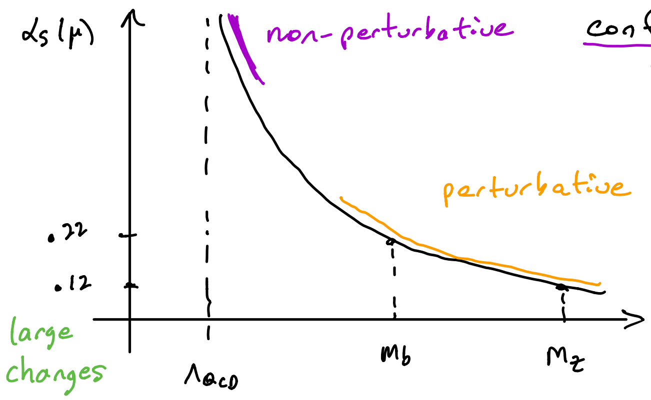
precisely what happens if both pert & non-pert physics is involved

- heavy particles decouple

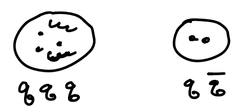


continuous at $\mu = m_t$ (at this order)

If this is unfamiliar see Hmwk http:www2.lns.mit.edu/~iains/registerEFTx Chapter 4



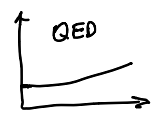
confinement, α_s so strong
q & g are confined in hadrons



$M_{proton} \propto \Lambda_{QCD}$
$(\text{size}) \propto \Lambda_{QCD}^{-1}$
Lattice (Giusti)

$\alpha_s(\mu \to \infty) = 0$ Asymptotic freedom
free quarks at short dist.

Physical Picture: large magnetic moments of charged spin-1 gluons make vacuum paramagnetic, screen mag. charge, antiscreen electric chg.



Factorization

Key tool to calculate cross sections is the ability to independently consider different parts of the process

$$d\sigma \sim \begin{pmatrix}\text{Prob. for} \\ \text{gluons taken} \\ \text{from protons}\end{pmatrix} \begin{pmatrix}\hat{\sigma}(gg \to H), \\ \hat{\sigma}(gg \to Hg), \\ \ldots\end{pmatrix} \begin{pmatrix}\text{Prob. for gluons} \\ \text{to produce} \\ \text{jets}\end{pmatrix}$$

Another Key idea is to exploit inclusive observables

$e^+e^- \to X$ (any hadrons)

$e^-p \to e^- X$ DIS

eg. Higgs Production via gluon fusion

$pp \to H + X$ (any hadrons or 0+1+2+... jets)

$$\sigma = \int dx_a\, dx_b\; f_g(x_a,\mu)\, f_g(x_b,\mu)\; \hat{\sigma}_{gg\to H+X}(x_a, x_b, \mu, m_H) \quad * (1)$$

universal — parton dist'n function (PDF)

discuss μ later (distinct time scales)

f_g = Prob. of finding g in proton with momentum fraction x_a = Probability Density (proton snapshot)

$(1) = \sum_i \text{Prob}(i)$ sum over everything that can happen to final state quarks & gluons, so we are not sensitive to this dynamics (jets etc)

Practical Limits on $\sum_i$ → cuts on jets to control background or enhance signals ($\geq N$ jets SUSY)

→ need for more exclusive events to determine expt. efficiencies etc.

Still sum over dynamics inside the jet & characterize it by a few variables: jet momentum $p_J^\mu = \sum_{i\in J} p_i^\mu$

angular size 

$e^+e^- \to X$ (hadrons)

$e^+e^- \underset{\gamma^*}{\to} q\bar{q},\ q\bar{q}g, \ldots$

massless quarks
ignore Z exchange

Tree Level: $\Big|$  $\Big|^2$

Often normalize to $e^+e^- \to \mu^+\mu^-$ "R-ratio"

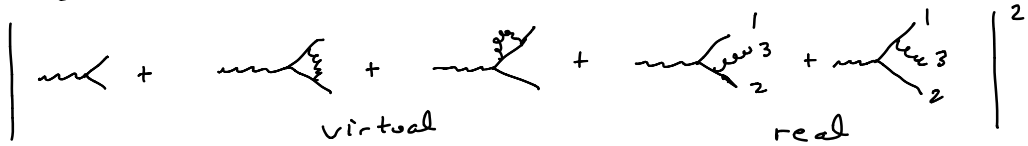
$$\sigma_0 = \frac{4\pi \alpha_{em}^2}{3q^2} N_c \sum_i Q_i^2$$

Q_i = active quark E&M charge $= \frac{2}{3}, -\frac{1}{3}$

$q^2 \geq m_i^2$

→ $\sum_i$ over active/massless quarks, transitions at quark thresholds

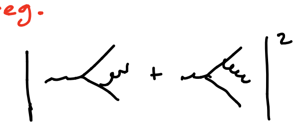
$\mathcal{O}(\alpha_s)$:



$$\sigma = \sigma_0 \left(1 + \hat{\sigma}_V + \hat{\sigma}_R \right)$$

Real first:

$d-1$, d for dim. reg.

$$\int d\Phi_3 \, |A^{real}|^2 = \int \prod_{i=1}^{3} \frac{d^3 p_i}{2 p_i^0} \, (2\pi)^4 \, \delta^{(4)}(q - p_1 - p_2 - p_3) \, \Big|$$  $$\Big|^2$$

$\uparrow$ $p_i^2 = 0$, $p_i^0 = |\vec{p}_i|$

cm frame: $q = (Q, \vec{0})$ $\quad x_i \equiv \frac{2 q \cdot p_i}{q^2} = \frac{2}{Q} p_i^0$ energy fractions $0 \le x_i \le 1$

$x_1 + x_2 + x_3 = 2$

$p_1^2 = 0 = (q - p_2 - p_3)^2 \implies 2 p_2 \cdot p_3 = Q^2 (x_2 + x_3 - 1) = Q^2 (1 - x_1)$

$= 2 E_2 E_3 (1 - \cos\theta_{23})$

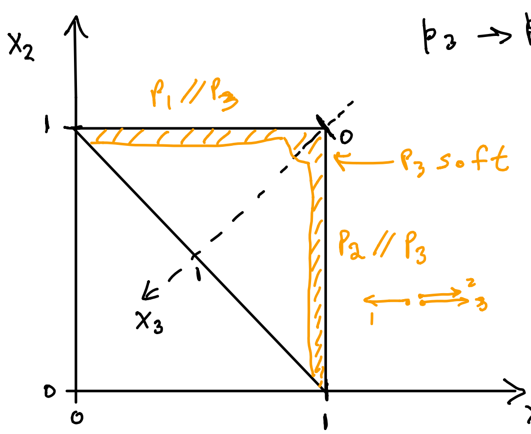
get $$\int_0^1 dx_1 \, dx_2 \, dx_3 \; \frac{\delta(2 - x_1 - x_2 - x_3)}{(1-x_1)^\epsilon (1-x_2)^\epsilon (1-x_3)^\epsilon} \left[\frac{x_1^2 + x_2^2 - \epsilon x_3^2}{(1-x_1)(1-x_2)} \right]$$

IR divergences :

$p_3 \to 0$ soft gluon $\quad x_3 \to 0$ so x_1 & $x_2 \to 1$

$p_3 \to p_1$ g collinear q $\quad p_1 \cdot p_3 = 0$, $x_2 \to 1$ $(\theta_{13} \to 0)$

$p_3 \to p_2$ g collinear $\bar{q}$ $\quad p_2 \cdot p_3 = 0$, $x_1 \to 1$ $(\theta_{23} \to 0)$



IR singularities at edges of phase space

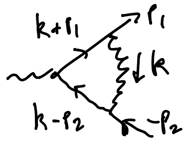
Regulate with dimensional regularization $d = 4 - 2\epsilon$

These are limits where we can't resolve the partons [hatched box] like 2-jets , rest [triangle] 3-jets

KLN Thm: singularities cancel if we sum over degenerate states

IR divergenes cancel with virtual graphs

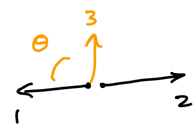
e.g.



most IR singular integral

$$\int \frac{d^dk \; p_1\cdot p_2}{k^2 (k+p_1)^2 (k-p_2)^2} \quad \underset{\text{soft}}{\overset{k\to 0}{\sim}} \quad \int \frac{d^4k \; p_1\cdot p_2}{k^2 \, p_1\cdot k \; p_2 \cdot k} \quad \text{IR singular} \sim \frac{d^4k}{k^4}$$

also collinear limits: $k \to p_1$, $k \to p_2$ IR singular



eg soft integrand

$$\int \frac{dk^0 \, d^{d-1}k \qquad E_1 E_2}{(k^0 - |\vec{k}| + i0)(k^0 + |\vec{k}| - i0)\, E_1 (k^0 - |\vec{k}|\cos\theta)\, E_2 (k^0 + |\vec{k}|\cos\theta)}$$

$$\sim \int \frac{d^{d-1}k}{|\vec{k}|^3 (1-\cos^2\theta)} \quad \sim \quad \underbrace{\int_0 \frac{dk}{k} k^{-2\epsilon}}_{\text{soft IR } 1/\epsilon} \; \underbrace{\int_{-1}^{1} d\cos\theta \, \frac{(\sin\theta)^{-2\epsilon}}{(1-\cos^2\theta)}}_{\text{collinear IR } 1/\epsilon}$$

<u>Full results</u>

$$\hat{\sigma}_V = \frac{\alpha_s(\mu) C_F}{\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \frac{\cos(\pi\epsilon) e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(\frac{-1}{\epsilon^2} - \frac{3}{2\epsilon} - 4 \right)$$

$$\hat{\sigma}_R = \frac{\alpha_s(\mu) C_F}{\pi} \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \frac{\cos(\pi\epsilon) e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4} \right)$$

Here: $\sigma_B \to \sigma_B$

$$\sigma = \sigma_0 \left(1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} \right) \quad \text{IR finite}$$

- At next order we find $\alpha_s^2 \ln(\mu^2/Q^2)$ so $\mu^2 \simeq Q^2$ is good scale choice

- Can we really compare q & g calculation with hadronic cross-section? (eg. # particles ~ 30 not 2 or 3)

- What happens if we restrict real radiation?