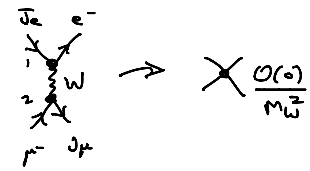
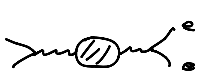
Operator Product Expansion for $e^+e^- \to$ hadrons

OPE : $\quad J_1(x)\, J_2(0) \xrightarrow{x\to 0} \sum_n C_n(x)\, O_n(0)$



unitarity (optical Thm) $\quad \sigma = \frac{1}{2q^2}\,\mathrm{Im}\Big($  $\Big)$

$$= -\frac{(4\pi\alpha)^2}{q^2}\,\mathrm{Im}\,\Pi_h(q^2)$$

$Q_i\, \bar\psi_i \gamma_\mu \psi_i$ EM quark current

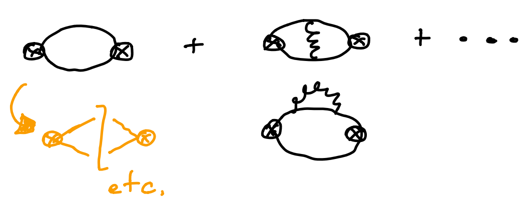
$$\Pi_h(q^2) = \underset{-2}{\frac{i}{3q^2}} \underset{-4}{\int d^4x}\, e^{iq\cdot x} \langle 0|\, T\, \underset{+3}{J^\mu(x)}\, \underset{+3}{J_\mu(0)}\, |0\rangle$$

$\boxed{\text{IF}}$ dominated by short distances $x \to 0$

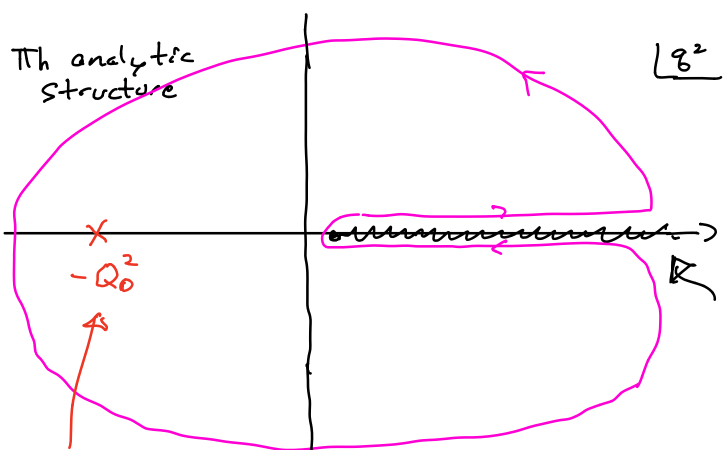
$$= -\Big[\, \underset{0}{C^1(q^2)}\, \mathbb{1} + \underset{-4}{C^{q\bar q}(q^2)}\, \underbrace{m\bar\psi\psi}_{m\Lambda_{QCD}^3} + \underset{-4}{C^{G^2}(q^2)}\, \underbrace{G^{\mu\nu}G_{\mu\nu}}_{\Lambda_{QCD}^4} + \cdots \Big]$$

$\langle 0|\mathbb{1}|0\rangle = 1$ $\qquad$ suppressed

$\mathrm{Im}\, C^1(q^2)$: $\quad \mathrm{Im}$



reproduces q & g pQCD calculation



Want $\Pi_h(q^2)$ for large time like q^2 where dominated by high E int. states with many hadrons

OPE valid for spacelike points $x_E \to 0$

$$\oint \frac{dq^2}{2\pi i} \frac{\Pi_h(q^2)}{(q^2+Q_0^2)^2} = \frac{d\Pi_h}{dq^2}\Big|_{q^2=-Q_0^2}$$

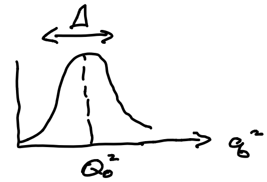
result we can calculate with OPE & pQCD

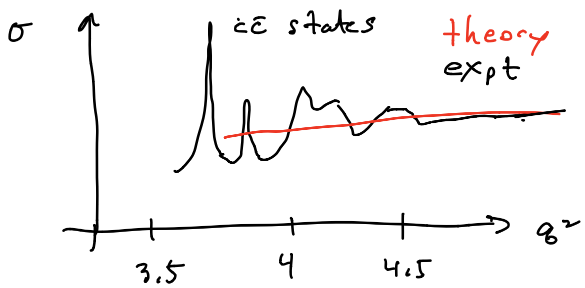
$$= \int \frac{dq^2}{2\pi i} \frac{1}{(q^2+Q_0^2)^2}\, \underbrace{\mathrm{Disc}\,\Pi_h(q^2)}_{2i\,\mathrm{Im}\,\Pi_h}$$

$$= \frac{-1}{(4\pi\alpha)^2}\int_{4m_\pi^2}^{\infty}\frac{dq^2}{\pi}\,\frac{q^2}{(q^2+Q_0^2)^2}\,\sigma(q^2) \longleftarrow$$ smeared hadronic cross-section

<u>Moral</u>: Need to average over enough states to get agreement between hadronic & pQCD results "quark-hadron duality"

Other smearing functions are possible

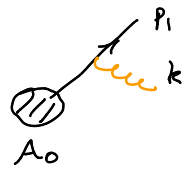




At large q^2 more states in given Δq^2 & can consider unaveraged comparison "local duality"

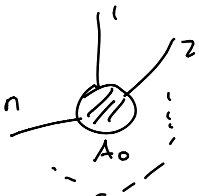
If we put cuts on phase space: $\frac{1}{\epsilon}$ poles in $\hat{\sigma}_R$ accompanied by logs of cutoff parameters.

Soft Approximation (Eikonal)



$$\underbrace{\bar{u}_1 \left(-ig \not{\epsilon}^a T^a\right) \frac{i(\not{p}_1+\not{k})}{(p_1+k)^2}} A_0 \qquad \bar{u}_1 \not{p}_1 = 0, \; p_1^2 = 0$$

$$\simeq g \frac{p_1 \cdot \epsilon^a}{p_1 \cdot k} T^a \qquad \text{Eikonal amplitude}$$



$$= \sum_i \frac{g\, p_i \cdot \epsilon^a \hat{T}_i^a}{p_i \cdot k} A_N$$

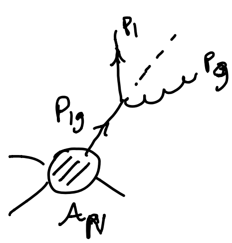
Factorizes with non-trivial color correlation

Applied to $\hat{\sigma}_R$ gives $\frac{p_1 \cdot p_2}{p_1 \cdot p_3 \; p_2 \cdot p_3}$ integrand, using $\int_0^\delta dx_3$, $\delta \ll 1$

$$\hat{\sigma}_R^{soft} = \frac{C_F \alpha_s}{\pi} \left[\frac{1}{\epsilon^2} - \frac{2}{\epsilon} \ln \delta + 2 \ln^2 \delta + \text{finite} \right]$$

↑ reproduces $\frac{1}{\epsilon^2}$

Collinear Approximation



$P_{1g} = p_1 + p_g$

$p^\mu = \left(|\vec{P}_{1g}|, \vec{P}_{1g}\right)$

$\bar{n}^\mu = \left(1, -\frac{\vec{P}_{1g}}{|\vec{P}_{1g}|}\right)$

Sudakov decomposition:

$$p_1^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{\bar{n}^\mu}{2\bar{n}\cdot p}$$

$$p_g^\mu = (1-z) p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z} \frac{\bar{n}^\mu}{2\bar{n}\cdot p}$$

fixed by $p_1^2 = 0$

$p_g^2 = 0$

p^μ & $\bar{n}^\mu$ light like $p^2 = \bar{n}^2 = 0$

$$z = \frac{p_1^0}{P_{1g}^0} \quad \text{energy fraction}, \qquad P_{1g}^2 = \frac{-k_\perp^2}{z(1-z)}$$

consider $k_\perp \to 0$: $\quad P_{13}^2 \to 0$ approx. on-shell

$$|A_{N+1}(p_g, p_1, \dots)|^2 \simeq \left[\frac{2C_F}{P_{1g}^2} g^2 P_{qq}(z,\epsilon) \right] |A_N(p_1+p_g, \dots)|^2$$

Amplitude factorizes into lower-pt amplitude times splitting function

$$P_{qq}(z,\epsilon) = \left(\frac{1+z^2}{1-z} - \epsilon(1-z) \right)$$

Applied to $\hat{\sigma}_R$ for $p_1 /\!/ p_3$, $z = 1 - x_3$ with

$$\int_{1-\delta_c}^{1} dx_2 \int_{\delta}^{1} dx_3 \qquad \text{gives}$$

collinear region

avoid soft region

$$\hat{\sigma}_R^{p_1/\!/p_3} = \frac{C_F \alpha_s}{\pi} \frac{1}{2} \left[\frac{3}{2\epsilon} + \frac{2\ln\delta}{\epsilon} - \ln^2\delta - \frac{3}{2}\ln\delta_c - 2\ln\delta\ln\delta_c + \text{finite} \right]$$

will be doubled by adding $p_2 /\!/ p_3$

reproduces $\frac{1}{\epsilon}$

Cancels with $\hat{\sigma}_R^{soft}$

Gives us an idea what a jet cross-section looks like

$$\sigma_{1\text{-loop}}^{total} = \sigma_{2\text{-jet}} + \sigma_{3\text{-jet}}$$

$\frac{1}{\epsilon_{IR}}$ cancel

$$\sigma_{2\text{-jet}} = \sigma_0 \left(1 + \hat{\sigma}_V + \hat{\sigma}_R^{soft} + \hat{\sigma}_R^{p_1/\!/p_3} + \hat{\sigma}_R^{p_2/\!/p_3} \right)$$

$$= \sigma_0 \left(1 + \frac{\alpha_s C_F}{\pi} \left[-2\ln\delta\ln\delta_c + \ln^2\delta - \frac{3}{2}\ln\delta_c + \text{finite} \right] \right)$$

$$\sigma_{3\text{-jet}} = \sigma_{1\text{-loop}}^{total} - \sigma_{2\text{-jet}} \qquad \text{"Exclusive Jets"}$$

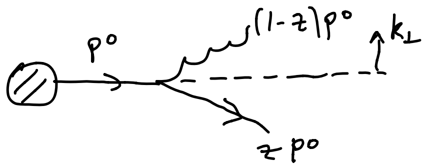
(analogous to 1st jet defn, Sterman-Weinberg Jet, 1977)

Inclusive Jets: ask for 1 jet in region away from edge of phase space, then $\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$ fine as leading result without considering δ, δ_c

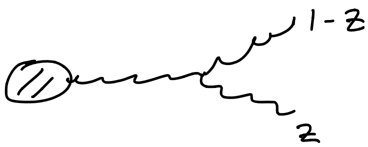
[Weinberg Story]

Jets — Why does QCD produce Jets ?

log enhancement from collinear singularities

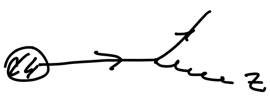


$$\sigma \propto \frac{\alpha_s C_F}{\pi} \frac{dk_\perp}{k_\perp} dz\, P_{qq}(z)$$

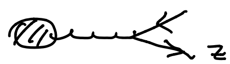


$$\sigma \propto \frac{\alpha_s C_A}{\pi} \frac{dk_\perp}{k_\perp} dz\, P_{gg}(z)$$

$$P_{gg}(z) = \frac{z}{1-z} + \frac{1-z}{z} + z(1-z)$$



$C_F P_{gq}(z)$ ← $\frac{1+(1-z)^2}{z}$



$\frac{1}{2} P_{qg}(z)$ ← $z^2+(1-z)^2$

prefer to split in collimated manner

Soft singularity also plays a role. Here the fact that soft gluons are prefferentially emitted within cone of collinear emissions (angular ordering) plays a role.

Leading contribution is strongly ordered

$$k_{1\perp} \gg k_{2\perp} \gg k_{3\perp} \cdots \gg k_{n\perp} \sim \Lambda_{QCD}$$

If $\alpha_s \ln\left(\frac{k_{i\perp}}{k_{i+1\perp}}\right) \sim 1$ no perturbative suppression

How do we define a Jet ?

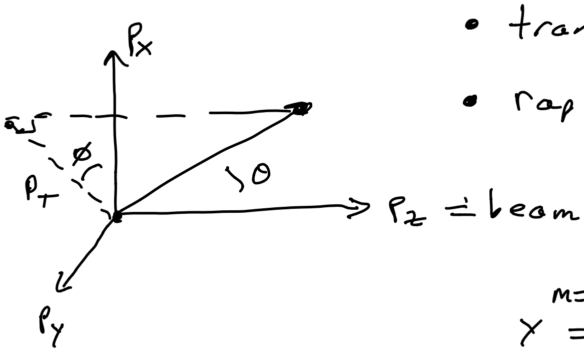
final state is collection of hadrons

which particles do we group together ? (not unique)

need IR safe algorithm: invariant under $p_i \to p_j + p_k$
if $p_j \parallel p_k$ or $p_j \to 0$

Hadron Collider Vars

know proton cm, not partonic collision's cm



- transverse momentum p_T
- rapidity $y = \frac{1}{2} \ln\left(\frac{E+p_z}{E-p_z}\right)$

$\Delta y = y_1 - y_2$ is invariant under $\hat{z}$ boosts

$$y \overset{m=0}{=} \ln \cot \frac{\theta}{2}$$

$\Delta R = \left[(\Delta y)^2 + (\Delta \phi)^2\right]^{1/2}$ is boost invariant angular distance

Recombination Algorithms:

consider set of particles L (hadrons, partons, calorimeter cells)

$$d_{ij} = \min\left(p_{Ti}^{2r}, p_{Tj}^{2r}\right) \frac{\Delta R_{ij}^2}{R^2} = \text{distance}(i,j)$$

$$d_{iB} = p_{Ti}^{2r} = \text{distance}(i, \text{beam})$$

$$\text{Find} \quad \min_{i,j \in L} \left(\{d_{ij}\}, \{d_{iB}\} \right)$$

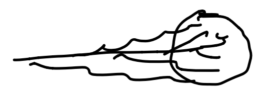
join i & j into new particle in L & repeat

call i a jet and remove it, & repeat

Stop when L is empty

- $r = 1$ k_T algorithm, clusters soft particles first (jet regions not circular)
- $r = 0$ Cambridge/Aachen, geometric

- $r = -1$ Anti-k_T, clusters harder particles first (circular jet regions)
  <u>default</u> ATLAS & CMS

R = jet radius parameter 

eg. $R = 0.5$, demand 1-jet with $p_T > 30$ GeV & all remaining jets having $p_T \le 30$ GeV
"1-jet event"

eg. $H + 0$-jets (used in Higgs coupling measurements)
all jets have $p_T \le 30\text{ GeV} = p_T^{cut}$

$$\sigma \sim \sigma_{incl} \left[1 - \frac{2 \alpha_s C_A}{\pi} \ln^2\left(\frac{p_T^{cut}}{m_H}\right) + \cdots \right]$$

large log series that must be summed to all orders

Leading Logs: $\sim 1 + \alpha_s L^2 + \alpha_s^2 L^4 + \cdots$ exponentiate

$$\sigma \sim \sigma_{incl} \exp\left[- \frac{2 \alpha_s C_A}{\pi} \ln^2\left(\frac{p_T^{cut}}{m_H}\right) \right]$$

example of Sudakov form factor from restricting radiation

Also Cone Algorithms 

which are no longer popular

Parton Shower

- construct an exclusive description of events at hadron level (needed for experimental analyses)
- Monte Carlo program to iterate collinear approximation
- LL shower + large N_c + model for hadronization

  simplify interference, planar color flow

(• improvements MC@NLO, POWHEG, ...)

Probability for parton i to branch between

$$q^2 \;\&\; q^2+dq^2 \;=\; d\mathcal{P}_i = \frac{\alpha_s}{2\pi}\frac{dq^2}{q^2}\int_{Q_0^2/q^2}^{1-Q_0^2/q^2} dz\, P_{ji}(z)$$

evolution var.

partons no longer resolved if $\Delta q^2 \leq Q_0^2$, cuts off z gives finite probability

Probability for no branching between Q^2 & q^2 is

$$\equiv \Delta_i(Q^2, q^2)$$

then

$$\frac{d\Delta_i(Q^2,q^2)}{dq^2} = \lim_{dq^2\to 0}\frac{\Delta_i(Q^2, q^2+dq^2) - \Delta_i(Q^2,q^2)}{dq^2}$$

$$= \Delta_i(Q^2,q^2)\,\frac{d\mathcal{P}_i}{dq^2}$$

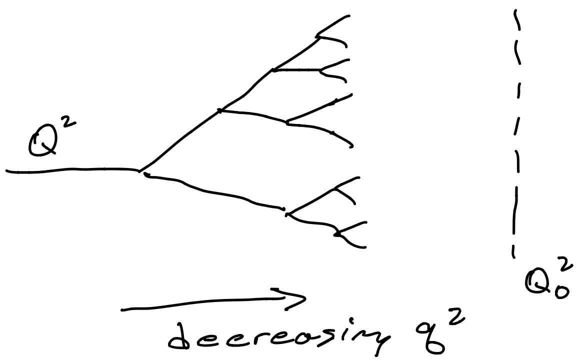
no branching to q^2

branching btwn q^2 & q^2+dq^2

Solution:

$$\Delta_i(Q^2, q^2) = \exp\left[-\int_{q^2}^{Q^2} \frac{dk^2}{k^2}\,\frac{\alpha_s}{2\pi} \int_{Q_0^2/k^2}^{1-Q_0^2/k^2} dz\; P_{ji}(z)\right]$$

note: $\Delta_i(Q^2, Q_0^2) \sim \exp\left[-C_F \frac{\alpha_s}{2\pi} \ln^2 \frac{Q^2}{Q_0^2}\right]$ Sudakov Form Factor



Implementation

- random number $\rho \in [0,1]$, solve $\Delta_i(Q^2, q_1^2) = \rho$
  - if $q_1^2 > Q_0^2$ choose z-value with $P_{ji}(z)$
  - if $q_1^2 < Q_0^2$ stop
- repeat on daughter branches with $\Delta_i(q_1^2, q_2^2)$

Pythia, Herwig, Sherpa, ...