Lecture 3 Outline

- DIS : Parton Dist'ns & factorization

  $pp \to H + X$

- $e^+e^- \to$ 2 jets, event shapes, factorization

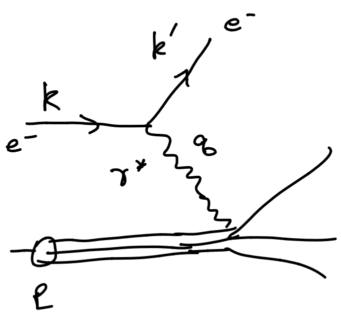
- $pp \to H + 0$-jets

Deep Inelastic Scattering (DIS) $e^- p \to e^- X$

– key process for foundations of QCD (quarks, asym. freedom)

$$s = (k+P)^2$$

$$q^2 = -Q^2 \qquad Q^2 > 0 \qquad y = \frac{Q^2}{x s}$$



$$x = \frac{Q^2}{2P\cdot q} \qquad 0 < x < 1$$

$$y = \frac{q\cdot P}{k\cdot P} \qquad 0 < y < 1 \qquad \left(y = 1 - \frac{k^{0\prime}}{k^0} \ \text{e}^- \text{ energy loss in proton rest frame} \right)$$

measurable with leptons

$Q^2 \gg \Lambda_{QCD}^2$

$$P_X^2 = (q+P)^2 = \frac{Q^2(1-x)}{x} \sim Q^2 \quad \text{large}$$

(proton blown apart)

$$\frac{d\sigma}{dx\,dQ^2} = \frac{8\pi\alpha^2}{Q^4}\left[(1+(1-y)^2)\, \underline{F_1(x,Q^2)} + \frac{(1-y)}{x}\left\{ \underline{F_2(x,Q^2)} - 2x F_1(x,Q^2) \right\} \right]$$

QCD/ hadronic dependence in dimensionless structure functions

hadronic tensor

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X (2\pi)^4 \delta^4(q+P-P_X) \langle P | J^\mu(0) | X \rangle \langle X | J^\nu(0) | P \rangle$$

$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1(x,Q^2) + \left(P^\mu + \frac{q^\mu}{2x} \right)\left(P^\nu + \frac{q^\nu}{2x} \right) \frac{F_2(x,Q^2)}{P\cdot q}$$

- uses current conservation $\partial^\mu J_\mu = 0 \Rightarrow q^\mu W_{\mu\nu} = 0$
- Parity & Time Reversal & hermiticity $J^\dagger = J$

actually $F_i = F_i\left(x, \frac{Q^2}{\Lambda_{QCD}^2}\right)$

Factorization Theorem

$$F_1\left(x, \frac{Q^2}{\Lambda_{QCD}^2}\right) = \sum_j \int_x^1 \frac{d\xi}{\xi} \; C_j\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right) f_j\left(\xi, \frac{\mu}{\Lambda_{QCD}}\right) + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

similar for F_2

depend on quark flavor

parton distribution functions f_j: f_{q_i} & f_g $\quad u, d, \bar{u}, \bar{d}, \ldots$

take snapshot of proton on short time scale $t \sim \frac{1}{Q}$

x = mom. fraction of struck quark, ξ = mom. fraction of parton j in proton

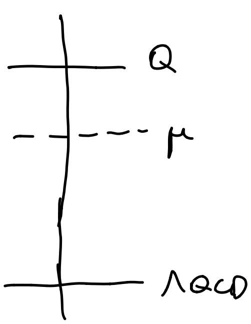
Proof:
- OPE (long), twist-2 operators
- IR structure of QCD eg. with Soft-Collinear Effective Theory (SCET) $\to$ extra reading

$$f_{q_i}\left(\xi, \frac{\mu}{\Lambda}\right) = \int \frac{dy}{2\pi} \, e^{-2i(\xi \bar{n}\cdot p)y} \langle p | \bar{\psi}_i(\bar{n}y) \, W(\bar{n}y, -\bar{n}y) \, \bar{\slashed{n}} \, \psi_i(-\bar{n}y) \, | p \rangle \quad (*)$$

- $\bar{n}^2 = 0$ light cone matrix element
  ($\to$ twist 2, symmetric & traceless, $\bar{n}^{\mu_1} \ldots \bar{n}^{\mu_k}$)
- $W = P\exp \int_{-y}^{y} ds \; \bar{n}\cdot A(\bar{n}s)$ for gauge invariance
  Wilson Line

a fundamental mom. distribution of proton

Scale Separation



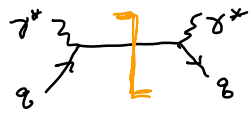
- μ divides long & short distance physics

$$\ln\left(\frac{Q}{\Lambda_{QCD}}\right) = \ln\left(\frac{Q}{\mu}\right) + \ln\left(\frac{\mu}{\Lambda_{QCD}}\right)$$

in C_j $\qquad$ in f_j

$f_j\left(\xi, \frac{\mu}{\Lambda}\right)$ depending on scale "μ" where we probe the parton j the distribution changes (more later)

Tree Level



$$C_j\left(\frac{x}{\xi}, \frac{Q^2}{\mu}\right) = \frac{Q_j^2}{2}\, \delta\left(1 - \frac{x}{\xi}\right)$$

$$\therefore \quad F_1\left(x, \frac{Q^2}{\Lambda^2}\right) = \sum_j \frac{Q_j^2}{2}\, f_{q_j}\left(x, \frac{\mu}{\Lambda}\right)$$

$\Rightarrow$ measurements of universal f_{q_j}'s

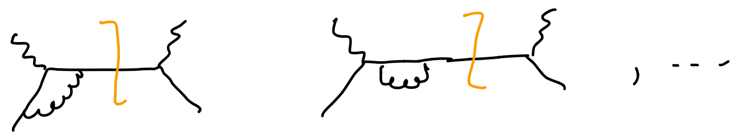
"parton model", independent of $Q \to$ scaling

Also $F_2 = 2x\, F_1 \Rightarrow$ spin-$\frac{1}{2}$ quarks

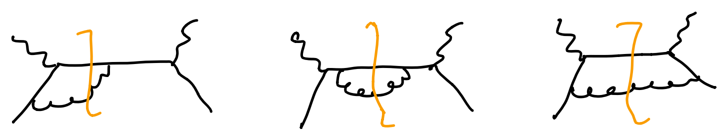
Scaling violation from $\ln Q$ dependence at higher orders (excellent agreement w data)

IR divergences

virtual



real



$$4F_1^V = \frac{\alpha_s C_F}{\pi} Q_f^2 \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left(-\frac{1}{\epsilon^2} - \frac{1}{2\epsilon} + \cdots\right) \delta(1-x)$$

$$4F_1^R = \frac{\alpha_s C_F}{\pi} Q_f^2 \left(\frac{\mu^2}{Q^2}\right)^{\epsilon} \left[-\frac{(1+x^2)}{2\epsilon\,(1-x)^{1+\epsilon}} + \frac{1}{4(1-x)^{1+\epsilon}} + \cdots\right]$$

Treat $(1-x)^{-1-\epsilon}$ as distribution, test fn $g(x)$

$$\int_0^1 dz\, \frac{g(x)}{(1-x)^{1+\epsilon}} = \int dz\, \frac{g(x) - g(1) + g(1)}{(1-x)^{1+\epsilon}} = -\frac{g(1)}{\epsilon} + \int dz\, \frac{g(x) - g(1)}{1-x}$$

$$\therefore \quad \frac{1}{(1-x)^{1+\epsilon}} = -\frac{1}{\epsilon}\, \delta(1-x) + \frac{1}{(1-x)_+} + \cdots$$

Now $1/\epsilon^2$ cancels $\quad -\frac{1}{\epsilon^2} + \frac{1}{\epsilon^2} = 0$

$$\text{sum} = \frac{\alpha_s C_F}{2\pi} Q_f^2 C_F \left[-\frac{1}{\epsilon} P_{qq}(x) - \ln\frac{\mu^2}{Q^2} P_{qq}(x) + \dots \right]$$

where $P_{qq}(x) = \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$ splitting function with distns

$\int_0^1 dx\, P_{qq}(x) = 0 \Rightarrow$ # quarks conserved

- left over $1/\epsilon$ collinear divergence $P_g \,/\!/\, P_{In}$

  which is part of $f_q(\xi)$

$$f_q(\xi,\mu)^{\text{partonic}} = \delta(1-\xi) - \frac{\alpha_s(\mu)}{2\pi\epsilon} P_{qq}(\xi) \qquad \overline{MS} \text{ defn}$$

Then

$$C_1\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right) = \frac{Q_f^2}{2}\left[\delta\left(1-\frac{x}{\xi}\right) - \frac{\alpha_s}{2\pi} \ln\frac{\mu^2}{Q^2} P_{qq}\left(\frac{x}{\xi}\right) + \dots \right]$$

indeed direct calculation from def'n above $\circledast$:

$$f_q(\xi)^{\text{bare}} = \delta(1-\xi) + \frac{\alpha_s}{2\pi}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right) P_{qq}(\xi)$$

UV renormalization gives RGE equation

$$\mu\frac{d}{d\mu} f_j(\xi,\mu) = \int_\xi^1 \frac{d\xi'}{\xi'} P_{jk}\left(\frac{\xi}{\xi'}\right) f_k(\xi',\mu)$$

DGLAP equations

Exercise (next page): Explore PDFs

- dist'n terms in splitting functions.
- action of this RGE

Lectures on Perturbative QCD

Iain Stewart, ICTP Summer School 2017

Problem: Splitting Functions

Infrared enhancements in the quark and gluon branching processes $q \to qg$, $g \to gg$, and $g \to q\bar{q}$ are key ingredient in the formation of jets. The structure of collinear enhancements is described by splitting functions P_{ab}, which to first order in the strong coupling α_s are:

$$\begin{aligned}
P_{qq}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} C_F \Big[\frac{1+x^2}{(1-x)_+} + a_q\, \delta(1-x)\Big] , \\
P_{qg}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} T_R \Big[x^2 + (1-x)^2\Big] , \\
P_{gq}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} C_F \Big[\frac{1+(1-x)^2}{x}\Big] , \\
P_{gg}^{(0)}(x) &= \frac{\alpha_s(\mu)}{2\pi} 2C_A \Big[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\Big] + a_g\, \delta(1-x) .
\end{aligned} \tag{1}$$

Here the color factors are $C_F = 4/3$, $T_R = 1/2$, and $C_A = 3$, and you will determine the constants a_q and a_g below. Each $P_{ab}^{(0)}(x)$ should be thought of as the probability of finding a parton of type a inside an initial parton b, with a having a fraction x of the parent b's momentum. These expressions include the familiar Dirac δ-function, and the less familiar $+$-function. The latter is defined by $1/(1-x)_+ = 1/(1-x)$ for any $x < 1$, and by the fact that the singularity at $x = 1$ is regulated such that

$$\int_0^1 dx \frac{1}{(1-x)_+} g(x) = \int_0^1 dx \frac{1}{(1-x)} \big[g(x) - g(1)\big] \tag{2}$$

for any function $g(x)$.

a) Derive results for the constants a_q and a_g such that quark number is conserved:

$$\int_0^1 dx\, P_{qq}^{(0)}(x) = 0 , \tag{3}$$

and momentum is conserved by the quark and gluon splittings:

$$\int_0^1 dx\, x \left[P_{qq}^{(0)}(x) + P_{gq}^{(0)}\right] = 0 , \qquad \int_0^1 dx\, x \left[P_{gg}^{(0)}(x) + 2n_f P_{qg}^{(0)}\right] = 0 . \tag{4}$$

Here n_f is the number of light quarks. Show that you can rewrite $P_{qq}^{(0)}$ as $P_{qq}^{(0)}(x) = (\alpha_s(\mu) C_F/2\pi) \left[(1+x^2)/(1-x)\right]_+$.

Given an initial distribution of quarks $q(\xi, \mu_0)$ and gluons $g(\xi, \mu_0)$ at a momentum scale μ_0, the distribution of quarks at a scale μ_1 is given by

$$q(x, \mu_1) = q(x, \mu_0) + \int_{\mu_0}^{\mu_1} \frac{2\, d\mu}{\mu} \int_x^1 \frac{d\xi}{\xi} \Big[P_{qq}^{(0)}\Big(\frac{x}{\xi}\Big) q(\xi, \mu) + P_{qg}^{(0)}\Big(\frac{x}{\xi}\Big) g(\xi, \mu)\Big] , \tag{5}$$

where the terms in the integral account for the possibility that the quark we observe came from a splitting rather than the initital distribution.

b) By iterative use of Eq. (5) derive a series in α_s that writes $q(x,\mu_1)$ in terms of terms only involving q's and g's at $\mu = \mu_0$. Draw Feynman diagrams to describe physically what is happening with the various terms in your infinite series.

The subtraction term from the plus function in $P_{qq}^{(0)}$ in Eq. (5) sets $\xi = x$, and is related to evolution to the scale μ_1 without branching, so strictly speaking Eq. (5) does not yet have a clean separation between branching and no-branching. To better distinguish the two possibilites we will rewrite this equation in a different way. To simplify the formulas below, we'll set $P_{qg}^{(0)} = 0$. The probability that a quark does not split when it evolves from μ_0 to μ_1 is then given solely by the quark Sudakov form factor:

$$\Delta_{qq}(\mu_1,\mu_0) \;=\; \exp\Big[-\int_{\mu_0}^{\mu_1} \frac{2\,d\mu}{\mu} \int dx\, \hat{P}_{qq}^{(0)}(x)\Big]\,. \tag{6}$$

Here $\hat{P}_{qq}^{(0)}(x) = (\alpha_s(\mu) C_F/2\pi)\,(1+x^2)/(1-x)$ and we will assume that the limits on the x integration keep us away from the singularity at $x=1$ (more on this in part d).

c) Taking $\mu_1 d/d\mu_1$ derive differential equations for $q(x,\mu_1)$ and $\Delta_{qq}(\mu_1,\mu_0)$. Next derive an equation for $\mu_1 d/d\mu_1(q/\Delta_{qq})$ and show that its solution yields

$$q(x,\mu_1) = \Delta_{qq}(\mu_1,\mu_0)\, q(x,\mu_0) + \int_{\mu_0}^{\mu_1} \frac{2\,d\mu}{\mu} \frac{\Delta_{qq}(\mu_1,\mu_0)}{\Delta_{qq}(\mu,\mu_0)} \int \frac{d\xi}{\xi} \hat{P}_{qq}^{(0)}\Big(\frac{x}{\xi}\Big) q(\xi,\mu)\,. \tag{7}$$

Since this result does not involve the +-function we can interpret the second term as the probability from splitting, and the first term as the probability of having no splitting. Thus the Sudakov form factor in the first term gives the no-splitting probability when we evolve from μ_0 to μ_1. Can you provide an interpretation for the presence of the ratio of Δ_{qq}'s in the second term? This result with its probabilistic interpretation is used in parton shower Monte Carlo programs that describe parton branching and QCD jets.

Next you will calculate the form of the exponent in $\Delta_{qq}(\mu_1,\mu_0)$. The result can be thought of as an infinite series in $\alpha_s(\mu_0)$, but to keep things simple for this calculation we'll freeze $\alpha_s(\mu) = \alpha_s(\mu_0)$ and approximate $P_{qq}^{(0)}(x) \simeq (\alpha_s(\mu_0) C_F/\pi)/(1-x)$ which will allow us to determine the dominant term for $\mu_1 \gg \mu_0$.

d) Lets identify the evolution scale parameter as the parton's virtual mass squared, $\mu^2 = p^2 \equiv t'$, and hence impose the corresponding kinematic limits on the x-integral: $\mu_0^2/\mu^2 < x < 1 - \mu_0^2/\mu^2$ (obtained for particles with large energy and expanding $\mu_0 \ll \mu$). With the approximations above and these limits perform the double integral in Eq. (6), and show that your result involves a $\ln^2(\mu_1/\mu_0)$. This double log is related to the presence in the branching and no-branching probabilities of the soft ($x \to 1$) singularity and the collinear singularity described by the splitting function equations.

$$F_1\left(x, \frac{Q^2}{\Lambda_{QCD}^2}\right) = \sum_j \int_x^1 \frac{d\xi}{\xi}\, C_j\Big(\frac{x}{\xi}, \underbrace{\frac{Q^2}{\mu^2}}_{\substack{\text{wants}\\ \mu \simeq Q}}\Big)\, f_j\Big(\xi, \underbrace{\frac{\mu}{\Lambda_{QCD}}}_{\substack{\text{wants}\\ \mu \sim \Lambda_{QCD}}}\Big)$$

Evolve PDF to appropriate scale:

$$\underset{\mu \simeq Q}{f_j(\xi, \mu)} = \underbrace{\int_\xi^1 \frac{d\xi'}{\xi'}\, U_{jk}\left(\frac{\xi}{\xi'}, \mu, \mu_0\right)} \underset{\mu_0 \simeq \Lambda_{QCD}}{f_k(\xi', \mu_0)}$$

perturbative evolution of PDFs

sums ∞ series of large

$L = \ln\left(\frac{\mu}{\mu_0}\right)$'s: $1 + \alpha_s L + \alpha_s^2 L^2 + \dots$ [like $\alpha_s(\mu)$]

(numerical solution here)

non-pert. boundary condition

$j \neq k$ PDF mixing

P_{gq} $\qquad$ P_{qg}

Note: μ dependence cancels order-by-order in expansion between $C_j\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right)$ & $f_j(\xi, \mu)$

Often use residual μ dependence to estimate higher order terms: $\mu = \frac{Q}{2}, Q, 2Q$

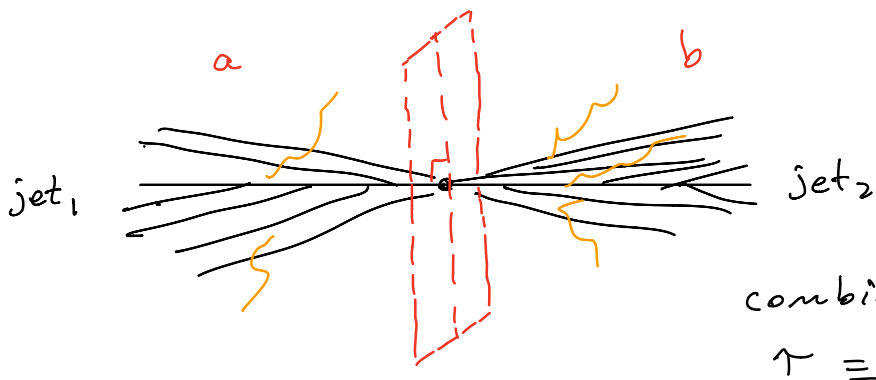
$\Rightarrow$ perturbative theory uncertainty

Same story for pp collisions:

$$\sigma = \int dx_a\, dx_b\, \underbrace{f_g(x_a, \mu)\, f_g(x_b, \mu)}_{\substack{\text{evolve from}\\ \mu_0 \sim \Lambda_{QCD} \text{ to}\\ \mu \simeq m_H}}\; \hat{\sigma}_{gg \to H+X}(x_a, x_b, \underbrace{\mu, m_H}_{\mu \simeq m_H})$$

$e^+e^- \to 2$-jets $\qquad e^+e^- \to \gamma^*(q) \to q\bar{q}$

- factorization theorems can also be derived for processes involving jets



measure hemisphere masses

$$M_a^2 = \Big(\sum_{i\in a} p_i^\mu\Big)^2$$

$$M_b^2 = \cdots$$

combine

$$\tau \equiv \frac{M_a^2 + M_b^2}{Q^2}$$

[related to "thrust"]

demanding $\tau \ll 1$ ensures 2-jets "event shape"

$\boxed{\text{collinear}}$ radiation with $p^0 \sim Q$ & $p_\perp \sim Q\sqrt{\tau}$

contributes $\to$ Jet Functions $\quad \sim p_\perp^2 \sim [Q\sqrt{\tau}]^2$

$\boxed{\text{soft}}$ radiation with $k^\mu \sim Q\tau$ contributes

$\to$ soft function $\quad (p+k)^2 \sim 2p\cdot k \sim (Q)(Q\tau)$

$$M^2 \simeq (p+k)^2 = p^2 + 2p\cdot k + \cancel{\mathcal{O}(k^2)} = s + Qk^+ \Rightarrow Q^2\tau \simeq s + s' + Qk$$

$$\frac{d\sigma}{d\tau} = \sigma_0\, H(Q,\mu) \int ds\, ds'\, \underbrace{J(s,\mu)\, J(s',\mu)}_{\text{jet functions}}\, \underbrace{S\Big(Q\tau - \frac{s+s'}{Q}, \mu\Big)}_{\text{soft fn}}$$

hard fn.
virtual corrections

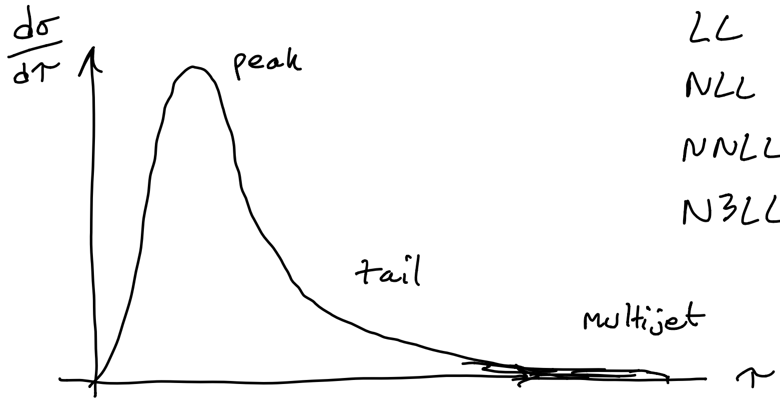
$$\mu^2 \sim Q^2 \quad \gg \quad \mu^2 \sim Q^2\tau \quad \gg \quad \mu^2 \sim Q^2\tau^2$$

renormalization group evolution in μ

sums $\alpha_s \ln^2\tau$ factors

RGE for $H(Q,\mu)$: solution is Sudakov Form Factor

$\longrightarrow$ exercise in EFTx course, chapter 13



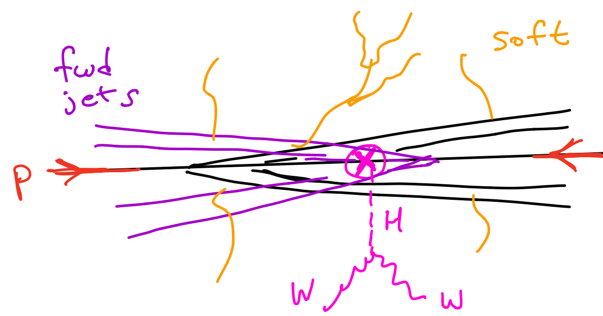
LL
NLL
NNLL
N3LL + $\mathcal{O}(\alpha_s^3)$ known

(derived using SCET)

1% precision

$\Rightarrow$ extract $\alpha_s(m_Z)$

$pp \to H + 0$-jets



anti-k_T with R

not jets with $p_T > p_T^{cut}$

$$\sigma(p_T^{cut}) = \sigma_0 \, H_{gg}(m_t, m_H, \mu) \int dY \; B_g(m_H, p_T^{cut}, R, x_a, \mu, \nu)$$

$$* \; B_g(m_H, p_T^{cut}, R, x_b, \mu, \nu) \; S_{gg}(p_T^{cut}, R, \mu, \nu)$$

extra rapidity scale parameter

$$x_{a,b} = \frac{m_H}{E_{cm}} e^{\pm Y}$$

with

$$B_g(m_H, p_T^{cut}, R, x, \mu, \nu) = \sum_j \int_x^1 \frac{d\xi}{\xi} \, \mathcal{I}_{gj}\left(m_H, p_T^{cut}, R, \frac{x}{\xi}, \mu, \nu\right) f_j(\xi, \mu)$$

usual PDFs

all other functions perturbative

Sum $\alpha_s \ln^2\left(\frac{p_T^{cut}}{m_H}\right)$ to higher orders

NNLL gives $\sim 7\%$ precision

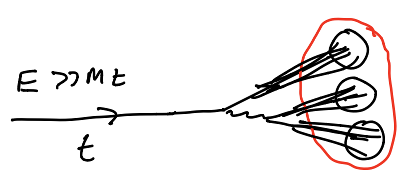
Active Areas in Collider Physics

- loop calculations, connection to Amplitudes, spinor/helicity techniques
- loops + legs, combining so that $1/\epsilon$'s cancel
  phase space slicing or subtractions
- Improving Parton Shower Monte Carlo
- Global Fits for determining PDFs
- Factorization (new formulas, new universal functions, factorization violation & MPI)
- Resummation, higher orders/precision, new types of logs (eg. log R) & multiple variables
- Jet Substructure

* boosted particles that decay hadronically can be identified by substructure

2 prong substructure

find new observables



3 prong substructure

* also techniques to "groom" jets, remove soft contamination inside jets to better probe the hard mother particle

References for Further Reading

- Effective Field Theory, including Soft-Collinear EFT for collider physics, see EFT_x course:

  http://www2.lns.mit.edu/~iains/registerEFTx

  (video lectures, SCET review notes, online problems)

- QCD Concepts (Renormalization Group, β - function, Fadeev-Popov, ...)

  http://www2.lns.mit.edu/~iains/talks/QFT3_Lectures_Stewart_2012.pdf

- Collider Physics: "QCD and Collider Physics" Book by Ellis, Stirling, and Webber

- Parton Shower Review, Buckley et al:

  https://arXiv.org/abs/1101.2599

- Review on Jets by Gavin Salam:

  https://arxiv.org/abs/0906.1833