Broken symmetries and lattice gauge theory (III): spontaneous symmetry breaking and the Banks–Casher relation

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Light pseudoscalar meson spectrum

- Octet compatible with SSB pattern

$$\mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}} \rightarrow \mathrm{SU}(3)_{\mathrm{L+R}}$$

and soft explicit symmetry breaking

$$m_u, m_d \ll m_s < \Lambda$$

- $m_u, m_d \ll m_s \Longrightarrow m_\pi \ll m_{\mathrm{K}}$

- A 9^{th} pseudoscalar with $m_{\eta'} \sim \mathcal{O}(\Lambda)$

I	I_3	S	Meson	Quark Content	Mass (GeV)
1	1	0	π^+	$u\bar{d}$	0.140
1	-1	0	π^-	$d\bar{u}$	0.140
1	0	0	π^0	$(d\bar{d} - u\bar{u})/\sqrt{2}$	0.135
$\frac{1}{2}$	$\frac{1}{2}$	+1	K^+	$u\bar{s}$	0.494
$\frac{1}{2}$	$-\frac{1}{2}$	+1	K^0	$d\bar{s}$	0.498
$\frac{1}{2}$	$-\frac{1}{2}$	-1	K^-	$s\bar{u}$	0.494
$\frac{1}{2}$	$\frac{1}{2}$	-1	$\overline{\mathrm{K}}^0$	$s\bar{d}$	0.498
0	0	0	η	$\cos\vartheta\eta_8 - \sin\vartheta\eta_0$	0.548
0	0	0	η'	$\sin\vartheta\eta_8 + \cos\vartheta\eta_0$	0.958

$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_0 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$
$$\vartheta \sim -10^\circ$$

QCD action and its (broken) symmetries

- QCD action for $N_F = 2$, $M^\dagger = M =$ diag(m, m)

$$S = S_G + \int d^4x \left\{ \bar\psi D \psi + \bar\psi_{\rm R} M^\dagger \psi_{\rm L} + \bar\psi_{\rm L} M \psi_{\rm R} \right\} , \qquad D = \gamma_\mu(\partial_\mu - i A_\mu)$$

- For $M = 0$ chiral symmetry

$$\psi_{\rm R,L} \to V_{\rm R,L} \psi_{\rm R,L} \quad \psi_{\rm R,L} = \left(\frac{1 \pm \gamma_5}{2}\right) \psi$$

Chiral anomaly: measure not invariant

SSB: vacuum not symmetric

$$\mathrm{SU(3)}_c \times \mathrm{SU(2)}_{\rm L} \times \mathrm{SU(2)}_{\rm R} \times \mathrm{U(1)}_{\rm L} \times \mathrm{U(1)}_{\rm R} \times \mathcal{R}_{\rm scale}$$

$\downarrow$ **(dim. transm., chiral anomaly)**

$$\mathrm{SU(3)}_c \times \mathrm{SU(2)}_{\rm L} \times \mathrm{SU(2)}_{\rm R} \times \mathrm{U(1)}_{\rm B=L+R}$$

$\downarrow$ **(Spont. Sym. Break.)**

$$\mathrm{SU(3)}_c \times \mathrm{SU(2)}_{\rm L+R} \times \mathrm{U(1)}_{\rm B}$$

$\downarrow$ (Confinement)

$$\mathrm{SU(2)}_{\rm L+R} \times \mathrm{U(1)}_{\rm B}$$

- Breaking due to non-perturbative dynamics. Precise quantitative tests are being made on the lattice

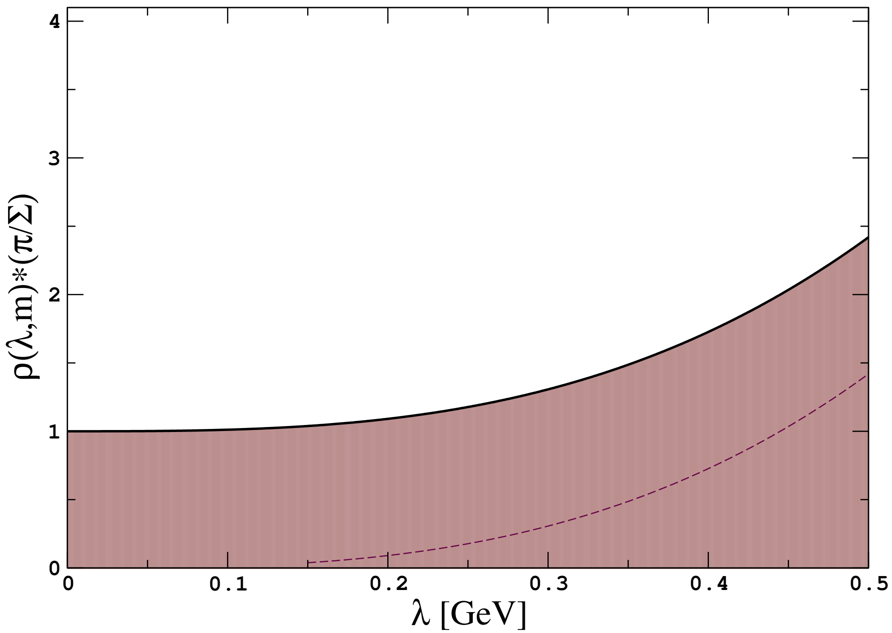
Banks–Casher relation [Banks, Casher 80]

- The spectral density of D is

[Banks, Casher 80; Leutwyler, Smilga 92; Shuryak, Verbaarschot 93]

$$\rho(\lambda, m) = \frac{1}{V} \sum_k \langle \delta(\lambda - \lambda_k) \rangle$$

where $\langle \dots \rangle$ indicates path-integral average



- The Banks–Casher relation

$$\lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m) = \frac{\Sigma}{\pi}$$

can be read in both directions: a non-zero spectral density implies that the symmetry is broken with a non-vanishing Σ and vice versa.

To be compared, for instance, with the free case $\rho(\lambda) \propto |\lambda^3|$

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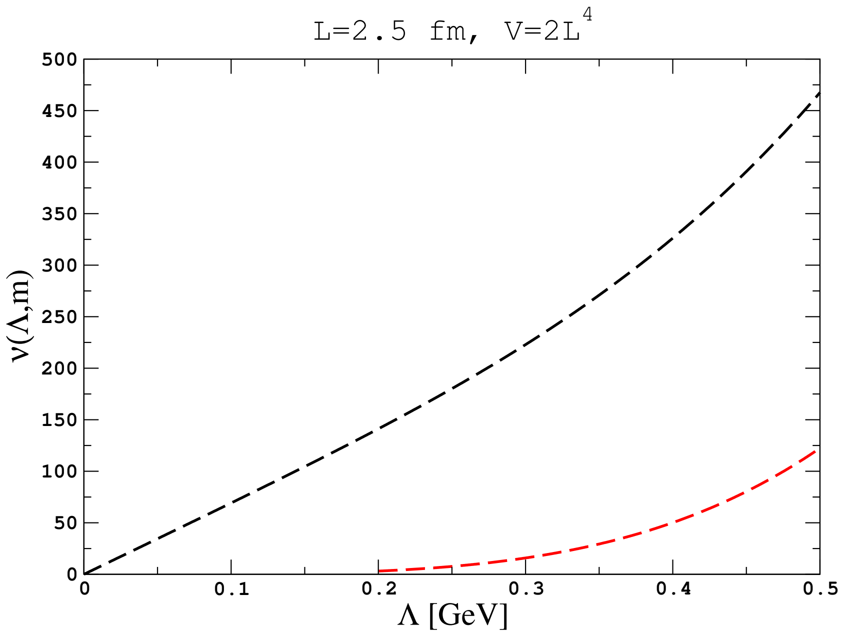


- The number of modes in a given energy interval

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda \, \rho(\lambda, m) \qquad \nu(\Lambda, m) = \frac{2}{\pi} \Lambda \Sigma V + \dots$$

grows linearly with Λ, and they condense near the origin with values $\propto 1/V$

In the free case $\nu(\Lambda, m) \propto V \Lambda^4$

Numerical computation for $N_{\rm f} = 2$ [Cichy et al. 13; Engel et al. 14]

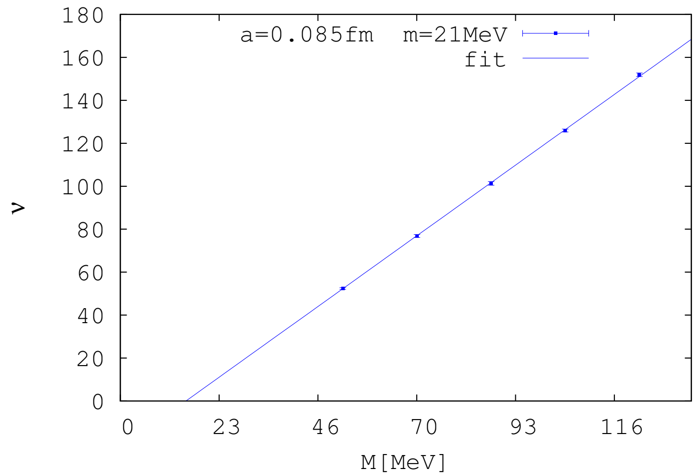
- Twisted-mass QCD [Cichy et al. 13]:

  $*\ a = 0.054$–0.085 fm

  $*\ m = 16$–47 MeV

  $*\ M = 50$–120 MeV

  $*\ M = \sqrt{\Lambda^2 + m^2}$



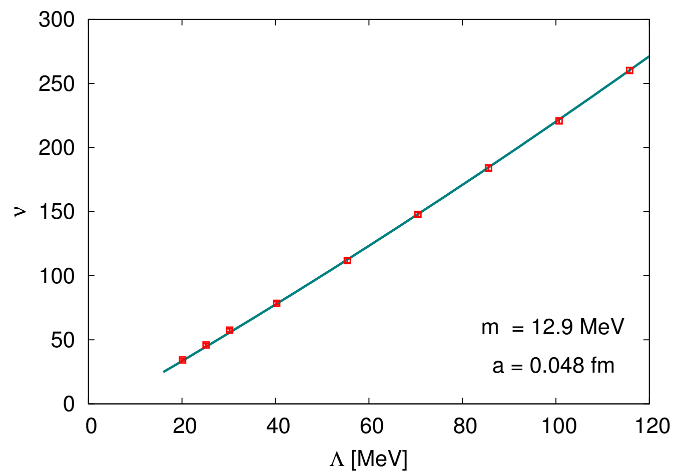
- $O(a)$–improved Wilson fermions [Engel et al. 14]:

  $*\ a = 0.048$–0.075 fm

  $*\ m = 6$–37 MeV

  $*\ \Lambda = 20$–500 MeV

  $*\ \nu = -9.0(13){+}2.07(7)\Lambda{+}0.0022(4)\Lambda^2$



- The mode number is a nearly linear function in Λ up to approximatively 100 MeV. The modes do condense near the origin as predicted by the Banks–Casher mechanism

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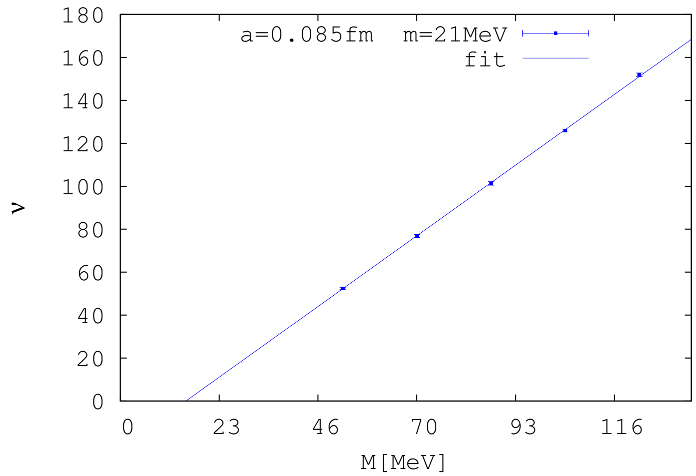

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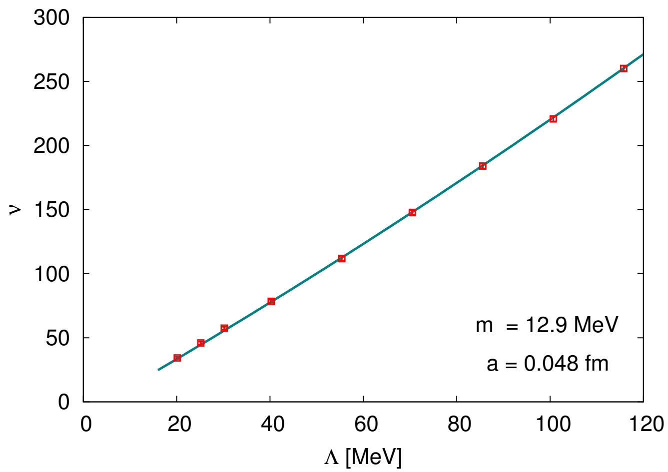


- At fixed lattice spacing and at the percent precision, however, data show statistically significant deviations from the linear behaviour of $O(10\%)$.

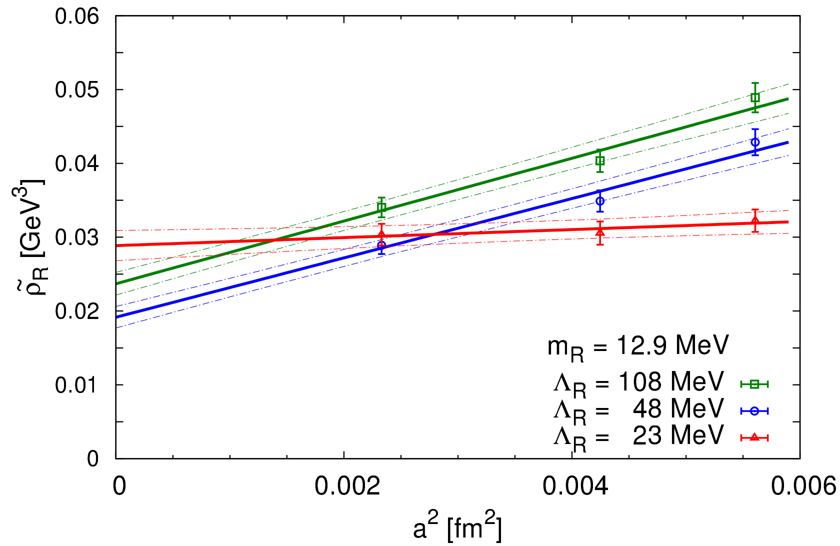
Continuum limit [Engel et al. 14]

- By defining

$$\widetilde{\rho}(\Lambda_1, \Lambda_2, m) = \frac{\pi}{2V} \frac{\nu(\Lambda_2) - \nu(\Lambda_1)}{\Lambda_2 - \Lambda_1}$$

the continuum limit is taken at fixed m, Λ_1 and Λ_2 [$\Lambda = (\Lambda_1 + \Lambda_2)/2$]

- Data are extrapolated linearly in a^2 as dictated by the Symanzik analysis



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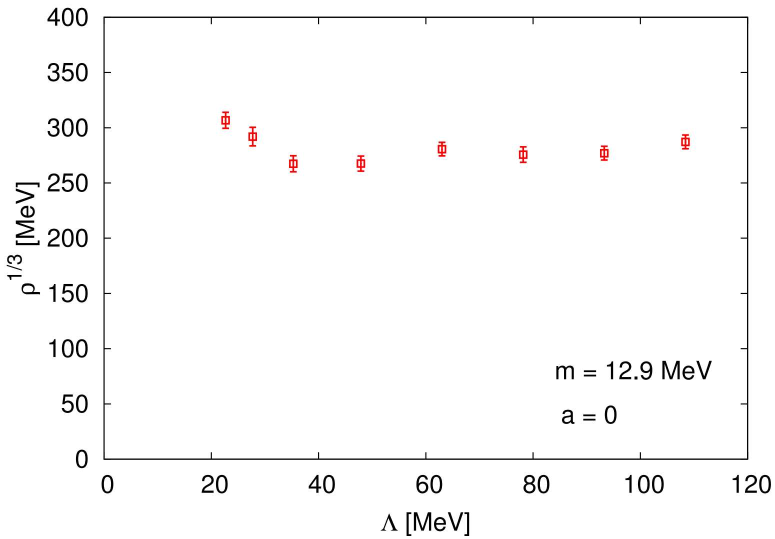

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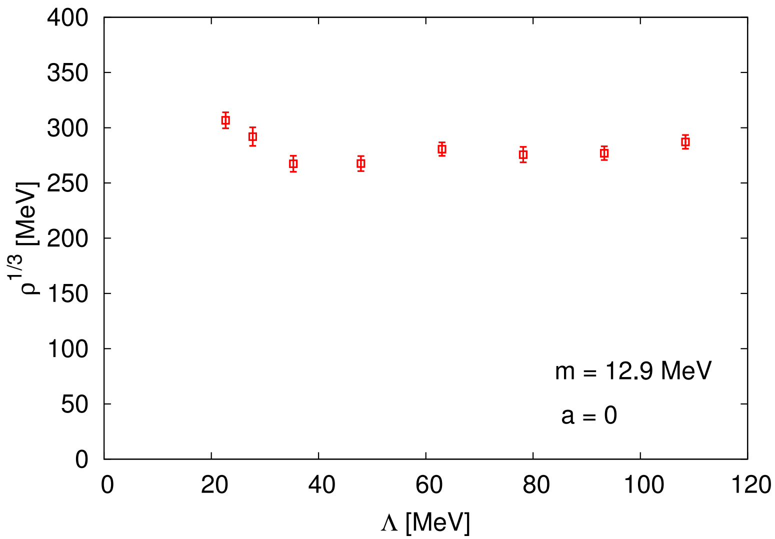


- It is noteworthy that no assumption on the presence of SSB was needed so far

- The results show that at small quark masses the spectral density is non-zero and (almost) constant in Λ near the origin

- Data are consistent with the expectations from the Banks–Casher mechanism in the presence of SSB. In this case NLO ChPT indeed predicts ($N_{\rm f} = 2$)

$$\widetilde{\rho}^{\,\rm nlo} = \Sigma\Big\{1 + \frac{m\Sigma}{(4\pi)^2F^4}\Big[3\,\bar{l}_6 + 1 - \ln(2) - 3\ln\Big(\frac{\Sigma m}{F^2\bar{\mu}^2}\Big) + \tilde{g}_\nu\Big(\frac{\Lambda_1}{m},\frac{\Lambda_2}{m}\Big)\Big]\Big\}$$

Spectral density in ChPT [Osborn et al. 99; LG, Lüscher 09; Damgaard, Fukaya 09; Damgaard et al. 10; Necco, Shindler 11]

- When chiral symmetry is spontaneously broken, the spectral density can be computed in ChPT. At the NLO

$$\rho^{\rm nlo}(\lambda, m) = \frac{\Sigma}{\pi}\Big\{1 + \frac{m\Sigma}{(4\pi)^2 F^4}\Big[3\,\bar{l}_6 + 1 - \ln(2) - 3\ln\Big(\frac{\Sigma m}{F^2\bar{\mu}^2}\Big) + g_\nu\Big(\frac{\lambda}{m}\Big)\Big]\Big\}$$

where $g_\nu(x)$ is a parameter-free known function

- The NLO formula has properties which can be confronted against the NP results:

  * at fixed λ no chiral logs are present when $m \to 0$

$$g_\nu(x) \xrightarrow{x\to\infty} -3\ln(x)$$

  * in the chiral limit $\rho^{\rm nlo}(\lambda, m)$ becomes independent of λ
    This is an accident of the $N_{\rm f} = 2$ ChPT theory at NLO [Smilga, Stern 93]

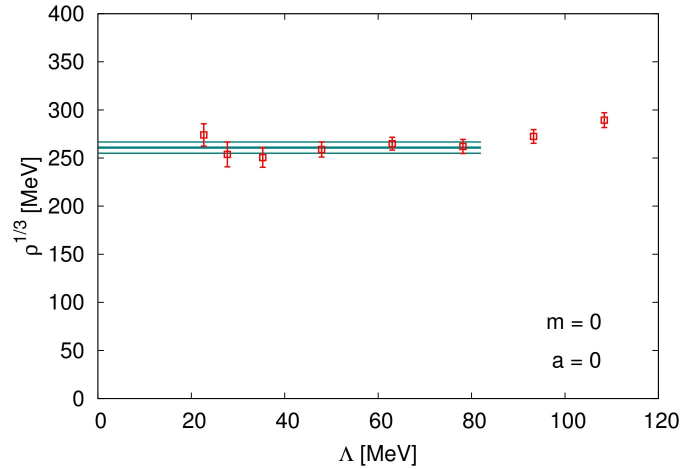
  * the λ dependence of $\rho^{\rm nlo}(\lambda, m)$ is a known function (up to overall constant).
    The spectral density is a slowly decreasing function of λ at fixed m

Chiral limit [Engel et al. 14]

- In the chiral limit NLO ChPT predicts $\widetilde{\rho}$ to be Λ-independent. By extrapolating to $m = 0$

$$[\widetilde{\rho}^{\,\overline{\rm MS}}]^{1/3} = [\Sigma^{\overline{\rm MS}}_{\rm BK}(2\,\text{GeV})]^{1/3} = 261(6)(8)\,\text{MeV}$$

where the spacing is fixed by introducing a quenched strange quark with $F_K = 109.6$ MeV



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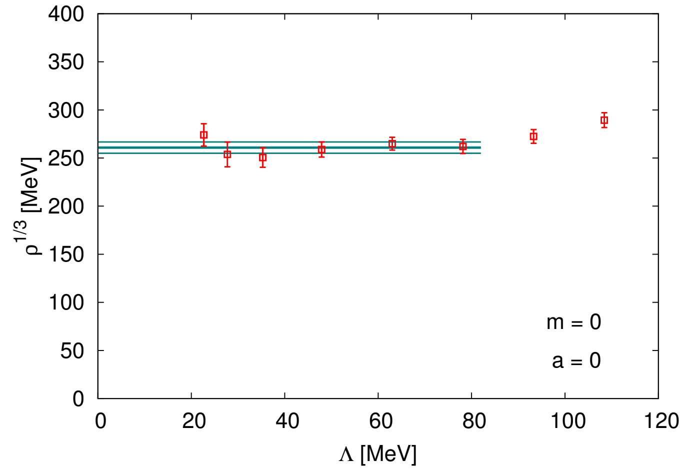
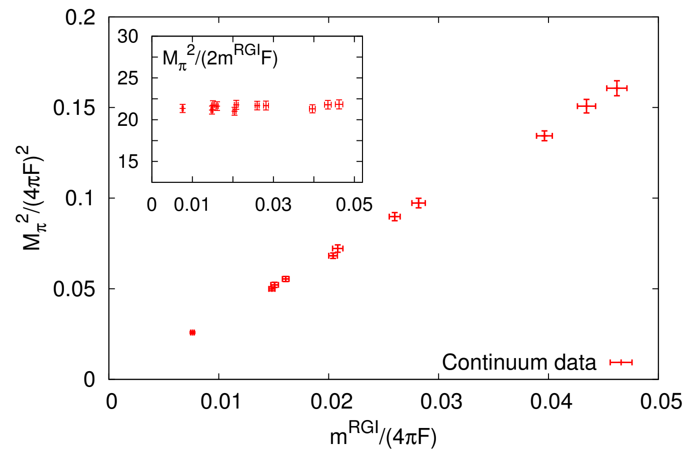


- The distinctive signature of SSB is the agreement between $\widetilde{\rho}$ and the slope of $M_\pi^2 F_\pi^2/2$ with respect to m in the chiral limit

- On the same set of configurations by fitting the data with NLO (W)ChPT for $M_\pi < 400$ MeV

$$[\Sigma^{\overline{\rm MS}}_{\rm GMOR}(2\,\text{GeV})]^{1/3} = 263(3)(4)\,\text{MeV}$$

to be compared with the previous result



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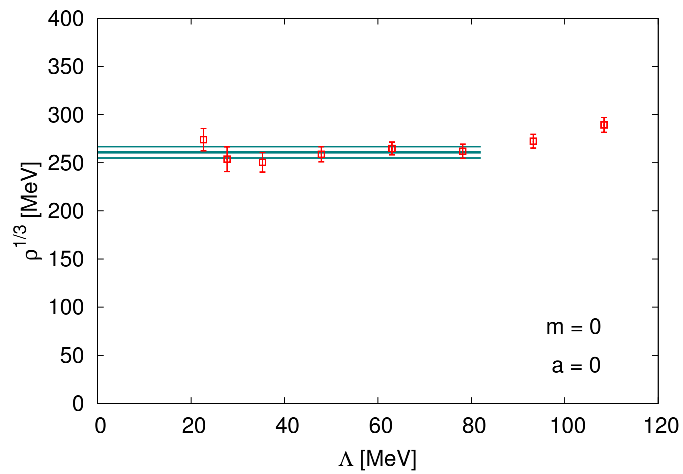

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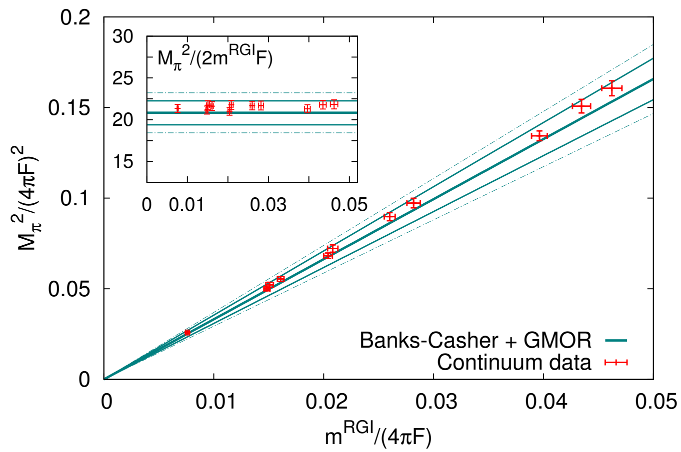
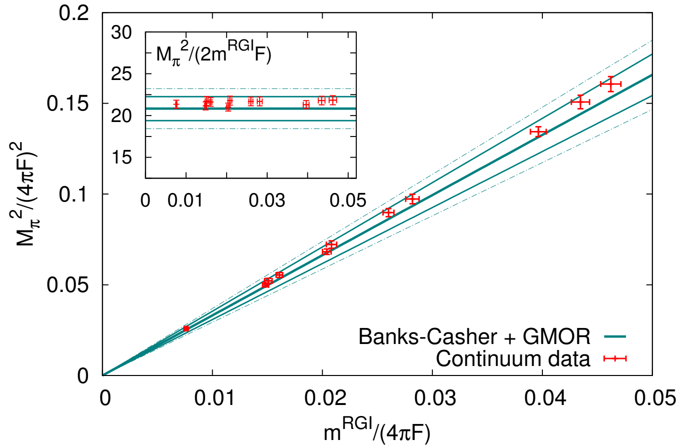


Gell-Mann–Oakes–Renner relation

- The spectral density of the Dirac operator in the continuum is $\neq 0$ at the origin for $m = 0$



- The low-modes of the Dirac operator do condense following Banks–Casher mechanism

- The rate of condensation agrees with the GMOR relation, and it explains the bulk of the pion mass up to $M_\pi \leq 500$ MeV

- The dimensionless ratios

$$[\Sigma^{\rm RGI}]^{1/3}/F = 2.77(2)(4)\ , \qquad \Lambda^{\overline{\rm MS}}/F = 3.6(2)$$

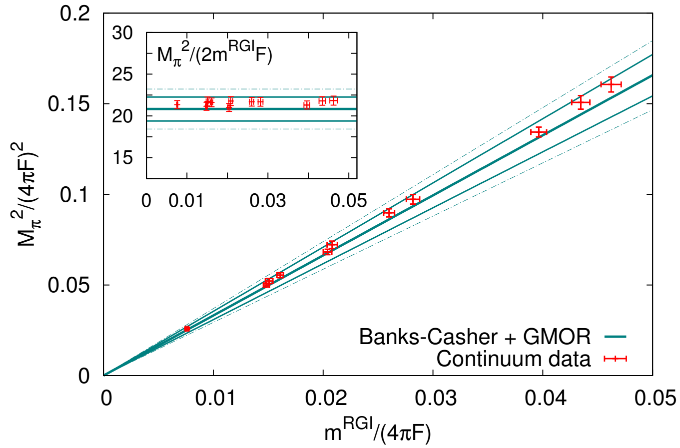
are "geometrical" properties of the theory. They belong to the category of unambiguous quantities in the two flavour theory that should be used for quoting and comparing results rather than those expressed in physical units

- They can be directly compared with your preferred approximation/model

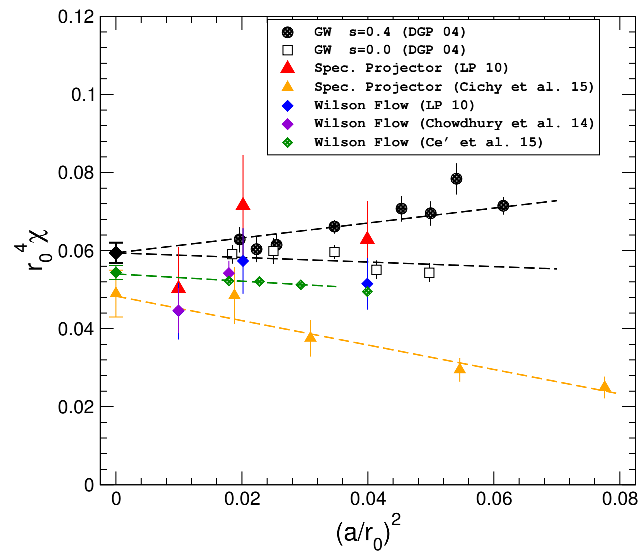
Summary

- An impressive global (lattice) community effort to reach a precise quantitative understanding of the behaviour of QCD in the chiral regime from first principles

- The spectral density of the Dirac operator in the continuum and chiral limits is $\neq 0$ at the origin. The rate of condensation explains the bulk of the pion mass up to $M_\pi \leq 500$ MeV



- All numerical results for $\chi^{\rm YM}$ are consistent with the conceptual progress made over the last decade. A percent precision reached. Universality is at work if χ is (properly) defined on the lattice!



- The (leading) QCD anomalous contribution to $M^2_{\eta'}$ supports the Witten–Veneziano explanation for its large experimental value