Topological Phases of Matter with Ultracold Atoms and Photons

Hannah Price

Currently*: INO-CNR BEC Center & University of Trento, Italy*

From October*: University of Birmingham, UK*

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Overview

Lectures 1 & 2

Introduction to Topological Phases of Matter

Lecture 3

Topological Phases of Matter with Ultracold Atoms

Lecture 4

Topological Phases of Matter with Photons

Overview

Lectures 1 & 2

Introduction to Topological Phases of Matter

Some good reviews and lecture notes:

Prange and Girvin, “The Quantum Hall Effect” (Springer, 1990).

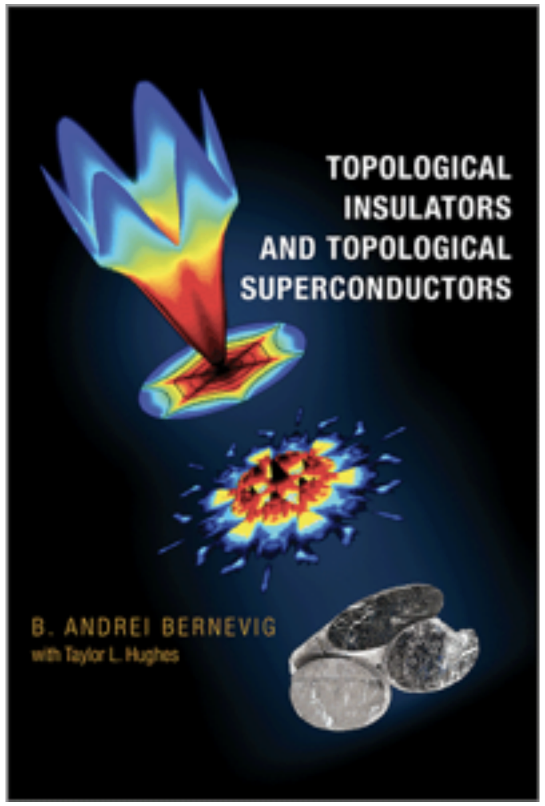
Hasan et al., Rev. Mod. Phys. 82, 3045 (2010).

Xiao et al, RMP, 82, 1959 (2010)

Qi et al, Rev. Mod. Phys. 83, 1057 (2011).

Chiu, et al., Rev. Mod. Phys. 88, 035005, (2016)

Bernevig with Hughes, “Topological Insulators and Topological Superconductors” (Princeton University Press (2013)).



Lecture notes on Quantum Hall: Tong, arXiv:1606.06687

Online course on topological physics: https://topocondmat.org/

Topology & Mathematics

In mathematics, topology is used to classify different surfaces




No holes: genus=0 1 hole: genus=1

The genus is an example of a "topological invariant"

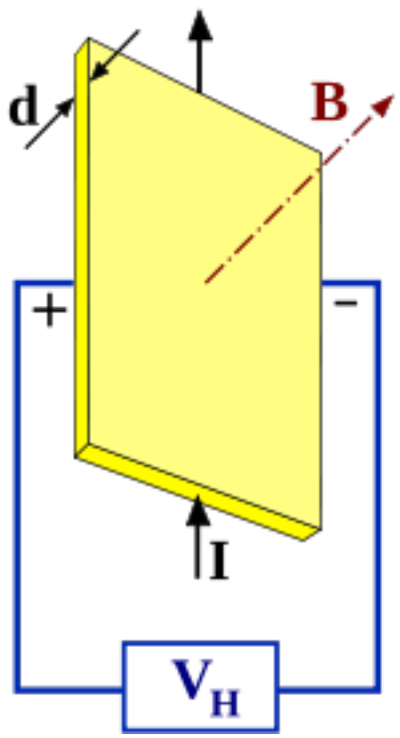
Key features of topological invariants:

- Global property
- Integer-valued
- Robust under smooth deformations

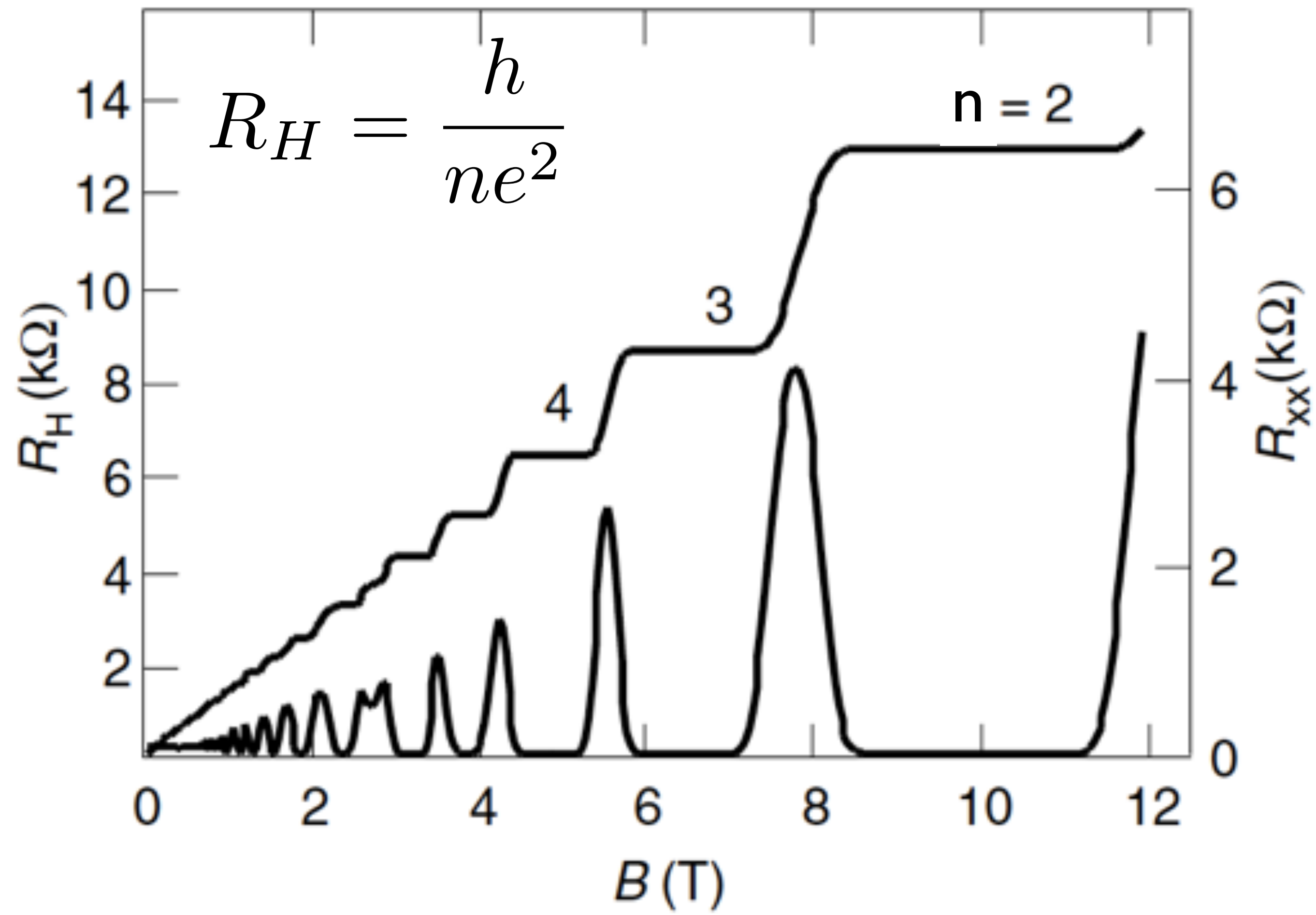
Topology & Phases of Matter

Discovery of the quantum Hall effect:

Cold, quasi-2D sample



$$V_H = R_H I$$



Very precisely-quantized response that is remarkably robust to disorder

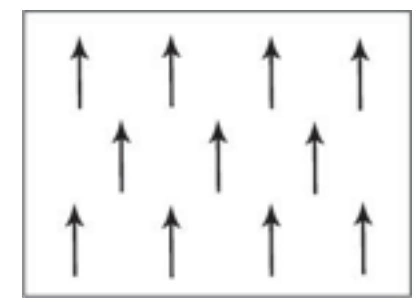
Why?

Because this is a **topological phase of matter**

Topology & Phases of Matter

Most phases of matter:

- Classified according to spontaneously broken symmetries, e.g.:
    - translational symmetry for solids
    - rotational symmetry for magnets…
- Characterised by *local* order parameter



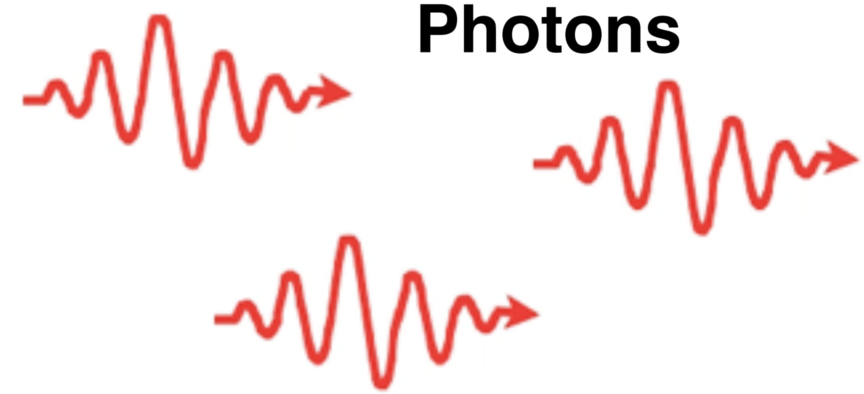
Topological phases of matter:

- Cannot be understood in terms of spontaneous symmetry breaking
- Characterised by *global* topological integer invariant

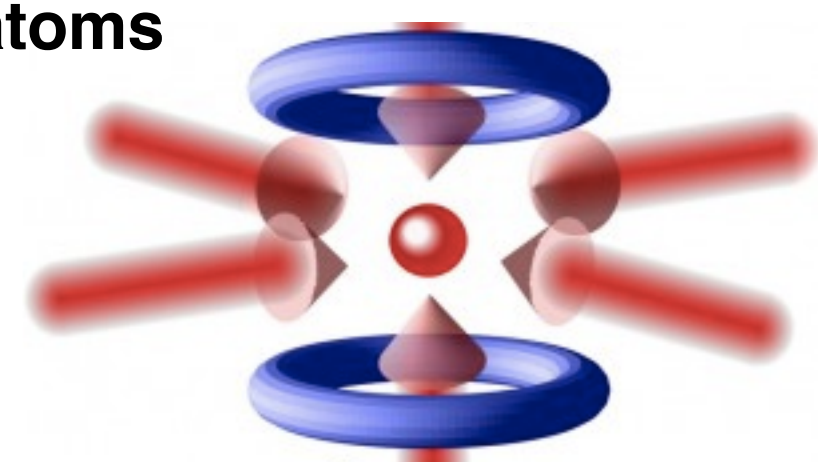
Topology & Phases of Matter

Not just electrons in solid-state systems, also

Photons



Cold atoms



- Greater controllability and tuneability
- Explore new topological phases of matter
- Different observables — learn more!

Fits into this summer school as:

- Very interdisciplinary area
- Fundamental quantum physics
- Quantum simulation
- Future quantum technologies?

ICTP
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Topological band theory

What is “topological” in a topological phase of matter?

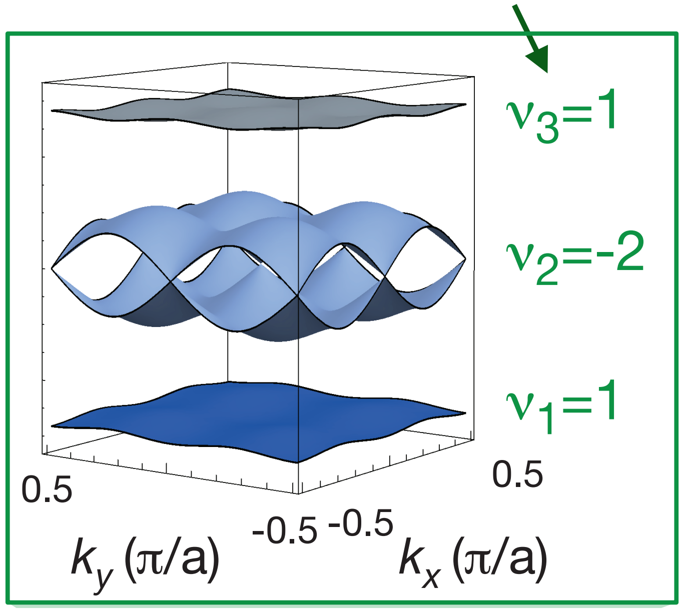
The wave-function

Figure from:
Aidelsburger et al.,
Nat. Phys. 11, 162,
(2015)

Topological invariants
(e.g. Chern numbers)

$\vdots$

$\mathcal{E}_1(\mathbf{k})$



We’ll focus on how single-particle energy bands are characterised by topological invariants

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,k}(\mathbf{r})$$

$$\hat{H}_{\mathbf{k}} u_{n,\mathbf{k}} = \mathcal{E}_n(\mathbf{k}) u_{n,\mathbf{k}}$$

Useful for systems well-described by **independent particles**:

- Integer quantum Hall effect, topological insulators…

or independent quasi-particles:

- topological superconductors and superfluids

Not useful for systems where **strong correlations** are important:

- Fractional quantum Hall effect…

Topological equivalence

Two bands are topologically-equivalent if one can be adiabatically-deformed into the other without closing the energy gap

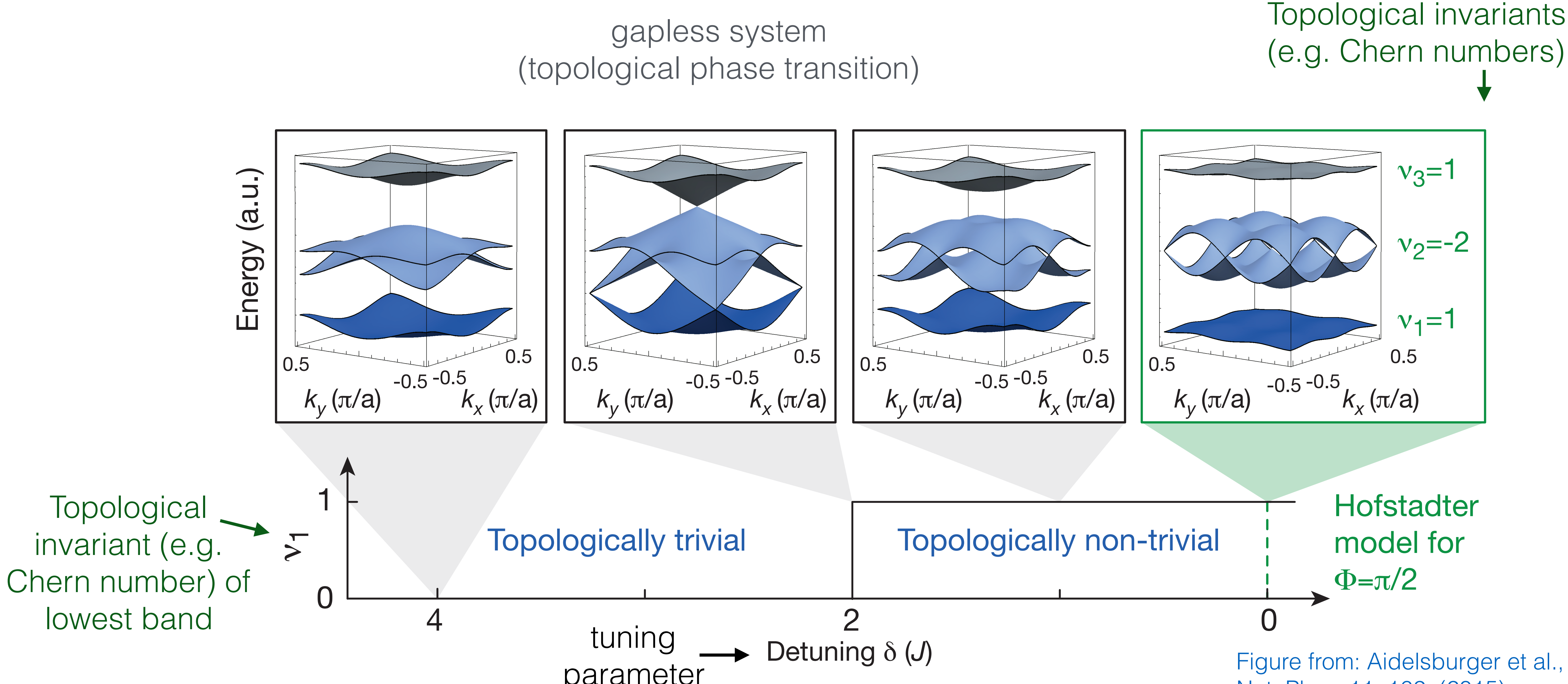


Figure from: Aidelsburger et al., Nat. Phys. 11, 162, (2015)

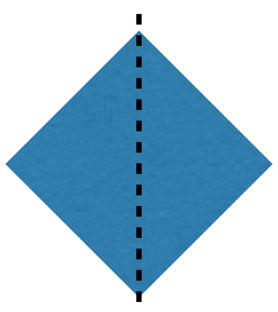
Bulk-boundary correspondence:
must be gapless modes at the boundary between two different topological phases

Symmetry & topological band theory

Symmetry is, as ever, a guiding physical principle!

Energy bands can be classified by *different types of topological invariants* depending on the symmetries and dimensionality of the system

Spatial symmetries



Reflection

Rotation…

Non-spatial symmetries

Time-reversal symmetry

$$\mathcal{T} H(k) \mathcal{T}^{-1} = H(-k)$$

Particle-hole symmetry

$$\mathcal{P} H(k) \mathcal{P}^{-1} = -H(-k)$$

Chiral symmetry

$$\mathcal{C} = \mathcal{T}\mathcal{P} \quad \mathcal{C} H(k) \mathcal{C}^{-1} = -H(k)$$

Find out more:

Kitaev, arXiv:0901.2686
Ryu et al., New J. Phys. 12, 065010 (2010)
Chiu, et al., RMP 88, 035005, (2016)

Symmetry & topological band theory

"Periodic table" of gapped phases of quadratic fermionic Hamiltonians with only non-spatial discrete symmetries

Class	Symmetry: Time-reversal	Particle-hole	Chiral	Dimensionality d: 1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Result of squaring the symmetry operator (0=symmetry is broken)

Possible values of topological invariant:

0 : always trivial
$\mathbb{Z}$: an integer
$\mathbb{Z}_2$: 0,1

N.B. Also topological classifications of:

(i) defects,

(ii) spatial symmetries,

(iii) gapless systems…

Find out more:

Kitaev, arXiv:0901.2686
Ryu et al., New J. Phys. 12, 065010 (2010)
Chiu, et al., RMP 88, 035005, (2016)

Symmetry & topological band theory

However, the presence/absence of certain symmetries is not a sufficient condition for nontrivial topology

$\boxed{0,}\,1, 2, 3... \in \mathbb{Z}$ $\boxed{0,}\,1 \in \mathbb{Z}_2$

N.B. Role of symmetry is very different to in spontaneous symmetry-breaking

So we still have to explicitly calculate the invariant to see if a system is nontrivial

Why are symmetries important? Tells us where to look &…

Topological robustness

Within a given symmetry class, topological invariants are typically very robust against small perturbations…

…but if the perturbation **breaks the symmetries**, the class changes and topological properties are no longer necessarily well-protected

These lectures

In these first two lectures, I will briefly introduce:

Class	Symmetry: Time-reversal	Particle-hole	Chiral	Dimensionality d: 1	2	3	4	5	6	7	8	
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	← **Quantum Hall**
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	← SSH Model
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	← Topological Superconductors
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	← Topological Insulators
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	

2D Quantum Hall Effect

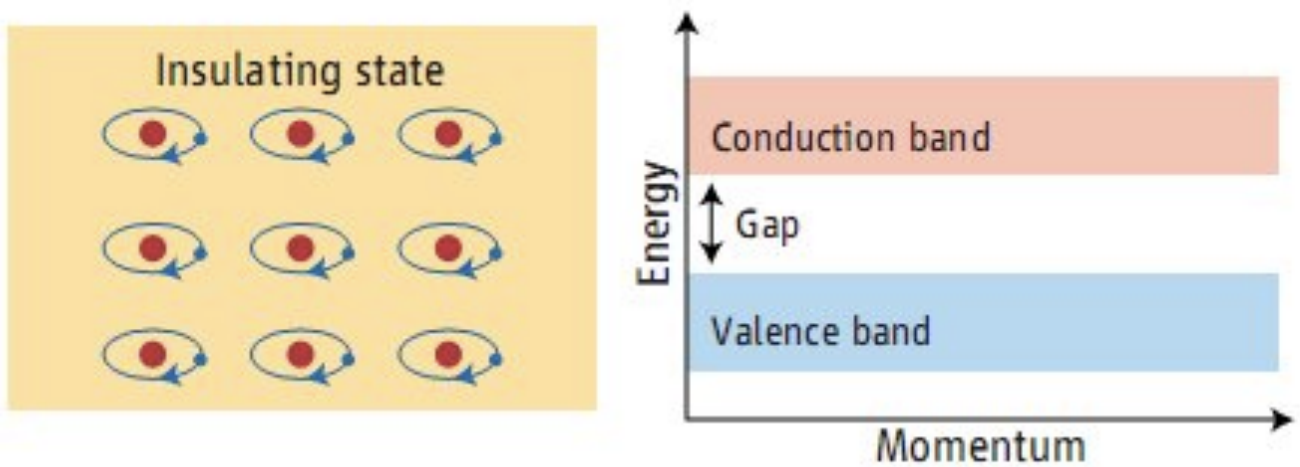
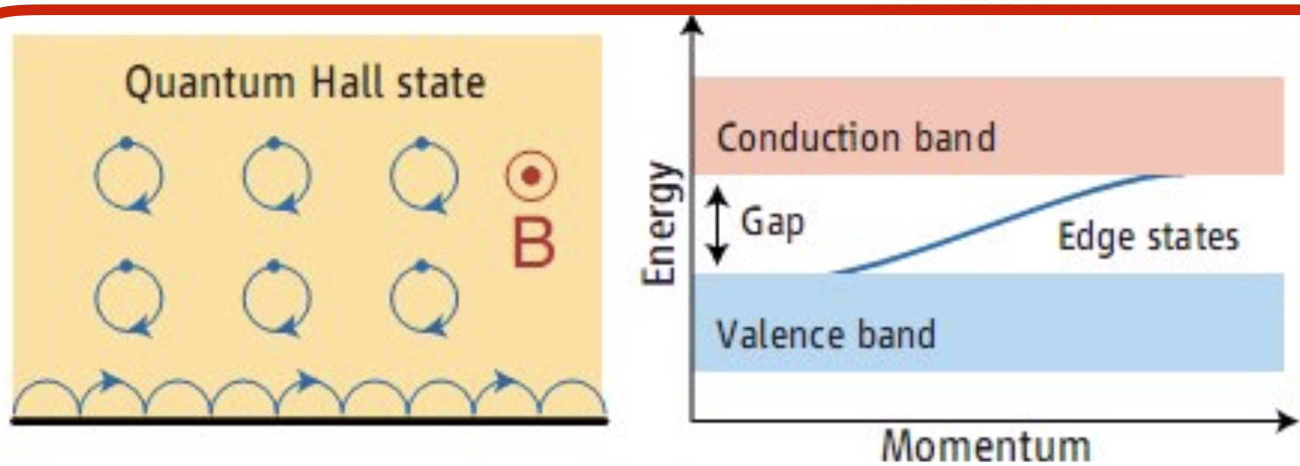


Figure from
C. L. Kane & E. J. Mele, Science 314, 5806, 1692 (2006)

Bands are topologically-trivial

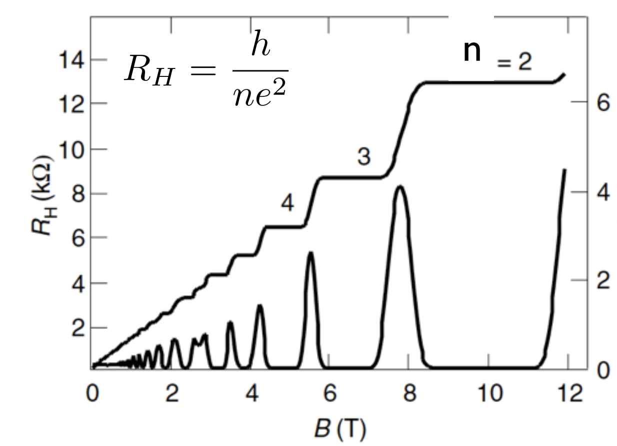


Bands have non-zero topological 1st Chern numbers ν_1^n

$$\sigma_{xy} = -\frac{e^2}{h} \sum_{n \in \text{occupied}} \nu_n$$

quantised Hall response of occupied bands

- Very robust as band invariants can only change if gap closes
- Bulk-boundary correspondence: one-way chiral edge states



Berry phase

M. V. Berry, Proc. R. Soc. A 392, 45 (1984)

Let's consider a general Hamiltonian with a set of parameters: $\mathbf{R} = (R_1, R_2..)$

$$H(\mathbf{R})|n(\mathbf{R})\rangle = E_n(\mathbf{R})|n(\mathbf{R})\rangle,$$

To keep it simple, we consider normalised non-degenerate eigenstates

A pure state $|n(\mathbf{R}(0))\rangle$ evolves under **adiabatic** variation of parameters as:

$$|\psi_n(t)\rangle = e^{-i\theta(t)}|n(\mathbf{R}(t))\rangle$$

Plug into the Schrodinger equation: $i\hbar\frac{\partial}{\partial t}|\psi_n(t)\rangle = H(\mathbf{R}(t))|\psi_n(t)\rangle,$

$$\hbar\frac{\partial\theta(t)}{\partial t}|n(\mathbf{R}(t))\rangle + i\hbar\frac{\partial}{\partial t}|n(\mathbf{R}(t))\rangle = E_n(\mathbf{R}(t))|n(\mathbf{R}(t))\rangle.$$

From the orthogonality and normalisation of eigenstates:

$$\hbar\frac{\partial\theta(t)}{\partial t} = E_n(\mathbf{R}(t)) - i\hbar\langle n(\mathbf{R}(t))|\frac{\partial}{\partial t}|n(\mathbf{R}(t))\rangle$$

$$\longrightarrow \quad \theta(t) = \frac{1}{\hbar}\int_0^t E_n(\mathbf{R}(t'))\mathrm{d}t' - i\int_0^t \langle n(\mathbf{R}(t'))|\frac{\partial}{\partial t'}|n(\mathbf{R}(t'))\rangle\mathrm{d}t'.$$

Berry phase

M. V. Berry, Proc. R. Soc. A 392, 45 (1984)

$$\theta(t) = \frac{1}{\hbar}\int_0^t E_n(\mathbf{R}(t'))\mathrm{d}t' - i\int_0^t \langle n(\mathbf{R}(t'))|\frac{\partial}{\partial t'}|n(\mathbf{R}(t'))\rangle \mathrm{d}t'.$$

Dynamical phase

and this?

Removing the explicit time-dependence

$$\gamma_n = i\int_0^t \langle n(\mathbf{R}(t'))|\frac{\partial}{\partial \mathbf{R}}|n(\mathbf{R}(t'))\rangle\frac{\partial \mathbf{R}}{\partial t'}\mathrm{d}t' = i\int_{\mathcal{C}}\langle n(\mathbf{R})|\frac{\partial}{\partial \mathbf{R}}|n(\mathbf{R})\rangle \mathrm{d}\mathbf{R}.$$

geometrical phase!

But what if we **gauge-transform** our wave-function? $|n(\mathbf{R})\rangle = e^{i\chi(\mathbf{R})}|n(\mathbf{R})\rangle$

smooth, single-valued function

$$\gamma_n' = i\int_{\mathcal{C}}\langle n(\mathbf{R})|\frac{\partial}{\partial \mathbf{R}}|n(\mathbf{R})\rangle \mathrm{d}\mathbf{R} - \int_{\mathcal{C}}\frac{\partial \chi(\mathbf{R})}{\partial \mathbf{R}}\mathrm{d}\mathbf{R} = \gamma_n - [\chi(\mathbf{R}(T)) - \chi(\mathbf{R}(0))],$$

Looks like we can choose the gauge-transformation so that

$$\gamma_n' = 0$$

Berry phase

M. V. Berry, Proc. R. Soc. A 392, 45 (1984)

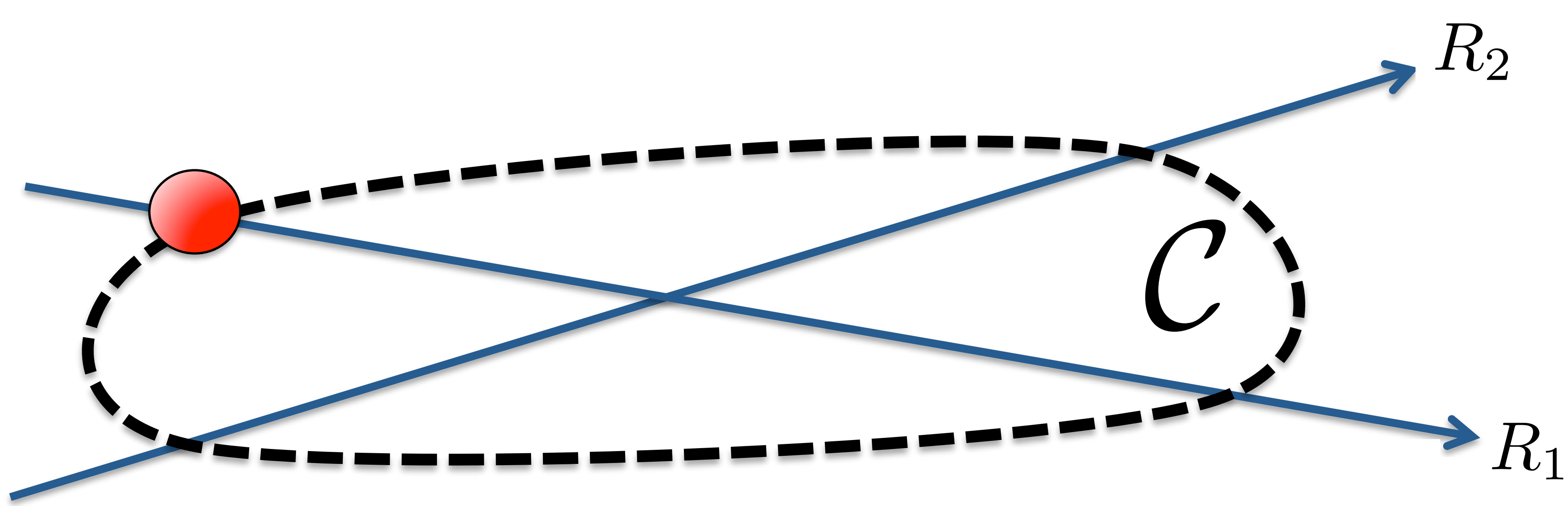
But if we consider **a closed contour** then:

$$|n(\mathbf{R})\rangle = e^{i\chi(\mathbf{R})}|n(\mathbf{R})\rangle$$

$$\chi(\mathbf{R}(T)) - \chi(\mathbf{R}(0)) = 2\pi \times \mathbb{Z},$$

and so $\gamma_n' = \gamma_n - 2\pi\mathbb{Z}$

i.e. the geometrical phase is physical and gauge-invariant (up to 2π)
for a closed contour : the **Berry phase**



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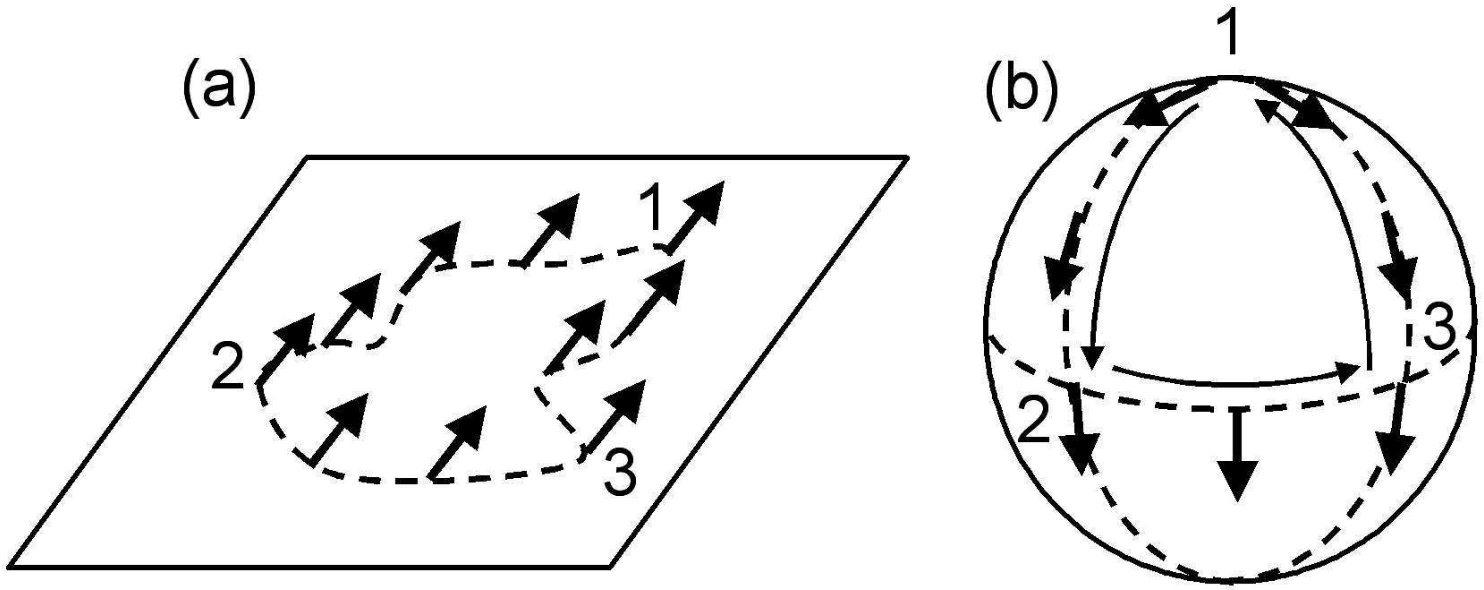
and so $\quad \gamma_n' = \gamma_n - 2\pi\mathbb{Z}$

i.e. the geometrical phase is physical and gauge-invariant (up to 2π)
for a closed contour : the **Berry phase**

Analogous to the rotation of a vector under parallel transport around a closed contour



$\kappa = 0$ $\qquad$ $\kappa = \frac{1}{r^2} > 0$

On a surface, the rotation of the vector depends on the Gaussian curvature enclosed

$$\kappa = \frac{1}{r_1 r_2}$$

Figure from: P. Bruno, arXiv:cond- mat/0506270

Berry phase, connection & curvature

Inspired by that geometrical analogy, let's define some more properties:

$$\gamma_n = \oint_{\mathcal{C}} \mathrm{d}\mathbf{R} \cdot \mathcal{A}_n(\mathbf{R}) = \int_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \Omega_n(\mathbf{R})$$

Berry phase	**Berry connection**	**Berry curvature**
$\gamma_n = i \oint_{\mathcal{C}} \langle n(\mathbf{R})\vert \frac{\partial}{\partial \mathbf{R}} \vert n(\mathbf{R})\rangle \mathrm{d}\mathbf{R}$	$\mathcal{A}_n(\mathbf{R}) = i\langle n(\mathbf{R})\vert \frac{\partial}{\partial \mathbf{R}} \vert n(\mathbf{R})\rangle$	$\Omega_n(\mathbf{R}) = \nabla \times \mathcal{A}_n(\mathbf{R})$
(modulo) gauge-invariant $\chi(\mathbf{R}(T)) - \chi(\mathbf{R}(0)) = 2\pi \times \mathbb{Z},$	gauge-dependent $\mathcal{A}_n(\mathbf{R}) \rightarrow \mathcal{A}_n(\mathbf{R}) - \frac{\partial \chi(\mathbf{R})}{\partial \mathbf{R}}$	gauge-invariant $\nabla \times \nabla \chi(\mathbf{R}) = 0$
Analogous to magnetic flux $\Phi = \int_{\mathcal{S}} \mathrm{d}\mathbf{S} \cdot \mathbf{B}(\mathbf{r})$	Analogous to a magnetic vector potential $\mathbf{A}(\mathbf{r})$	Analogous to a magnetic field $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$

From geometry to topology

Let's go back to that geometrical analogy again

In fact, geometry and topology are connected, e.g. :

$$\int_{\mathcal{S}_{\rm tot}} \kappa \mathrm{d}S = 4\pi(1-g)$$

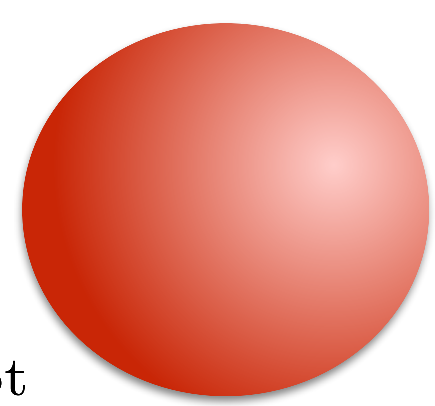
Gauss-Bonnet theorem

$g = 0$

$g = 1$

Analogously, we can relate the geometry and topology of eigenstates over a *closed* parameter space, e.g.:

$\mathcal{S}_{\rm tot}$



closed two-dimensional surface

topological
1st Chern number

$$\nu_n = \frac{1}{2\pi}\int_{\mathcal{S}_{\rm tot}} \mathrm{d}\mathbf{S}\cdot\Omega_n(\mathbf{R})$$

For much more about this, see "Geometry, Topology and Physics" by M. Nakahara, IOP Publishing, (2003)

1st Chern Number

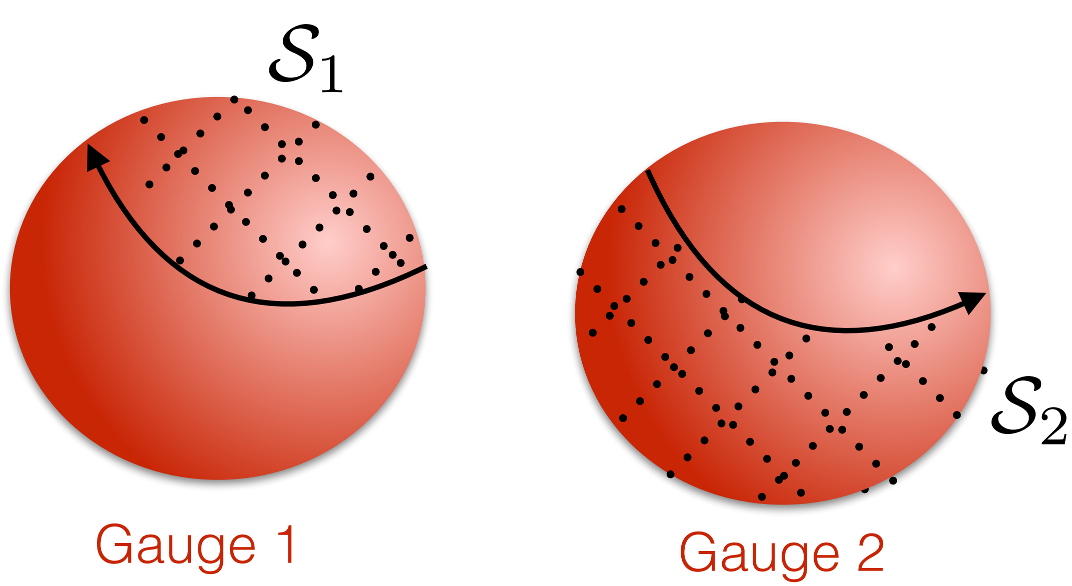
But wait a minute….

$$\nu_n = \frac{1}{2\pi}\int_{\mathcal{S}_{\rm tot}} \mathrm{d}\mathbf{S}\cdot\Omega_n(\mathbf{R}) \to \frac{1}{2\pi}\oint_{\mathcal{C}} \mathrm{d}\mathbf{R}\cdot\mathcal{A}_n(\mathbf{R}) = 0$$

So this is always zero?

Actually, Stokes' theorem assumes the Berry connection has **no singularities** inside the contour. If this is true, then yes the 1st Chern number is zero.

If not, we can get rid of singularities by defining different gauges over different patches



$$\gamma_n = \oint_{\mathcal{C}_1} \mathrm{d}\mathbf{R}\cdot\mathcal{A}_n = \int_{\mathcal{S}_1} \mathrm{d}\mathbf{S}\cdot\Omega_n(\mathbf{R}),$$

$$\mathcal{C}_1 = -\mathcal{C}_2$$

$$\gamma_n' = \oint_{\mathcal{C}_2} \mathrm{d}\mathbf{R}\cdot\mathcal{A}_n' = \int_{\mathcal{S}_2} \mathrm{d}\mathbf{S}\cdot\Omega_n(\mathbf{R}),$$

But $\quad \gamma_n = -\gamma_n' + 2\pi\nu_n,$

As Berry phase is gauge-dependent modulo 2π

$$\int_{\mathcal{S}_{\rm tot}} \mathrm{d}\mathbf{S}\cdot\Omega_n(\mathbf{R}) = \int_{\mathcal{S}_1} \mathrm{d}\mathbf{S}\cdot\Omega_n(\mathbf{R}) + \int_{\mathcal{S}_2} \mathrm{d}\mathbf{S}\cdot\Omega_n(\mathbf{R}) = \gamma_n + \gamma_n' = 2\pi\nu_n.$$

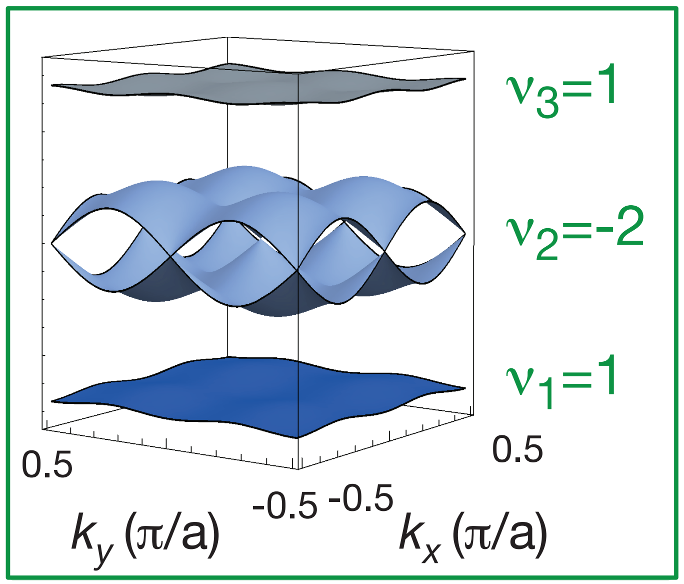
1st Chern numbers & Bloch bands

So far we have considered a general parameter space *[come back to in Lecture 3]*, but let's now return to **topological band theory**

Figure from: Aidelsburger et al., Nat. Phys. 11, 162, (2015)

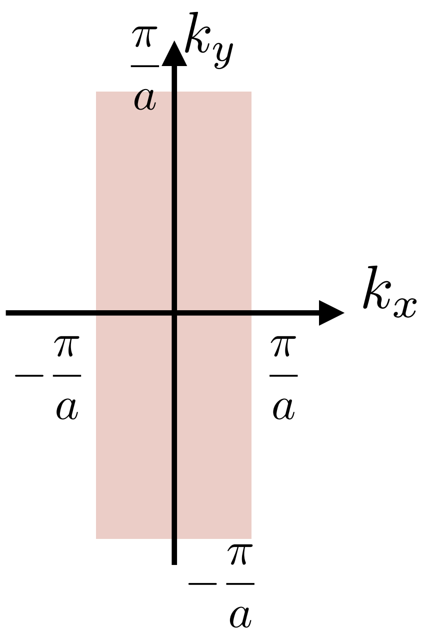
$\vdots$

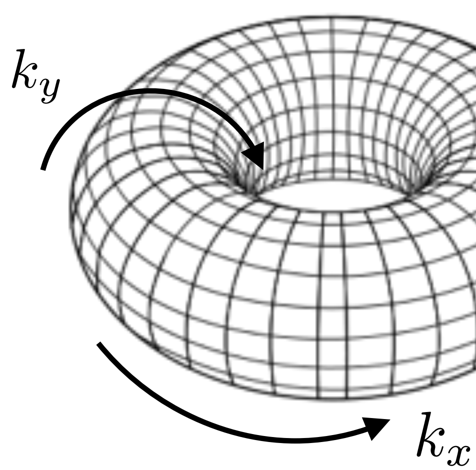
$\mathcal{E}_1(\mathbf{k})$



$$\psi_{n,\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,k}(\mathbf{r})$$

$$\hat{H}_{\mathbf{k}} u_{n,\mathbf{k}} = \mathcal{E}_n(\mathbf{k}) u_{n,\mathbf{k}}$$





$$\mathcal{A}_n(\mathbf{k}) = i\langle u_{n,\mathbf{k}} | \frac{\partial}{\partial \mathbf{k}} | u_{n,\mathbf{k}} \rangle$$

$$\Omega_n(\mathbf{k}) = \nabla \times \mathcal{A}_n(\mathbf{k})$$

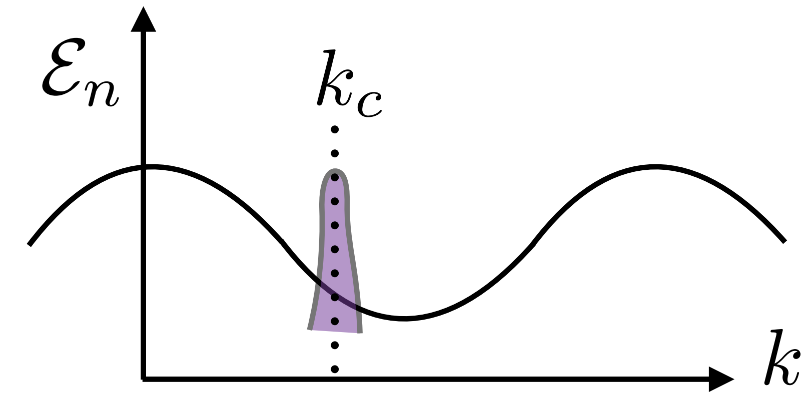
$$\nu_n = \frac{1}{2\pi} \int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k} \cdot \Omega_n(\mathbf{k}) \qquad \in \mathbb{Z}$$

N.B. the above can be generalised to bands with degeneracies

Crucially, the Brillouin zone defines "a closed surface"

2D Quantum Hall Effect

Semiclassical dynamics of a wavepacket in a lattice



$$\dot{\mathbf{r}}_c = \frac{1}{\hbar}\frac{\partial \mathcal{E}_n(\mathbf{k}_c)}{\partial \mathbf{k}_c} \boxed{- \dot{\mathbf{k}}_c \times \Omega_n(\mathbf{k}_c)}$$

$$\hbar \dot{\mathbf{k}}_c = -e\mathbf{E}$$

"Anomalous velocity": analogous to a Lorentz force with roles of position and momentum swapped

Karplus & Luttinger Phys. Rev. 95, 1154 (1954)…
Chang & Niu, PRL, 75, 1348 (1995)…
Review: Xiao et al, RMP, 82, 1959 (2010)

Now consider a band insulator

$$\mathbf{j} = -\frac{e}{(2\pi)^2}\sum_{n\in\text{occupied}}\int_{BZ} d^2\mathbf{k}\cdot\left[\frac{1}{\hbar}\frac{\partial \mathcal{E}_n(\mathbf{k})}{\partial \mathbf{k}} + \frac{e}{\hbar}\mathbf{E}\times\Omega_n(\mathbf{k})\right]$$

$$\longrightarrow \quad j_x = -\frac{e^2}{h}\frac{E_y}{(2\pi)}\sum_{n\in\text{occupied}}\int_{BZ} d^2\mathbf{k}\cdot\Omega_n(\mathbf{k})$$

[Better derivation with Kubo formula]

Thouless et al., Phys. Rev. Lett. 49, 405,1982

Consequences of the Chern number:

- Quantized Hall conductance (TKNN result) $\sigma_{xy} = -\frac{e^2}{h}\sum_{n\in\text{occupied}}\nu_n$
- Chiral edge modes (Bulk-Boundary Correspondence) : $n_{\text{edge}} = \sum_{n\in\text{occupied}}\nu_n$

Time-reversal symmetry breaking

Under time-reversal symmetry (TRS) $\Omega_n(\mathbf{k}) = -\Omega_n(-\mathbf{k})$ so $\nu_n = \dfrac{1}{2\pi}\displaystyle\int_{\mathrm{BZ}} \mathrm{d}^2\mathbf{k}\cdot\Omega_n(\mathbf{k}) = 0$ if TRS is present

How to break TRS physically?

- Apply a magnetic field to a charged particle: $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$
- But also many other approaches…

Breaking TRS for cold atoms & photons in Lectures 3 and 4

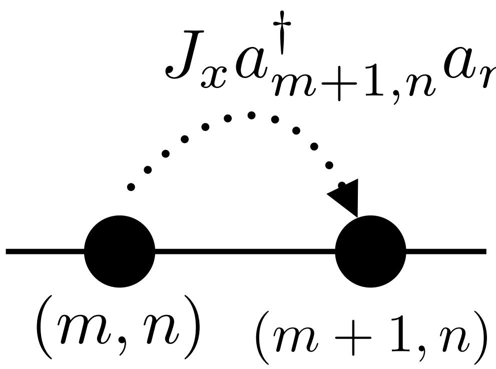
Seminal theoretical models with broken TRS:

- <u>Continuum</u>: Landau levels
- <u>Chern insulators [lattice models]</u>: Harper-Hofstadter model & Haldane model

	Symmetry			d							
	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Chern Insulator: Harper-Hofstadter Model

Charged particle hopping on a tight-binding 2D square lattice in a uniform magnetic field

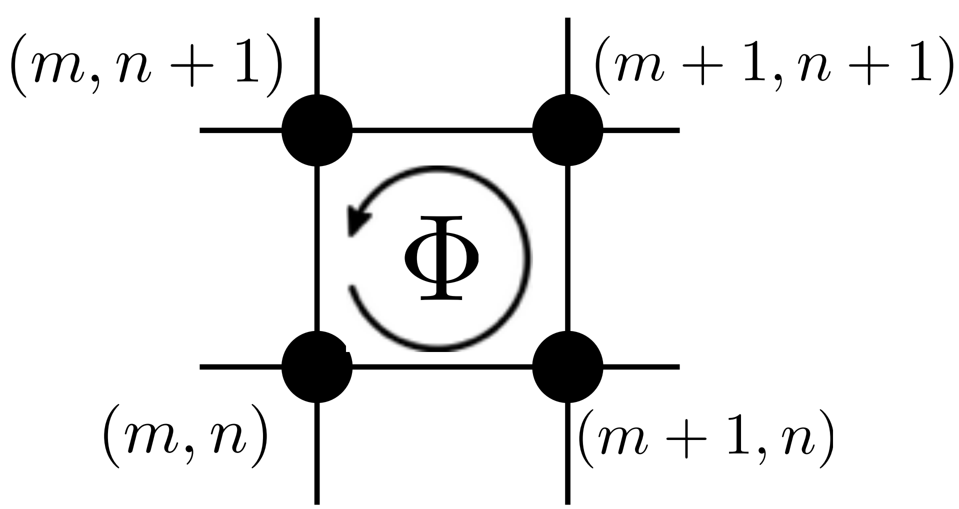


Peierls substitution

$$J_x \to J_x e^{i\theta^x_{m,n}}, \qquad \theta^x_{m,n} = -\frac{e}{\hbar}\int_{\mathbf{r}_{m,n}}^{\mathbf{r}_{m+1,n}} \mathbf{A}\cdot d\mathbf{x}$$

and similarly for y

Then hopping around a plaquette:



in the Landau gauge $\mathbf{A} = (0, Bx, 0)$

$$\begin{aligned}
\theta^y_{m+1,n} - \theta^y_{m,n} - \theta^x_{m,n+1} + \theta^x_{m,n} \\
&= -\frac{e}{\hbar}[B(m+1)a^2 - Bma^2] \\
&= 2\pi\frac{\Phi}{\Phi_0} = 2\pi\alpha
\end{aligned}$$

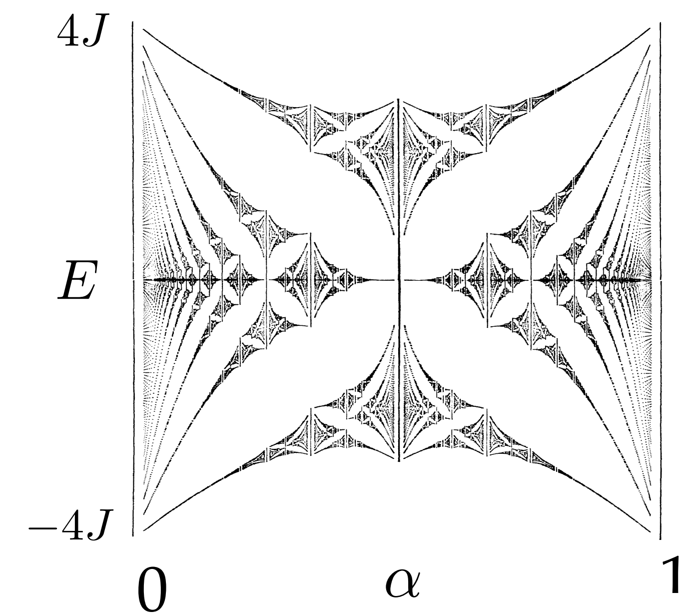
No/ flux quanta per plaquette
$\alpha = -eBa^2/h$

$$\mathcal{H} = -J\sum_{m,n}[a^\dagger_{m+1,n}a_{m,n} + e^{i2\pi\alpha m}a^\dagger_{m,n+1}a_{m,n} + \mathrm{h.c.}]$$

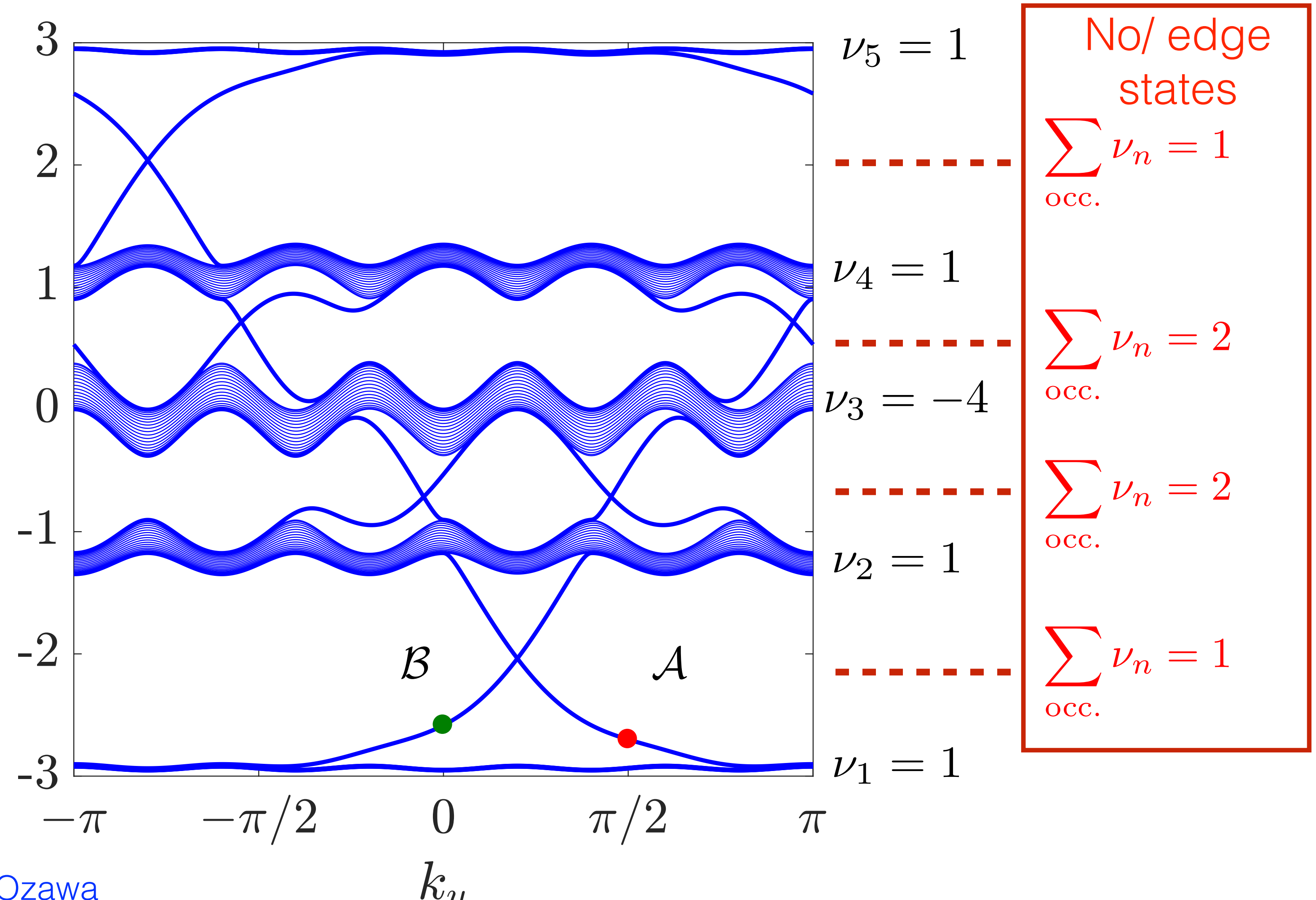
Hofstadter, PRB 14, 2239, (1976)

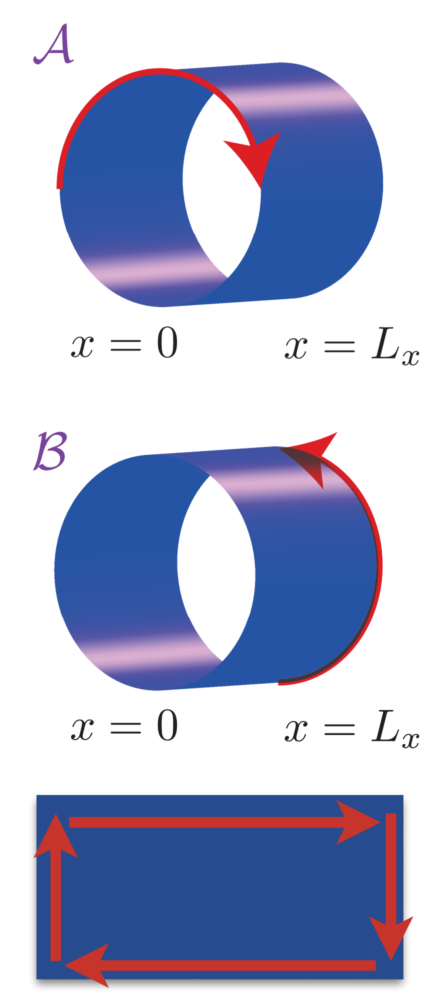
Chern Insulator: Harper-Hofstadter Model

Interplay between lattice and magnetic field gives fractal energy spectrum : **the Hofstadter butterfly**



Non-zero Chern numbers and topological edge modes:





Numerical diagonalisation on a cylinder

$\alpha = 1/5$

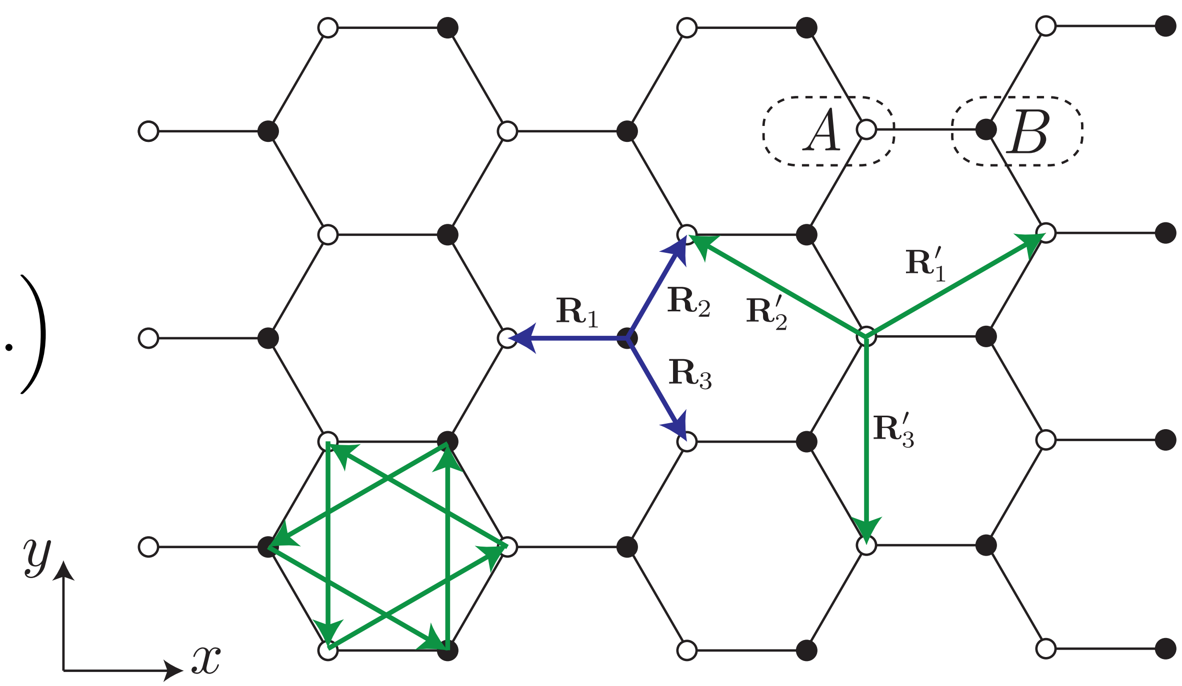
Figures courtesy of T. Ozawa

Chern Insulator: Haldane Model

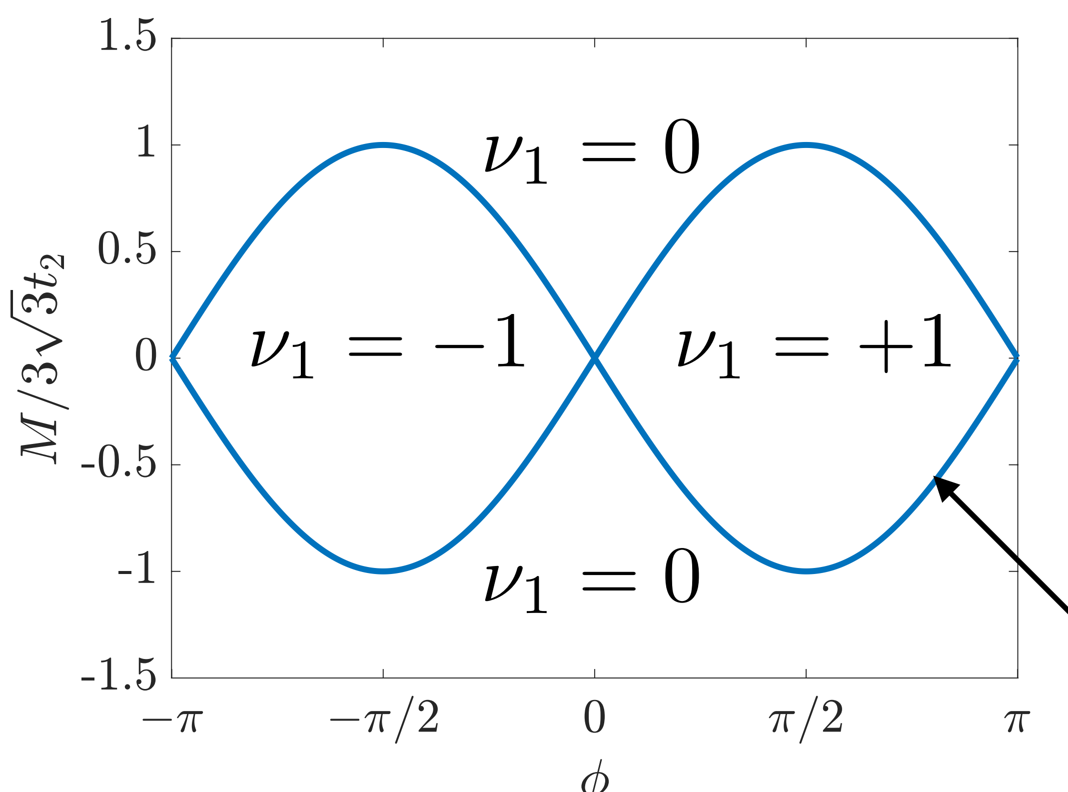
Charged particle hopping on a tight-binding 2D **honeycomb** lattice

Haldane, PRL, 61, 18, (1988)

$$H_{\text{Haldane}} = t_1 \sum_{\mathbf{r},j} \left(a^\dagger_{\mathbf{r}+\mathbf{R}_j} b_\mathbf{r} + \text{h.c.} \right) + t_2 \sum_{\mathbf{r},j} \left(e^{i\phi} a^\dagger_{\mathbf{r}+\mathbf{R}'_j} a_\mathbf{r} + e^{-i\phi} b^\dagger_{\mathbf{r}+\mathbf{R}'_j} b_\mathbf{r} + \text{h.c.} \right) + M \sum_{\mathbf{r}} \left(a^\dagger_\mathbf{r} a_\mathbf{r} - b^\dagger_\mathbf{r} b_\mathbf{r} \right)$$



Topological phase diagram



Topological phase transition (gapless Dirac points)

- Complex NNN hoppings break TRS
- First example of a Chern insulator model with zero average magnetic flux

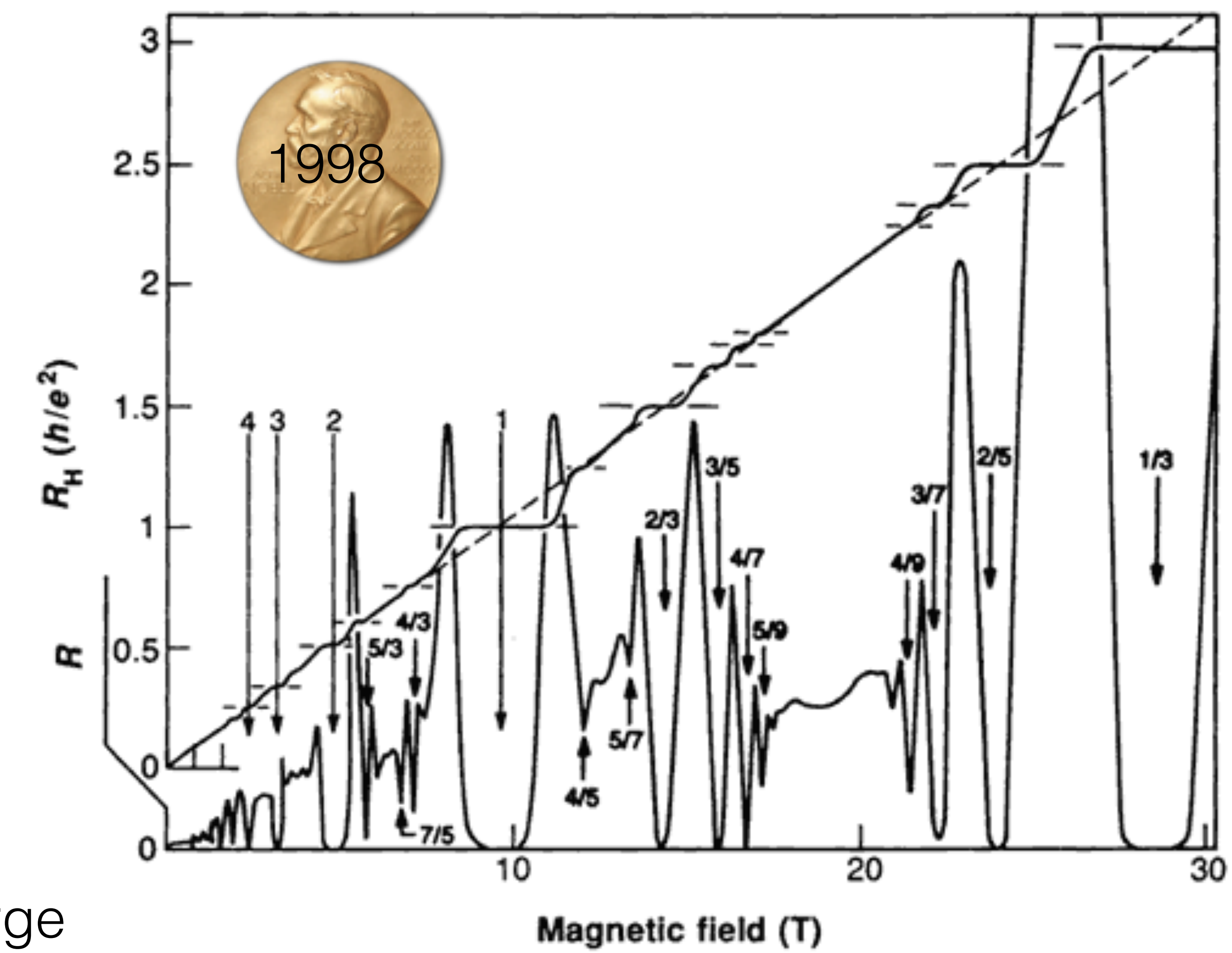
Figures courtesy of T. Ozawa

Fractional Quantum Hall Effect

Ultimate goal for why ultracold atoms and photons want to engineer artificial quantum Hall states

QH plateaux at non-integer filling fractions

$$\frac{N}{N_\phi} = \frac{p}{q}$$



Unlike in the integer quantum Hall effect, strong electron interactions play a crucial role

Important features include:

- topological degeneracy
- quasiparticles with fractional charge
- quasiparticles with fractional statistics & maybe even **non-abelian statistics…**

Find out more in lecture notes: Tong, arXiv:1606.06687

Anyons & Non-Abelian Anyons

In 2D, exchange statistics of 2 particles can be interesting:

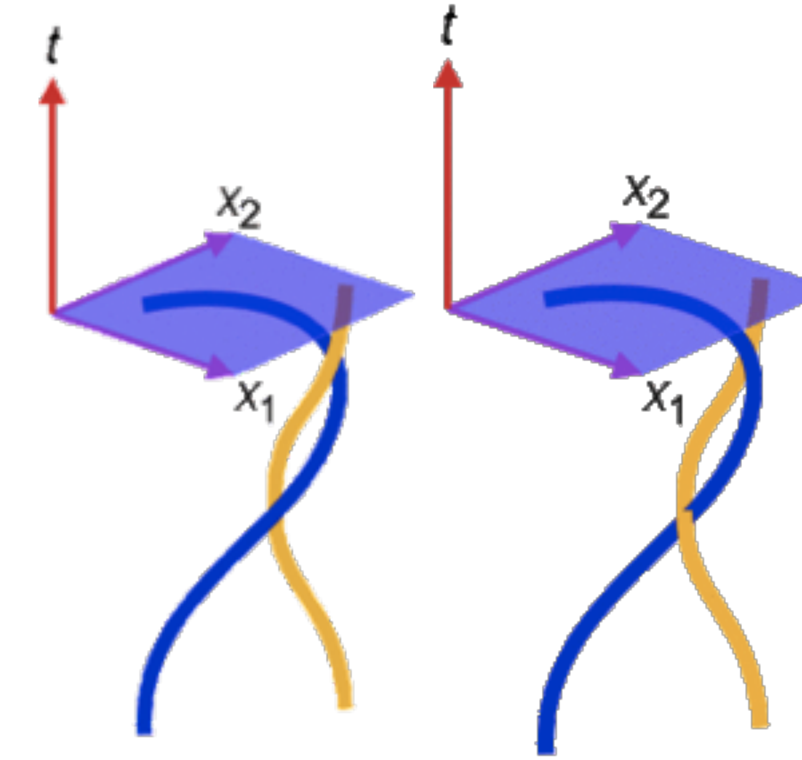
$$|\psi_1\psi_2\rangle = e^{i\theta}|\psi_2\psi_1\rangle$$ **Abelian** anyons

$\theta = \pi$ Fermions

$\theta = 0$ Bosons

$$|\psi_1\psi_2\rangle = \mathbf{M}|\psi_2\psi_1\rangle$$ **Non-Abelian** anyons

When different "Braiding" operations to exchange particles do not commute
(possible when states have some degeneracy)



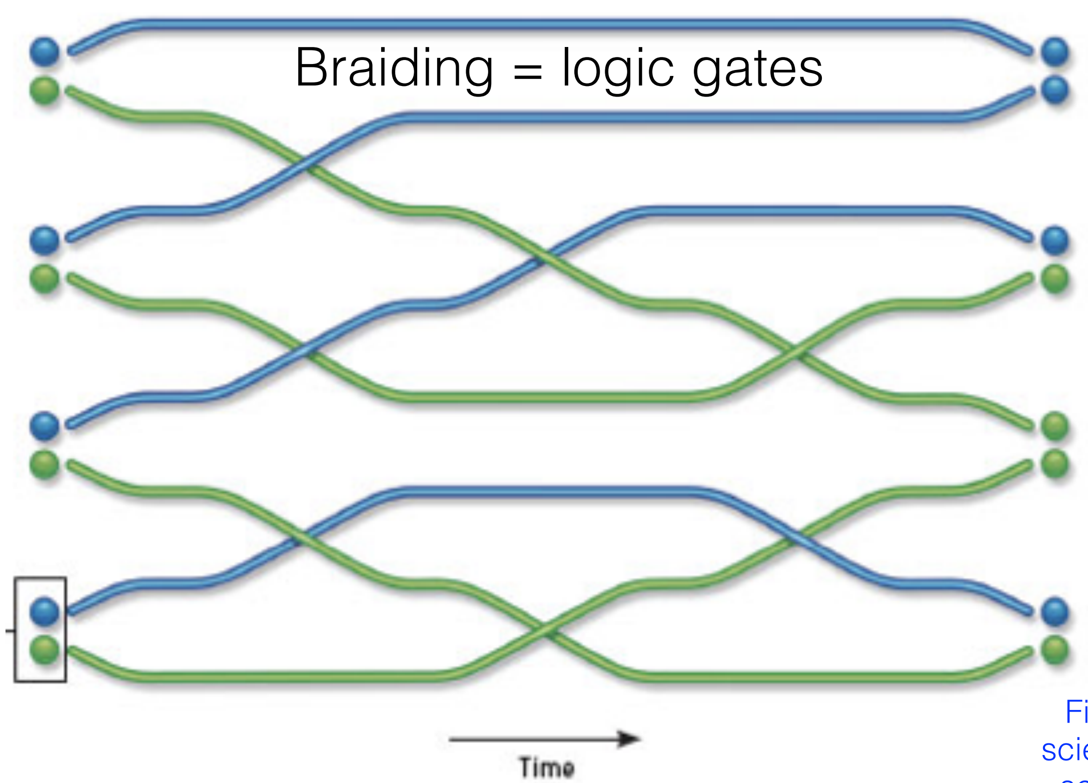
Anticlockwise Clockwise

Figures: CC0 wikipedia User:maschen

Non-Abelian anyons could be used for **topological quantum computing**

C. Nayak et al. Rev. Mod. Phys., 80,1083, 2008.

Qubits



Small perturbations do not disrupt braiding — very robust!

Figure from: http://www.nature.com/scientificamerican/journal/v294/n4/box/scientificamerican0406-56_BX3.html

Before we move on,
here are a few more interesting things about
quantum Hall systems

(that are relevant to cold atoms and photonics)

Higher dimensions

	Symmetry			*d*							
	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

in 4D: **2nd Chern Number**

$$\nu_n^{(2)} = \frac{1}{8\pi^2}\int_{\mathrm{4DBZ}} \Omega_n \wedge \Omega_n \in \mathbb{Z}\,,$$
$$= \frac{1}{4\pi^2}\int_{\mathrm{4DBZ}} [\Omega_n^{xy}\Omega_n^{zw} + \Omega_n^{wx}\Omega_n^{zy} + \Omega_n^{zx}\Omega_n^{yw}]\mathrm{d}^4k$$

in 6D: 3rd Chern number and so on…

Leads to a *nonlinear* quantized Hall response

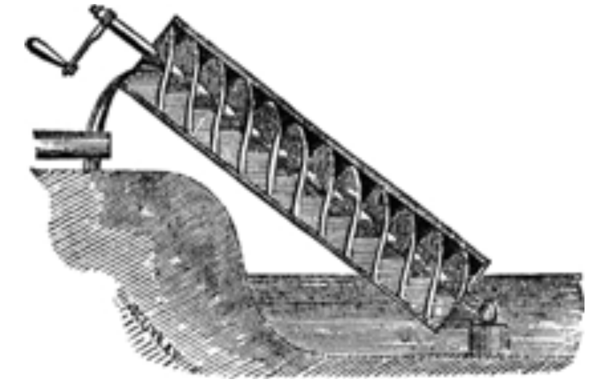
$$j_x = \frac{q^3}{h^2} E_y B_{zw} \sum_{n \in occ.} \nu_n^{(2)}$$

weak magnetic perturbation in z-w plane

Ultracold atoms and photons could be used to explore higher-dimensional topology (Lectures 3 & 4)

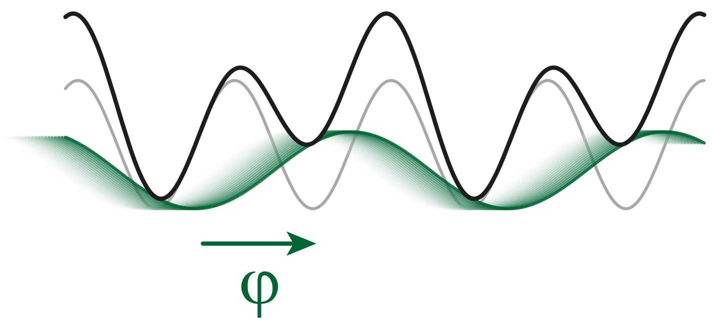
S.-C. Zhang and J. Hu, Science 294, 823 (2001).

Topological Pumping



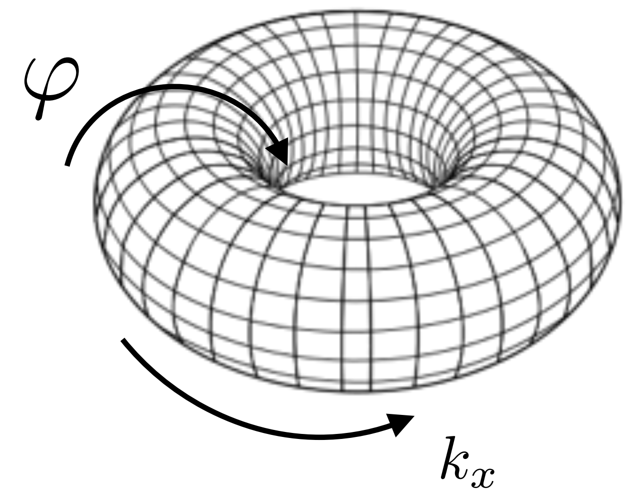
Consider $\mathcal{H}(x, \varphi(t)) = \mathcal{H}(x, \varphi(t+T))$

Fig adapted from Lohse, M et al. Nat. Phys. 12, 350–354 (2016).



pumping parameter

$\varphi(t+T) = \varphi(t) + 2\pi$



can define:

$$\nu_n = \frac{1}{2\pi}\int_0^{2\pi}\int_{BZ}\Omega_n^{k_x,\varphi}dk_x d\varphi \qquad \Omega_n^{k_x,\varphi} = i\left[\langle\frac{\partial u_n}{\partial k_x}|\frac{\partial u_n}{\partial \varphi}\rangle - \langle\frac{\partial u_n}{\partial \varphi}|\frac{\partial u_n}{\partial k_x}\rangle\right]$$

Physical consequences? Adiabatically pump a wavepacket

D. J. Thouless, Phys. Rev. B 27, 6083 (1983)

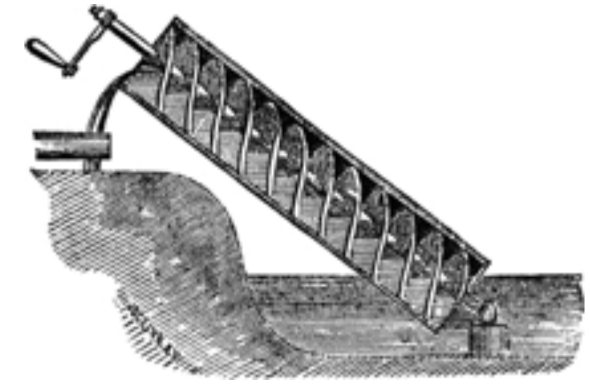
anomalous velocity $\dot{x}_n = \Omega_n^{\varphi,k_x}\partial_t\varphi$

Then for a filled band insulator: Quantized COM shift $x(T) = \nu_n$

BUT remember the pump parameter is external (not a dynamical variable)

—> quantised transport **only** after a full pump cycle

Topological Pumping



How to make a topological pump?

1. Start from a 2D QH system, e.g. HH model

$$\mathcal{H} = -J\sum_{m,n}[a^\dagger_{m+1,n}a_{m,n} + e^{i2\pi\alpha m}a^\dagger_{m,n+1}a_{m,n} + \mathrm{h.c.}]$$

2. Fourier-transform with respect to one dimension $a_{m,n} = \sum_{k_y} e^{ik_y n} a_{m,k_y}$

$$\mathcal{H} = -J\sum_{m,k_y}\left[a^\dagger_{m,k_y}a_{m+1,k_y} + \mathrm{h.c.} + 2\cos\left(2\pi\alpha m + k_y\right)a^\dagger_{m,k_y}a_{m,k_y}\right].$$

Harper model

3. Take a single Fourier component and relabel the momentum as the pumping parameter

$$\mathcal{H}_{1D} = -J\sum_{m}\left[a^\dagger_{m}a_{m+1} + \mathrm{h.c.} + 2\cos\left(2\pi\alpha m + \varphi\right)a^\dagger_{m}a_{m}\right].$$

This is just a 1D hopping model with an onsite-potential that depends on position

Pumping corresponds to shifting the on-site potential

[NB Can do similar procedure in 4D]

Kraus et al., Phys. Rev. Lett. 111, 226401 (2013).

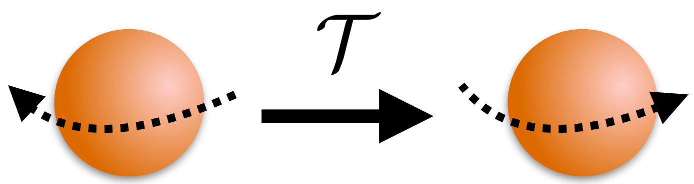
These lectures

In these first two lectures, I will briefly introduce:

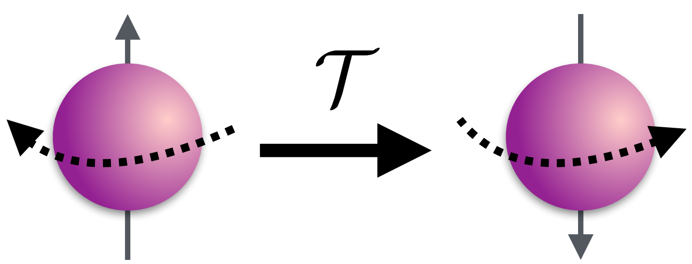
Class	Symmetry: Time-reversal	Particle-hole	Chiral	Dimensionality d: 1	2	3	4	5	6	7	8	
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	← Quantum Hall
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	← SSH Model
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	← Topological Superconductors
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	← **Topological Insulators**
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	

Time-Reversal Symmetry

For spinless particles: $\mathcal{T} = \mathcal{K}$ (charge conjugation) $\longrightarrow$ $\mathcal{T}^2 = \mathcal{K}^2 = +1$



For spin-1/2 particles: $\mathcal{T} = -i\sigma_y\mathcal{K}$ $\longrightarrow$ $\mathcal{T}^2 = (-i\sigma_y\mathcal{K})^2 = -1$



as need TR to also flip the spin:

$\mathcal{T}\sigma_x\mathcal{T}^{-1} = -\sigma_x$ and similarly for the other Pauli matrices

In general

bosons (integer spin) have $\mathcal{T}^2 = +1$

fermions (half-integer spin) have $\mathcal{T}^2 = -1$

Kramer's Theorem

Let's consider a Hamiltonian with TRS

$$H|\psi\rangle = E|\psi\rangle \qquad \text{TRS: } \mathcal{T}^{-1}H\mathcal{T} = H$$

Then from $H\mathcal{T}|\psi\rangle = \mathcal{T}\mathcal{T}^{-1}H\mathcal{T}|\psi\rangle = \mathcal{T}H|\psi\rangle = \mathcal{T}E|\psi\rangle = E\mathcal{T}|\psi\rangle.$

i.e. $\mathcal{T}|\psi\rangle$ is also an eigenstate at energy E

Is this the same state? if so: $\mathcal{T}|\psi\rangle = e^{i\alpha}|\psi\rangle$

some real number

For fermions $\mathcal{T}^2 = -1$

$$-|\psi\rangle = \mathcal{T}^2|\psi\rangle = \mathcal{T}e^{i\alpha}|\psi\rangle = e^{-i\alpha}\mathcal{T}|\psi\rangle = e^{-i\alpha}e^{i\alpha}|\psi\rangle = |\psi\rangle,$$

contradiction!
these must be different states

Kramer's theorem:
For a TRS fermionic system, all eigenstates are at least two-fold degenerate

(For bosons $\mathcal{T}^2 = +1$ so eigenstates can be non-degenerate)

Kramer’s Theorem

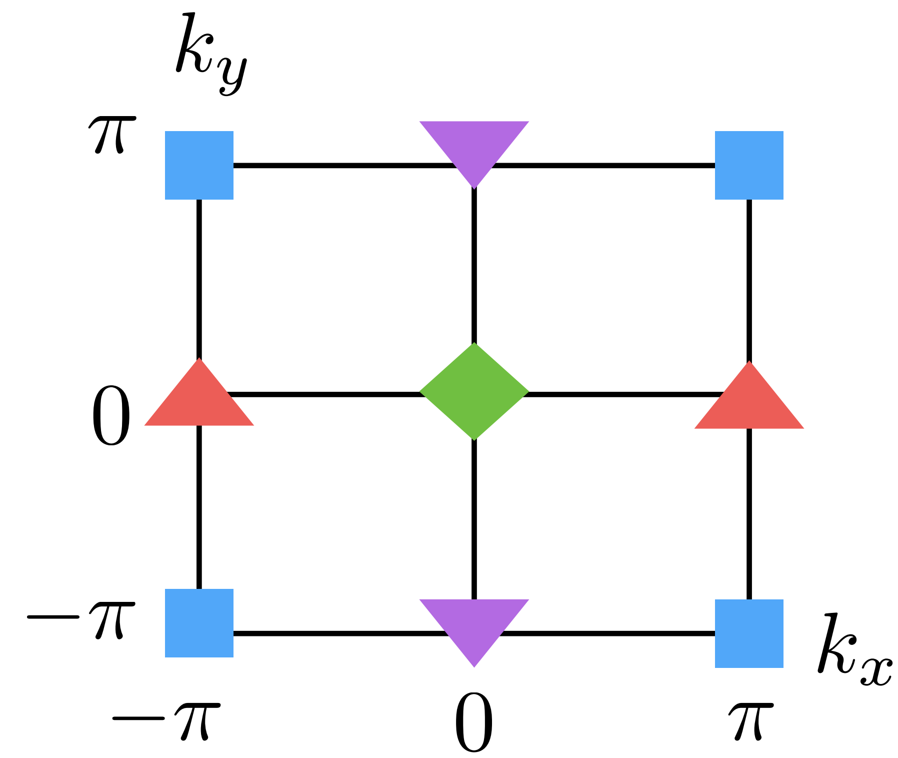
In momentum-space, TRS means

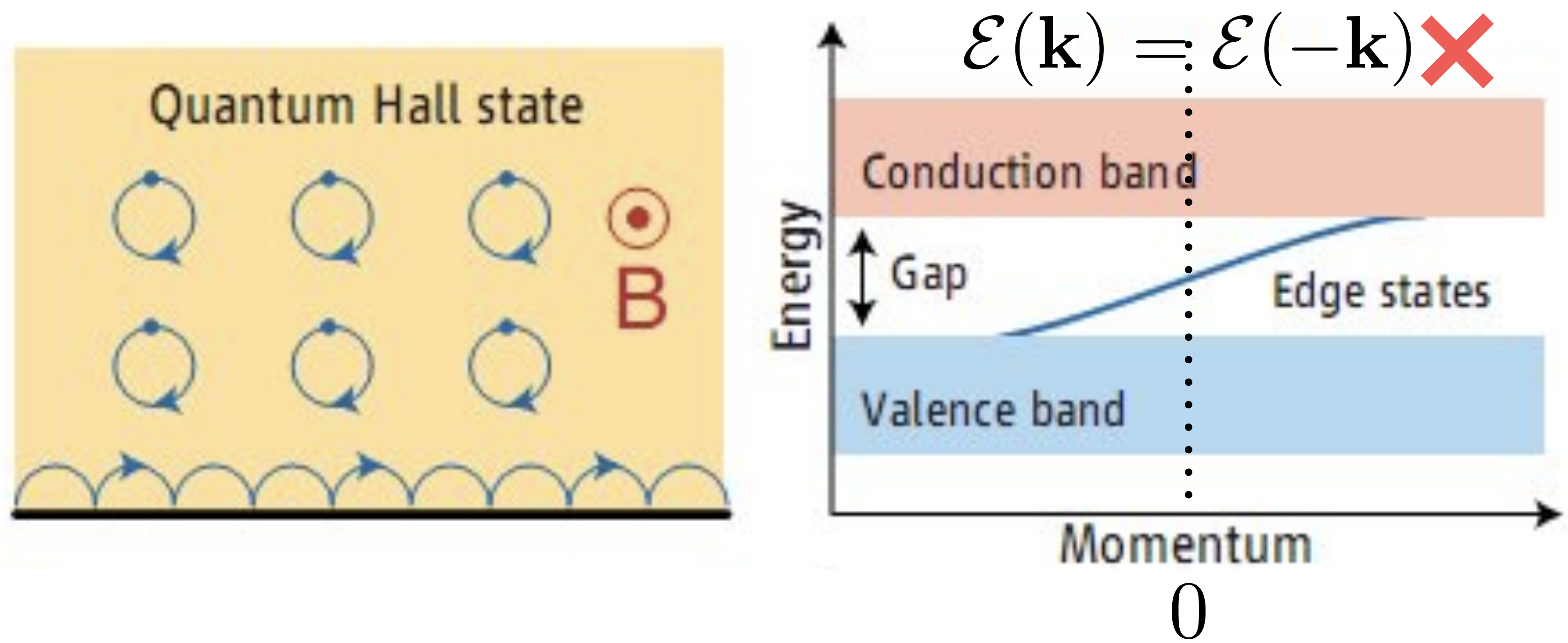
$$\mathcal{T} H(k) \mathcal{T}^{-1} = H(-k)$$

So $|\psi(\mathbf{k})\rangle$ & $|\psi(-\mathbf{k})\rangle$ have the same energy

i.e. $\mathcal{E}(\mathbf{k}) = \mathcal{E}(-\mathbf{k})$

and the 2D BZ has 4 TRS-invariant points where fermionic states must be doubly-degenerate





Going back to Lecture 1, chiral edge states are not possible with TRS

Figure adapted from
C. L. Kane & E. J. Mele, Science 314, 5806, 1692 (2006)

Kramer’s Theorem

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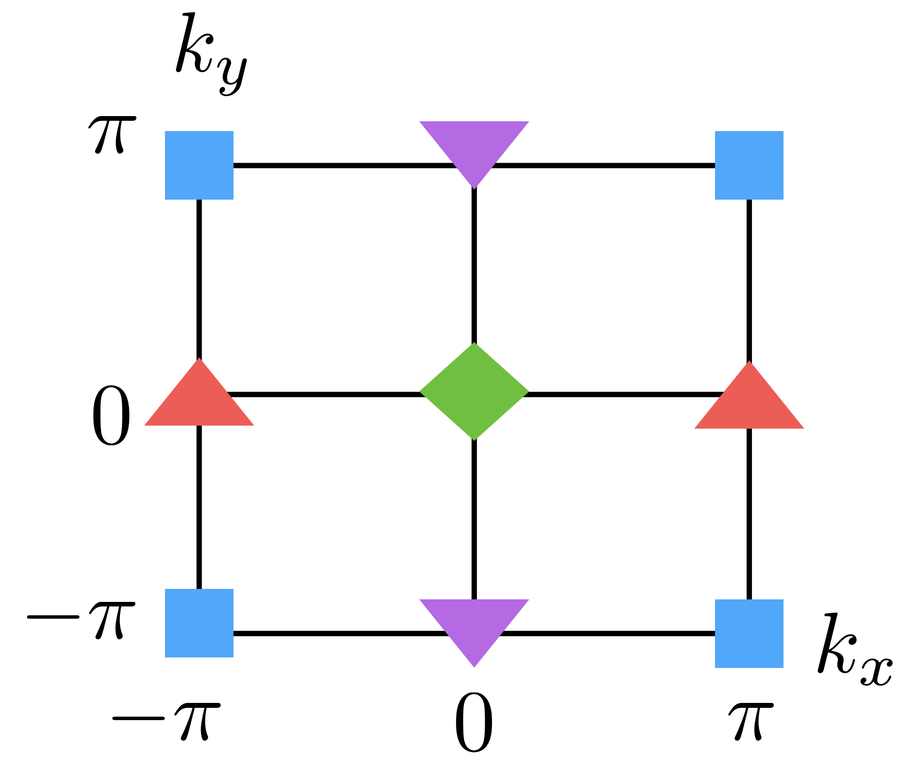
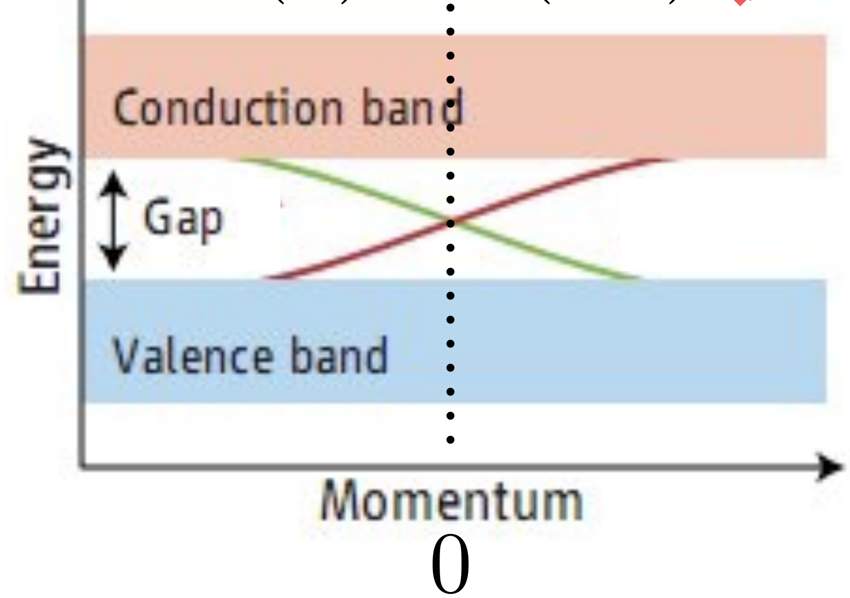


and the 2D BZ has 4 TRS-invariant points where fermionic states must be doubly-degenerate

How about two counter-propagating edge states on the same edge?

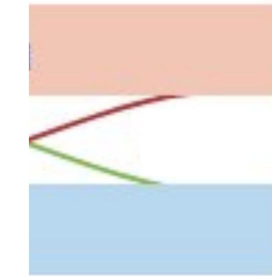
$\mathcal{E}(\mathbf{k}) = \mathcal{E}(-\mathbf{k})$ ✓



Usually a small perturbation would hybridise and mix the edge states

but with TRS, these are a Kramers pair and so can’t be mixed
—> robust states

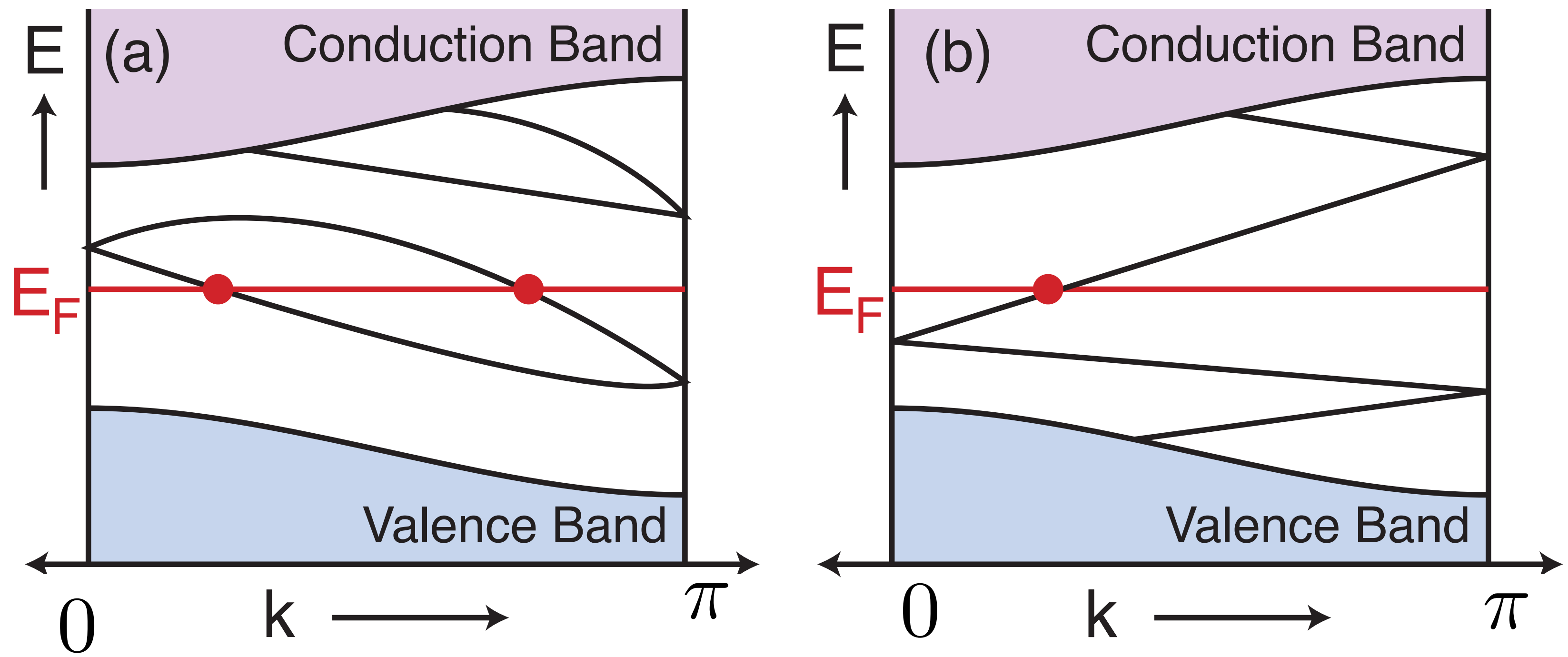
Figure adapted from
C. L. Kane & E. J. Mele, Science 314, 5806, 1692 (2006)



Z_2 Topological invariant in 2D

Figure from: Hasan et al., Rev. Mod. Phys. 82, 3045 (2010).

More generally, there are two possibilities



How many Kramers pairs are at the Fermi level?

even number → perturbations can mix pairs of Kramers pairs and push them out of the gap

$$\nu_{\mathrm{TI}} = 0$$

odd number → robust edge states

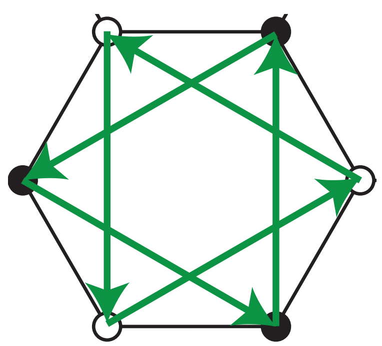
$$\nu_{\mathrm{TI}} = 1$$

Here we exploit the *bulk-boundary correspondence* — ν_{TI} can also be calculated from bulk states, but this is generally quite complicated…

	Symmetry			d							
	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

2D Topological Insulators

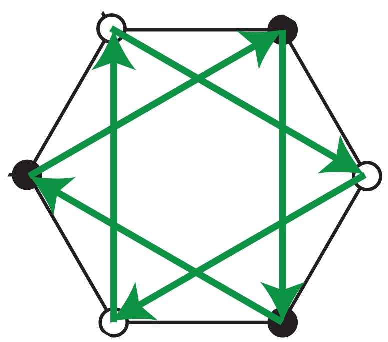
Simplest example is two copies of a Chern insulator, e.g.:



$$H_{\text{Haldane}}(+\phi)$$

for spin-up

$$\nu_{\uparrow} = +1$$



$$H_{\text{Haldane}}(-\phi)$$

for spin-down

$$\nu_{\downarrow} = -1$$

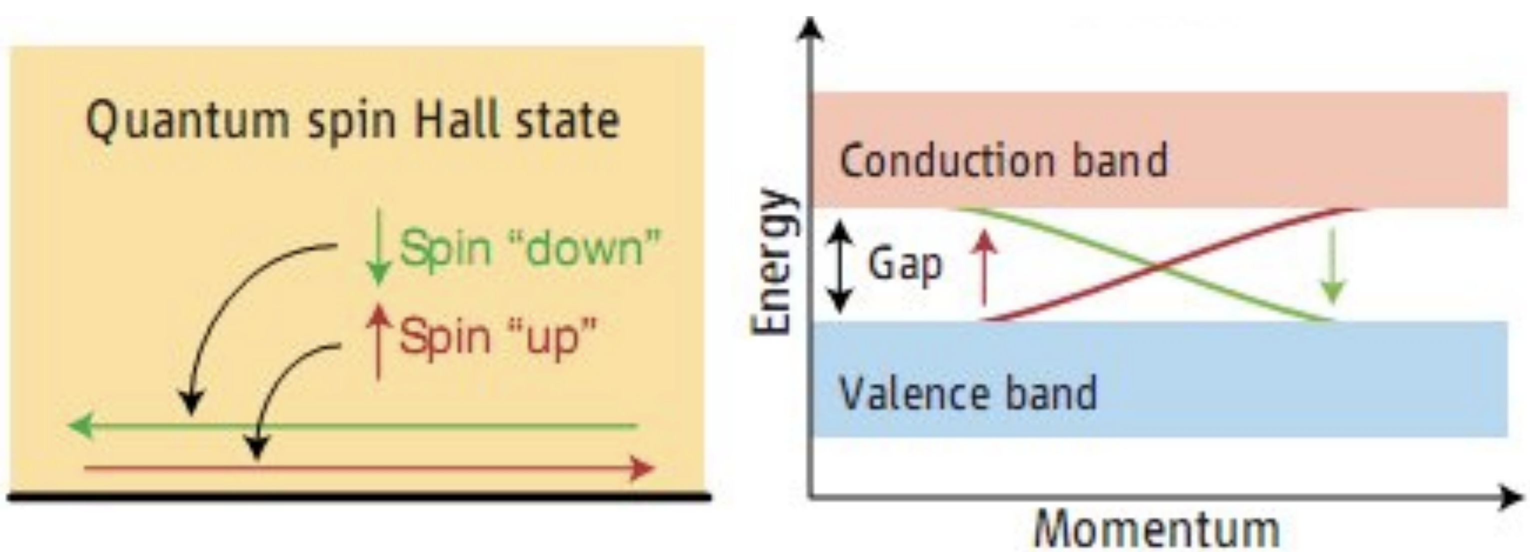
if have spin-conservation can calculate the Z_2 invariant simply as:

$$\nu_{\text{TI}} = \nu_{\uparrow,\downarrow} \text{mod} 2$$

Also, robust to spin-mixing terms preserving TRS e.g. Kane-Mele model…

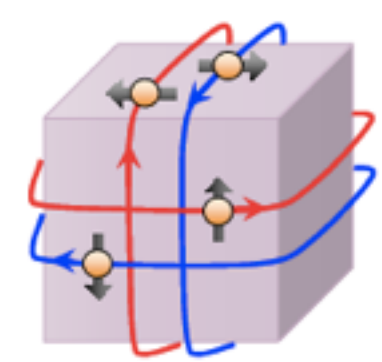
(but the Z_2 invariant is harder to calculate)

TIs found in materials with **strong spin-orbit coupling** e.g. HgCdTe quantum wells



[Quantum spin Hall state is another name for a 2D Topological Insulator]

Also 3D TIs!



These lectures

In these first two lectures, I will briefly introduce:

Class	Symmetry			Dimensionality d								
	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8	
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	← Quantum Hall
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	← **SSH Model**
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	← Topological Superconductors
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	← Topological Insulators
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	

1D Chiral Topological Phases

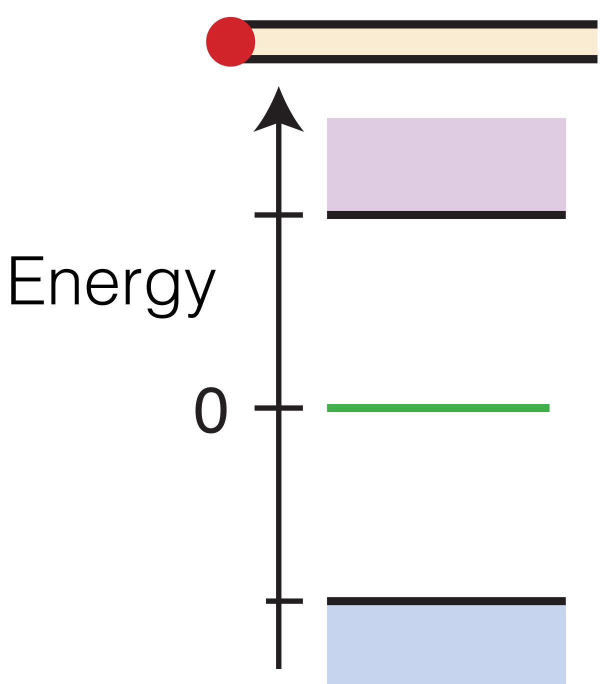
	Symmetry			d							
	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Let's consider a system with chiral symmetry: $\mathcal{C}H\mathcal{C}^{-1} = -H$

$$H|\psi\rangle = E|\psi\rangle$$

$$H\mathcal{C}|\psi\rangle = -\mathcal{C}H|\psi\rangle = -E\mathcal{C}|\psi\rangle.$$

so the spectrum must be symmetric around E=0



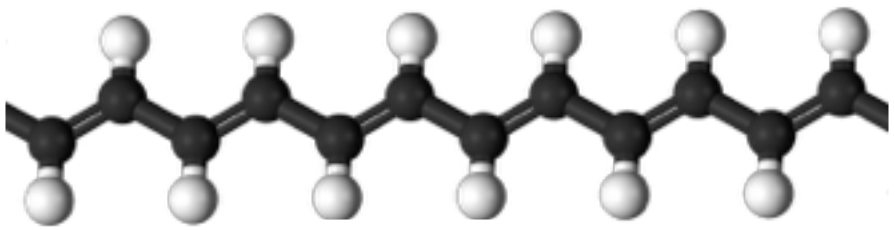
At E=0, we can have edge states which satisfy $|\psi\rangle = \mathcal{C}|\psi\rangle$ these are **topologically-protected** as they can't be perturbed away from E=0

Figure from: Hasan et al., Rev. Mod. Phys. 82, 3045 (2010).

Su-Schrieffer-Heeger Model

Probably the simplest model with nontrivial topology

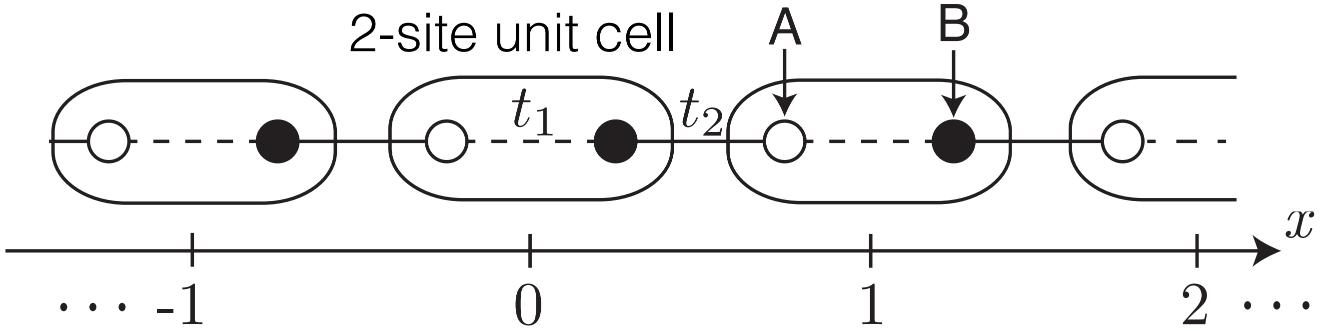
Originally proposed as a model for polyacetylene

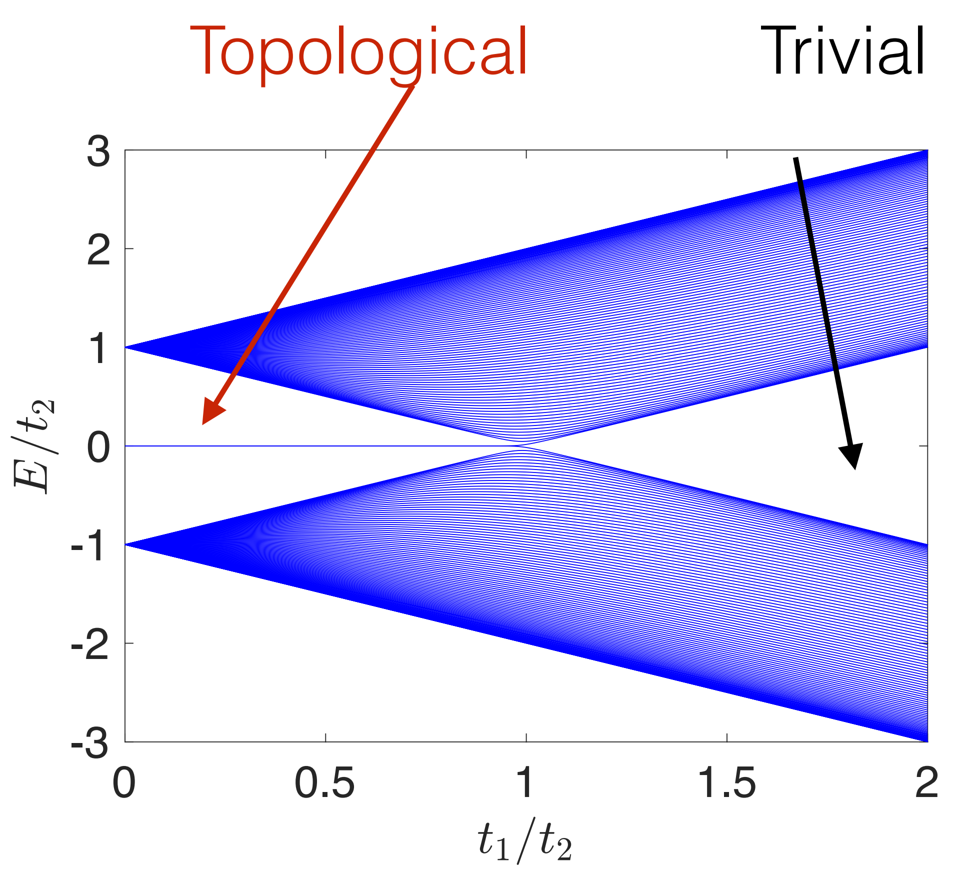


Su, Schrieffer, & Heeger, PRL. 42, 1698 (1979), ibid, PRB 22, 2099 (1980).

Bi-partite chain:

$$H = \sum_x \left(t_1 b_x^\dagger a_x + t_2 a_{x+1}^\dagger b_x + h.c. \right)$$

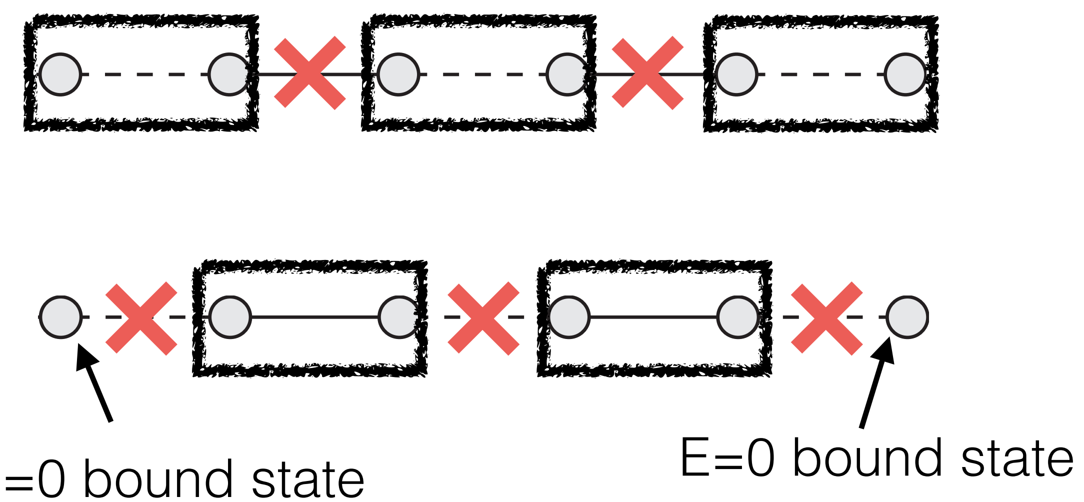




Spectrum for termination with A (B) on the left (right) edge

If $t_2 = 0$

If $t_1 = 0$



[Chiral symmetry here flips the sign of the wave-function on one sub lattice]

Figures courtesy of T. Ozawa

These lectures

In these first two lectures, I will briefly introduce:

	Symmetry			Dimensionality d								
Class	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8	
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	← Quantum Hall
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	← SSH Model
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	← **Topological Superconductors**
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	← Topological Insulators
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	

Topological Superconductors

In BCS theory,
Bogoliubov de-Gennes Hamiltonian

$$H - \mu N = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}} & c_{-\mathbf{k}} \end{pmatrix} \mathcal{H}_{BdG}(\mathbf{k}) \begin{pmatrix} c_{\mathbf{k}} \\ c^\dagger_{-\mathbf{k}} \end{pmatrix}$$

$$\mathcal{H}_{\mathrm{BdG}} = \begin{pmatrix} H_0(\mathbf{k}) - \mu & \Delta(\mathbf{k}) \\ \Delta(\mathbf{k})^* & -H_0(\mathbf{k}) + \mu \end{pmatrix}$$

mean-field pairing potential

excitation spectrum has a superconducting **energy gap**

Particle-hole (PH) symmetry: $\mathcal{P}\mathcal{H}_{\mathrm{BdG}}(\mathbf{k})\mathcal{P}^{-1} = -\mathcal{H}_{\mathrm{BdG}}(-\mathbf{k})$

where

$$\mathcal{P} = \sigma_x K$$

$$\mathcal{P}^2 = 1$$

every eigenstate at energy E has a partner at energy -E

particle-hole redundancy: $\Gamma^\dagger_E = \Gamma_{-E}$

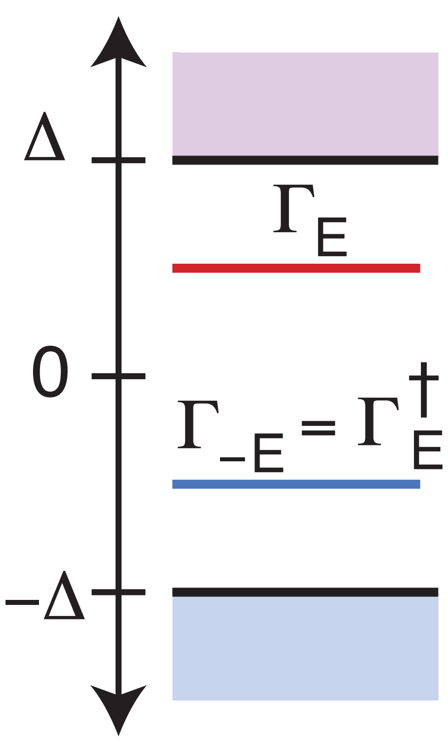
i.e. creating a quasiparticle in state +E has the same effect as removing one from state –E

1D Topological Superconductors

Gapped phase of a fermionic quadratic Hamiltonian with PH-symmetry

	Symmetry: Time-reversal	Particle-hole	Chiral	d = 1	2	3	4	5	6	7	8
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$

What happens in a 1D topological superconductor?

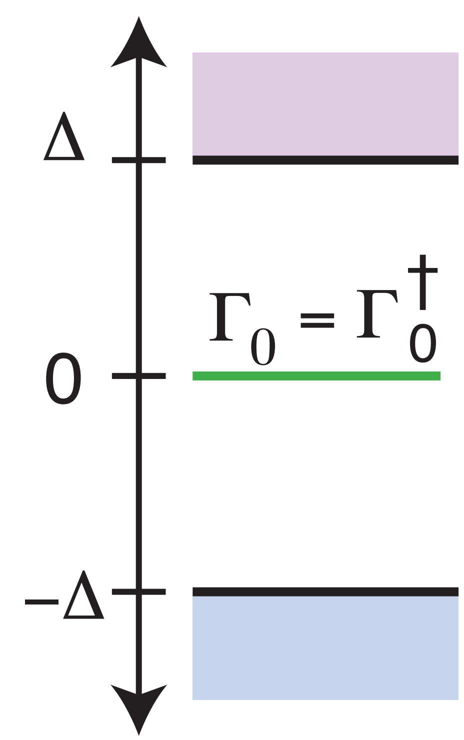


Pair of states at the edge:

Perturbations can push the pair out of the gap

Not topological

$\nu_{\text{1D SC}} = 0$



Topological

$\nu_{\text{1D SC}} = 1$

Single unpaired state at the edge

By PHS, state can't be moved from E=0

Majorana fermion

(like half of a usual Dirac fermion)

Non-Abelian anyons : topological quantum computing?

Figure from: Hasan et al., Rev. Mod. Phys. 82, 3045 (2010).

1D Kitaev Chain

A. Y. Kitaev. Physics-Uspekhi, 44:131, 2001.

$$\mathcal{H}_{\rm chain} = -\mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} \left(t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + h.c. \right) ,$$

spinless p-wave superconductor on a chain

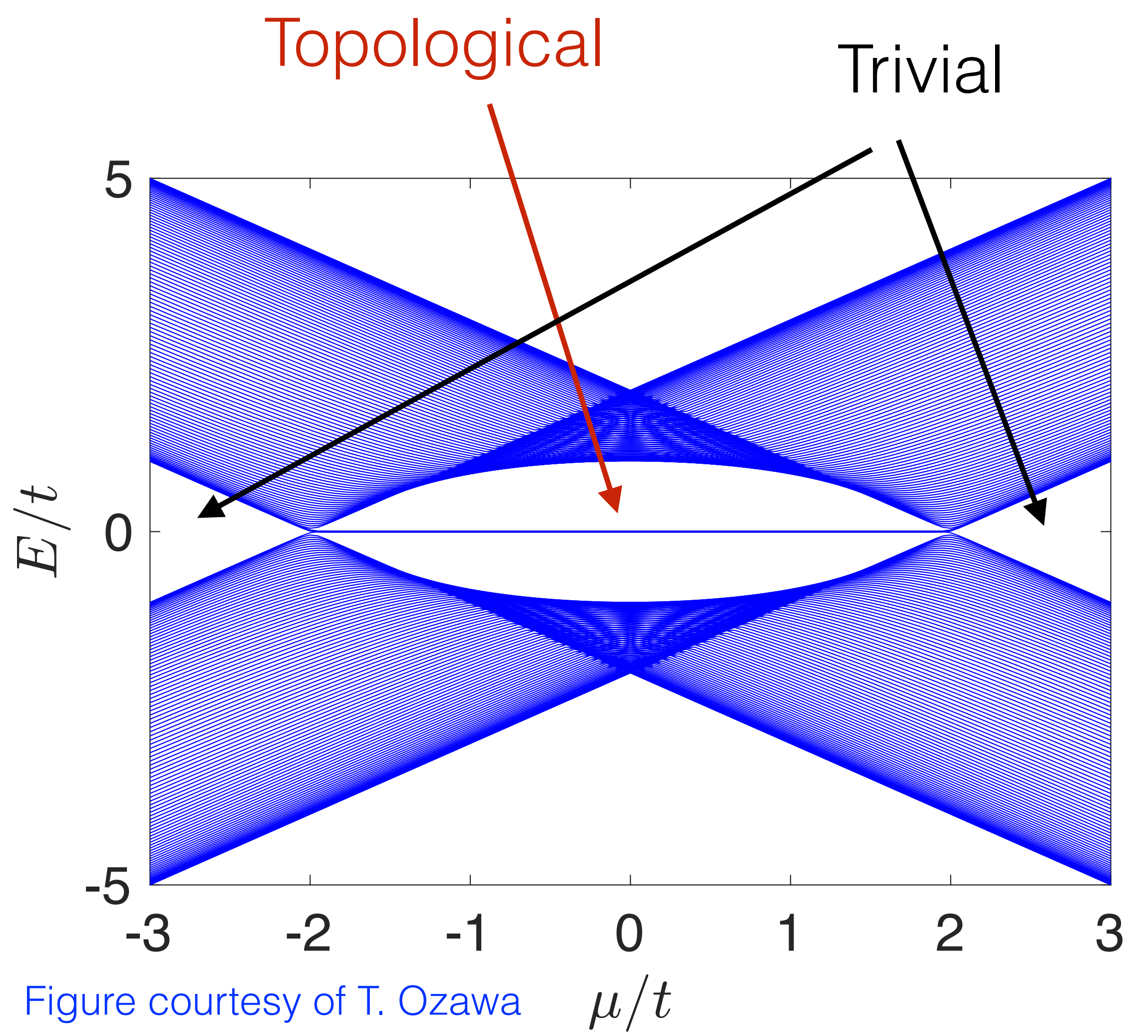
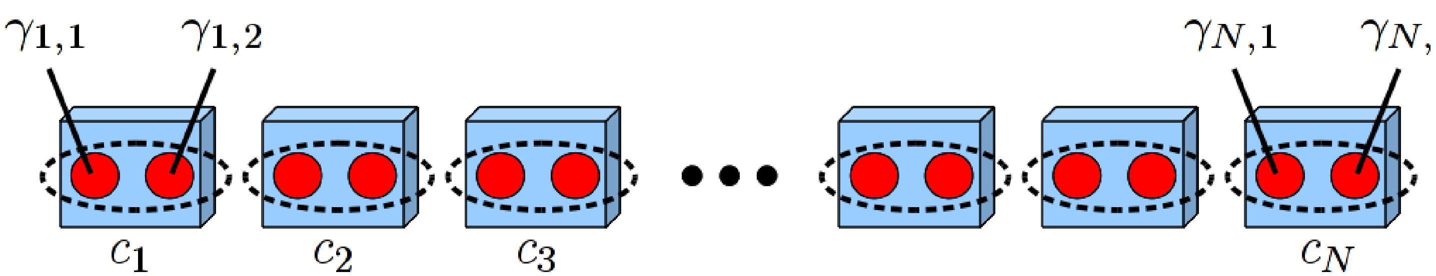


Figure courtesy of T. Ozawa

Spectrum for N=100 and $\Delta = t$

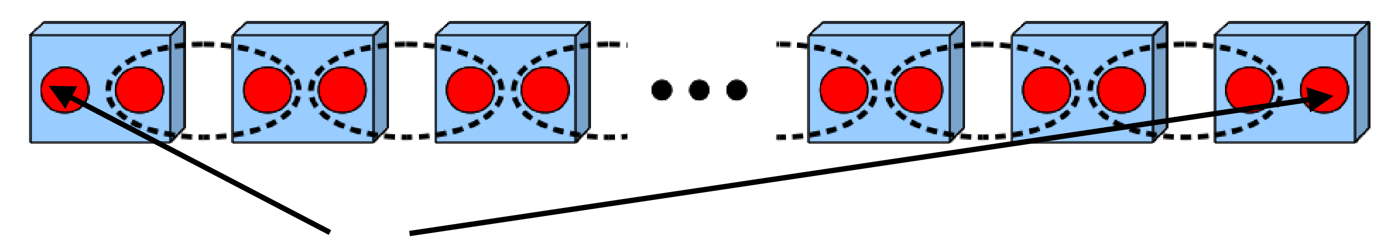
“Splitting” the Dirac fermions into Majorana fermions

$$c_i = \frac{1}{2}\left(\gamma_{i,1} + i\gamma_{i,2}\right), \qquad c_i^\dagger = \frac{1}{2}\left(\gamma_{i,1} - i\gamma_{i,2}\right),$$



At $\mu = 0, \quad \Delta = t$

$$\mathcal{H}_{\rm chain} = -it \sum_{i=1}^{N-1} \gamma_{i,2}\gamma_{i+1,1}.$$



Don’t appear in the Hamiltonian —> E=0 edge states

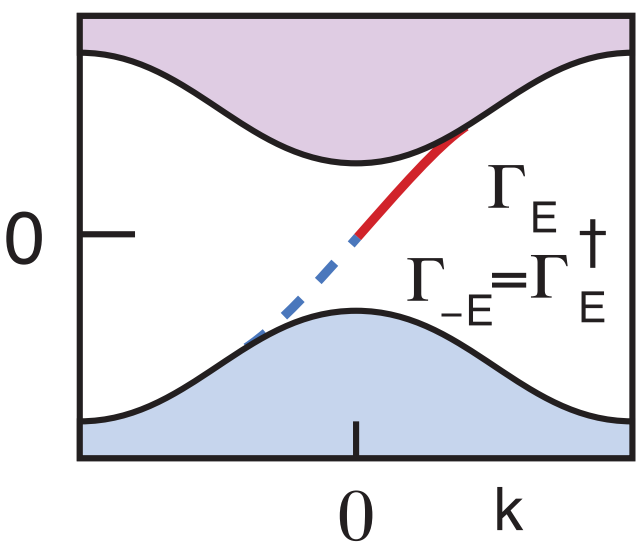
The end Majorana fermions define a degenerate 2-state system: a qubit!
An electron in this qubit will be in a **non-local** superposition of the two edge modes.

Review: Leijnse et al., Semicond. Sci. Technol. 27, 124003 (2012)

2D Topological Superconductors

	Symmetry			d							
	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$

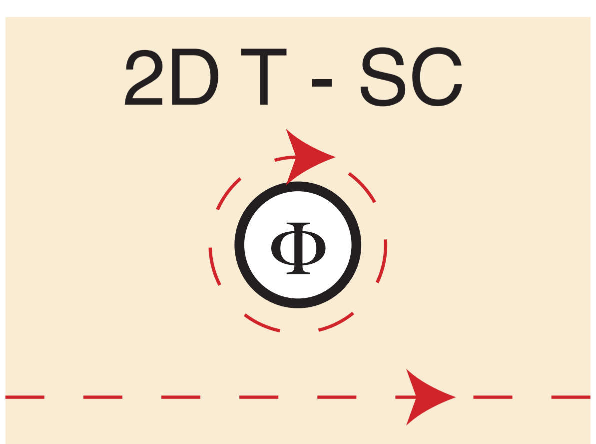
What happens in a 2D topological superconductor?



$\nu_{\mathrm{2D\ SC}} \in \mathbb{Z}$ gives no/ chiral Majorana edge modes

[Reminiscent of Chern number and QH chiral edge states in Class A]

Simple model: spinless superconductor with $p_x + ip_y$ pairing



Vortices in a topological superconductor can host **bound Majorana zero-modes**

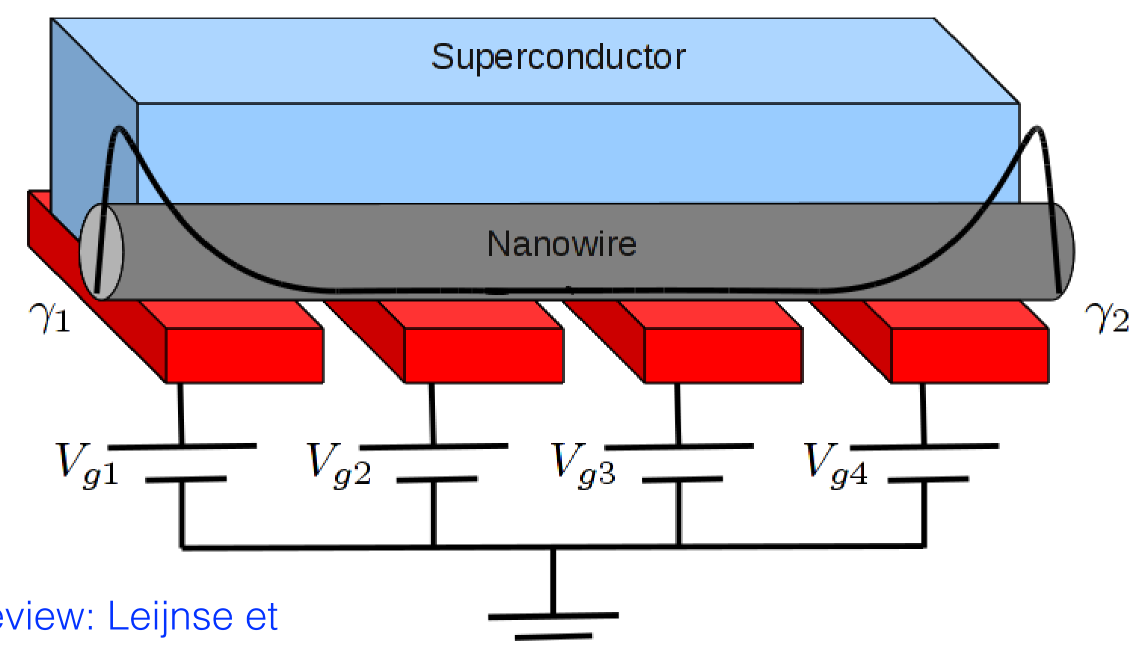
Figure from: Hasan et al., Rev. Mod. Phys. 82, 3045 (2010).

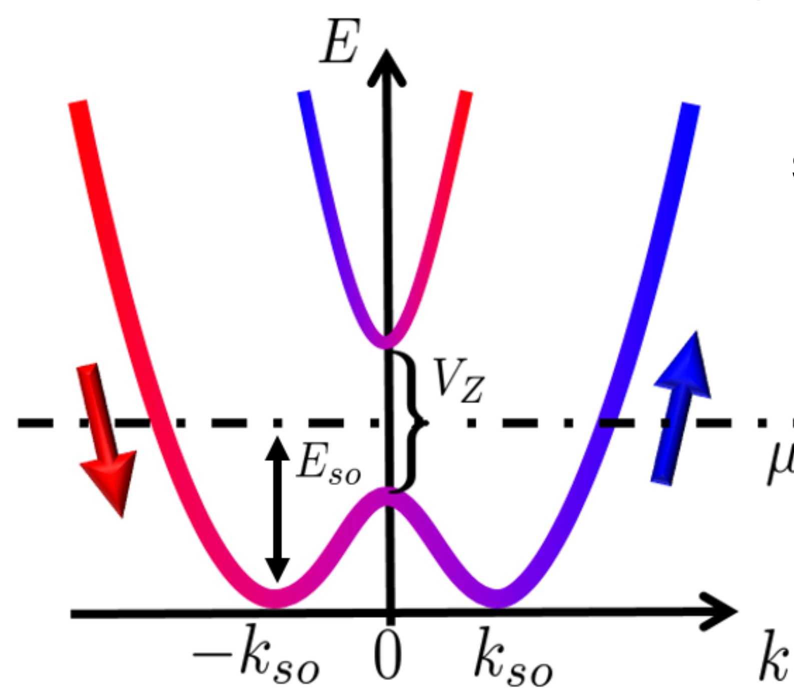
Hunting for Majorana Fermions

Majorana fermions are not so easy to realise experimentally:

- Quasiparticles in some fractional quantum Hall states?
- Unconventional superconductors, e.g. Sr_2RuO_4?
- Proximity effect devices (superconductors coupled to TIs, semiconductors or magnetic atoms…)

spin-orbit coupled nanowire with proximity induced s-wave pairing





spin-orbit coupling can provide a route to "spinless" superconductivity

Review: Leijnse et al., Semicond. Sci. Technol. 27, 124003 (2012)

Mounting experimental evidence, see recent review : Lutchyn et al., arXiv:1707.0489

And now…

	Symmetry			Dimensionality d								
Class	Time-reversal	Particle-hole	Chiral	1	2	3	4	5	6	7	8	
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	← Quantum Hall
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	← SSH Model
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	← Topological Superconductors
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	← Topological Insulators
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	

Can we realise these topological phases in ultracold atoms and photonics?
Lecture 3 & 4