“Quantum Bounds, Estimation, and Metrology”

lecture III

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QUANTUM METROLOGY

“ how to use quantum effects in order to improve our ability in probing reality ...“

parameter estimation

Hellstrom
Holevo ...

remote detection

hypothesis testing

Outlook

- Quantum Measurements ....
- State Discrimination problem ....
- Process Discrimination problem ....
- Parameter Estimation ....
- Heisenberg vs. Shot Noise limit ....

Quantum Measurements

In quantum mechanics a Quantum Measurement (QM) is any process which allows us to acquire (classical) information on the state of a (quantum) system.

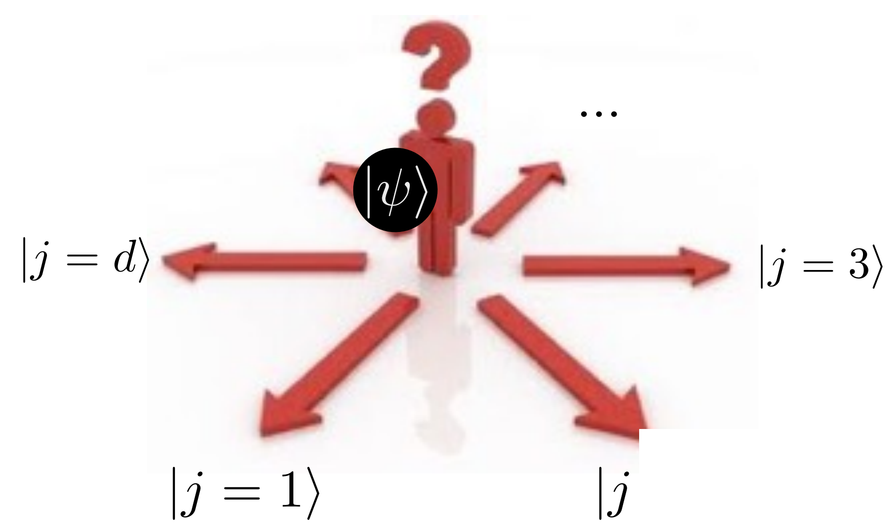
"ability of discriminating among different alternatives"

"present condition of a system"

PROJECTIVE (or von Neumann) MEASUREMENTS: (simplest and more fundamental form of quantum measurements)

A Projective Measurement (PM) *tries* to identify the state $|\psi\rangle$ of the system among a collection of orthonormal configurations (basis of S):

$$\{|j\rangle\}_{j=1,\cdots,d} \qquad\qquad \langle j|j'\rangle = \delta_{jj'} \ .$$

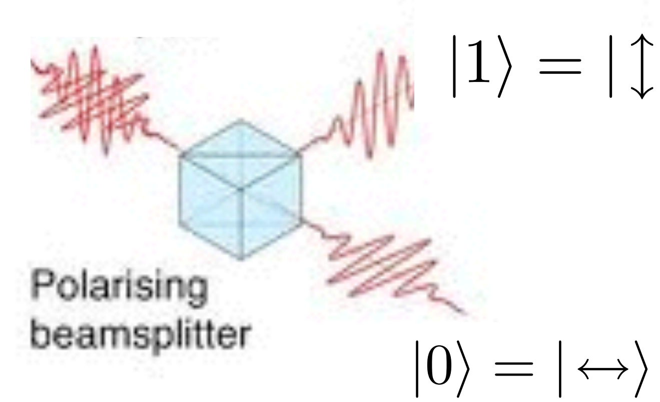


BORN RULE

$$p(j|\psi) = |\langle j|\psi\rangle|^2 \ ,$$

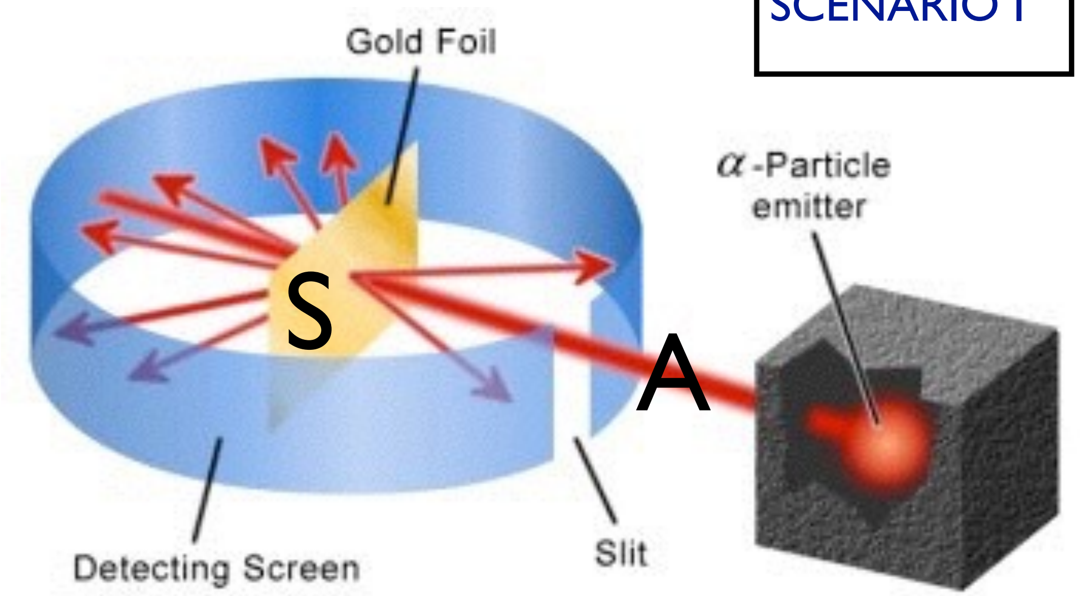
(for mixed state $p(j|\rho) = \langle j|\rho|j\rangle$).

Example: *polarized single photon*

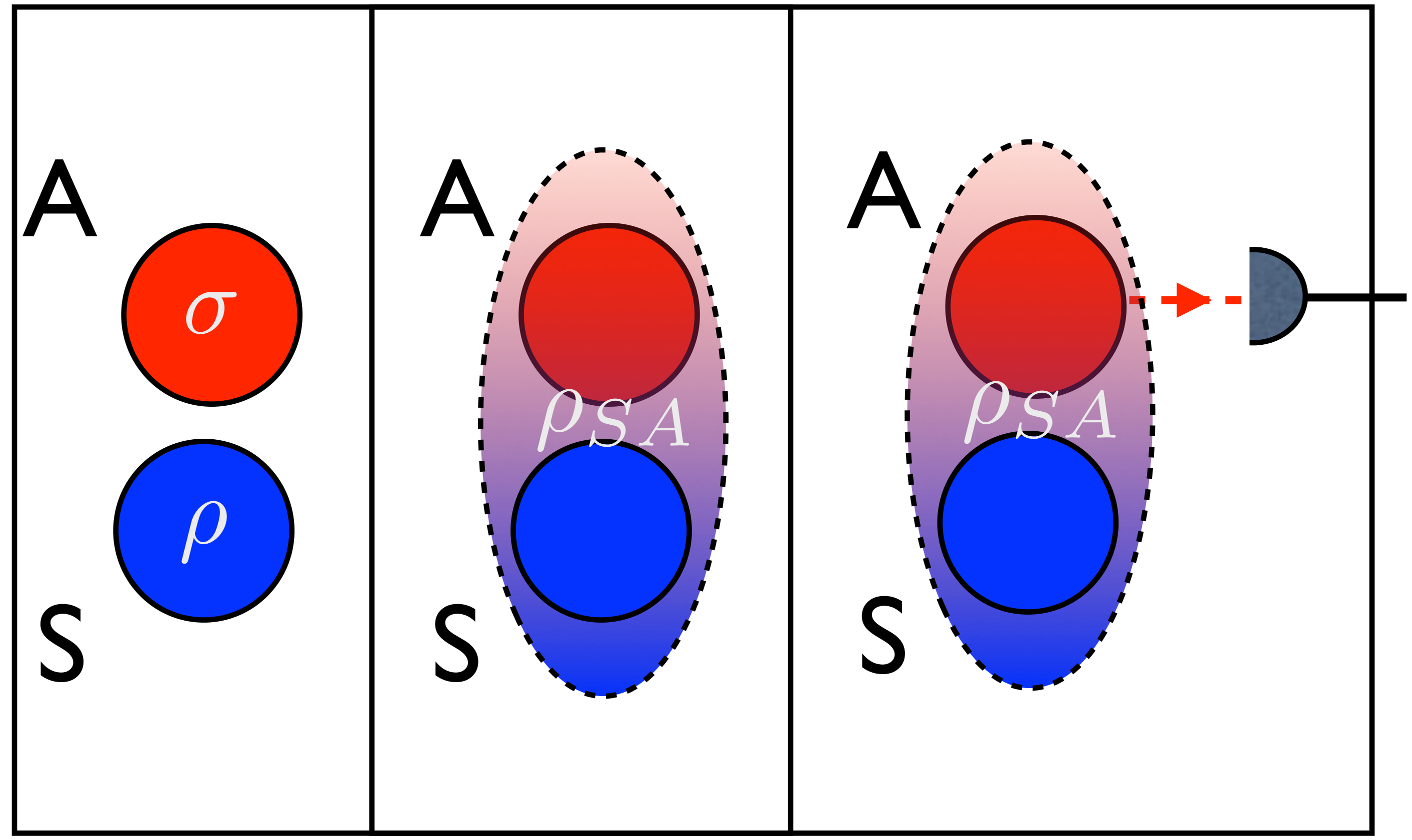


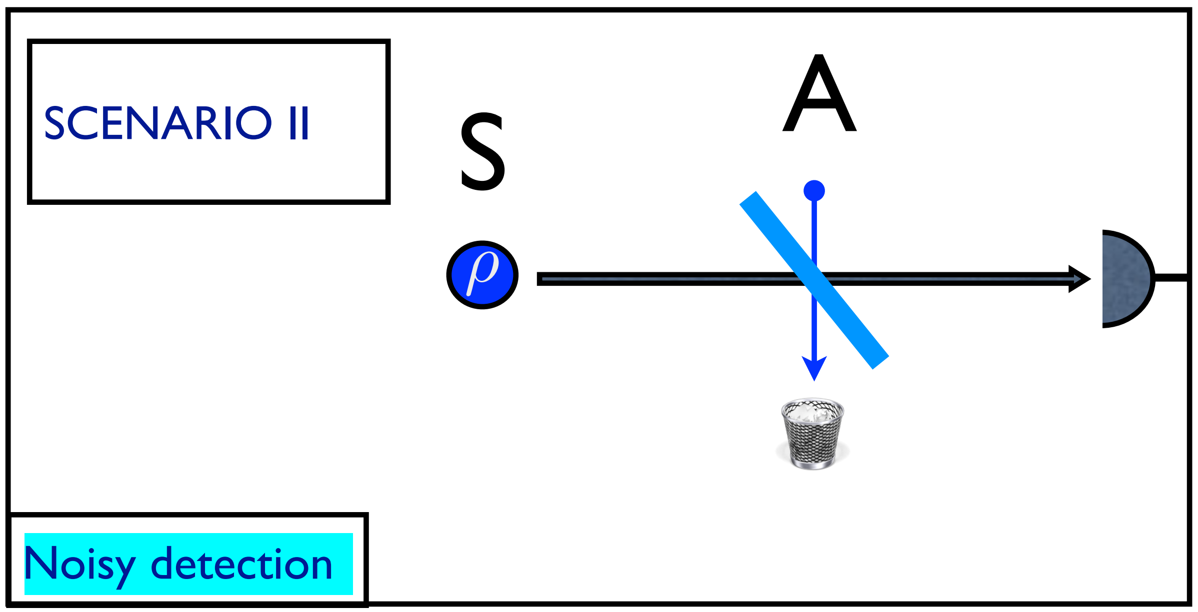
Beyond PROJECTIVE MEASUREMENTS:

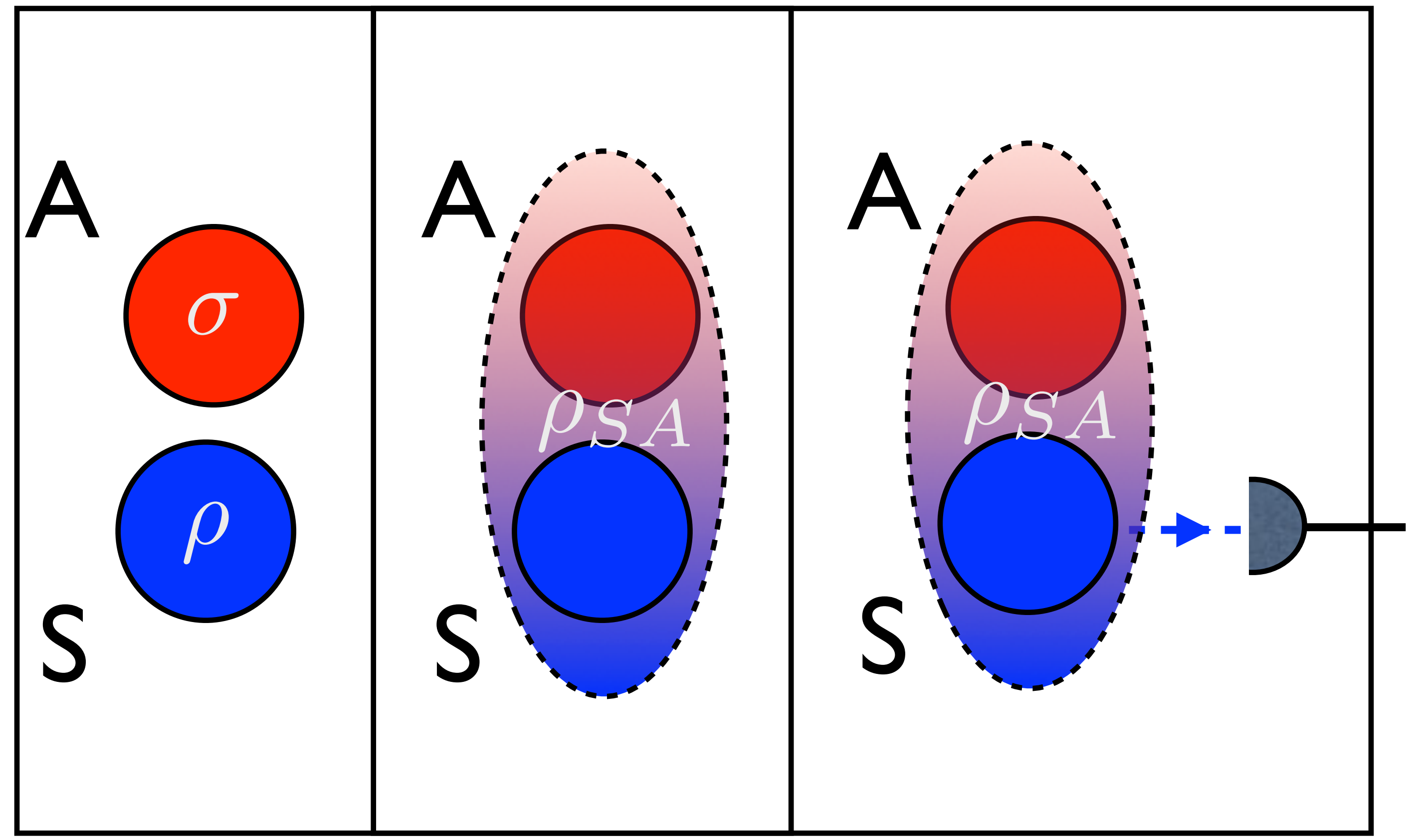
SCENARIO I

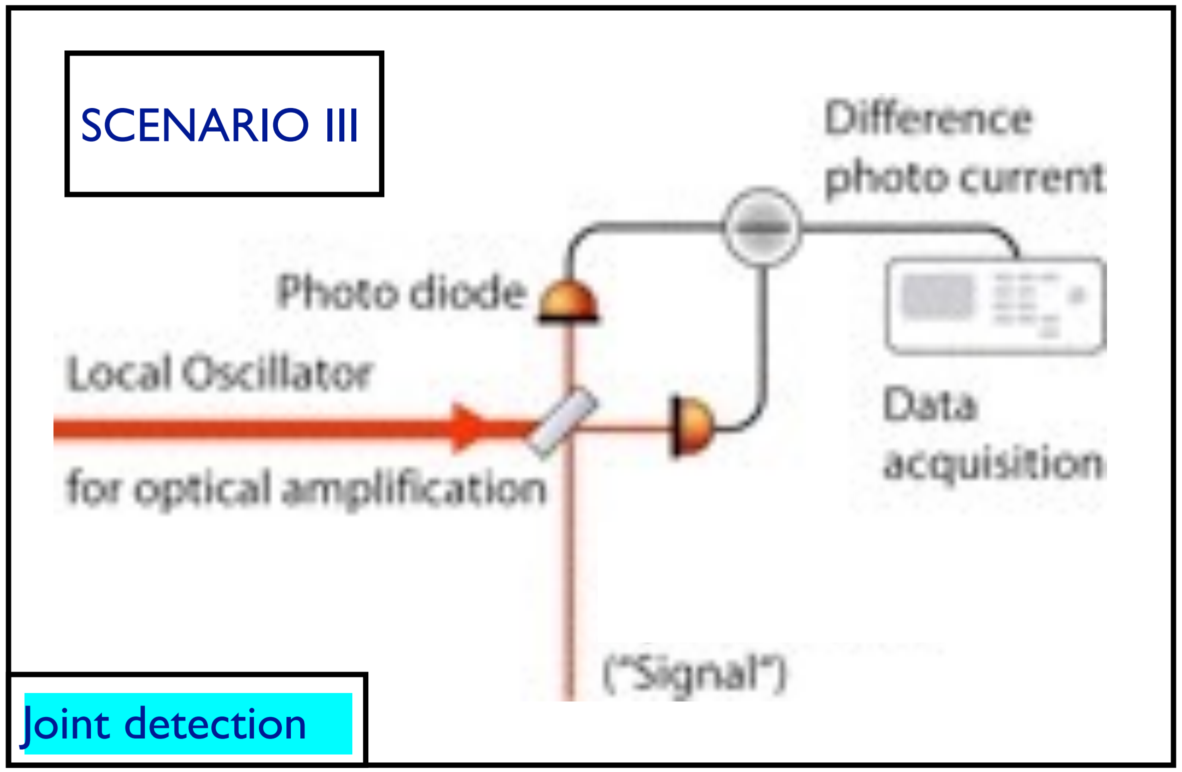


Indirect or mediated measurements

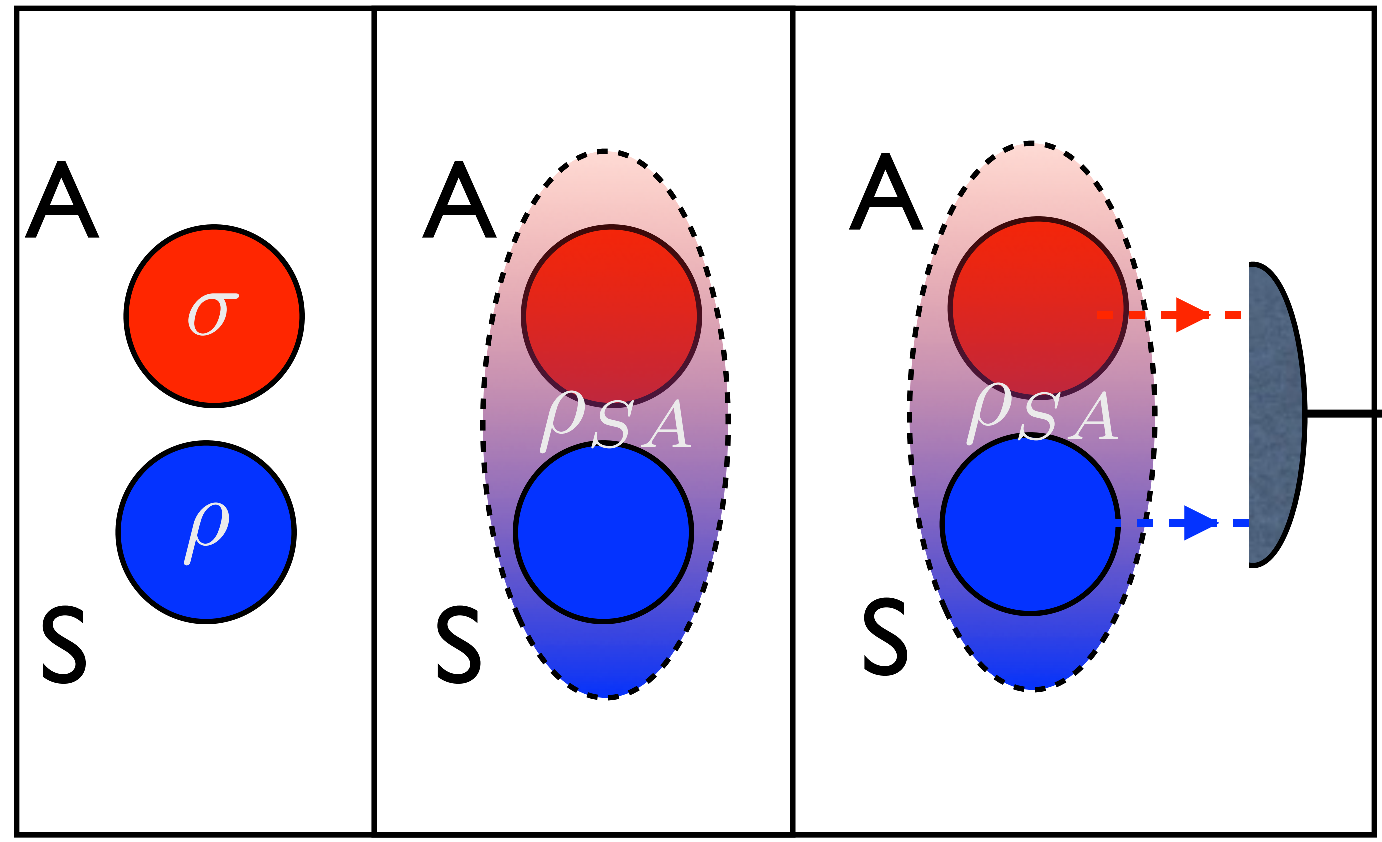



SCENARIO II
S
A
ρ
Noisy detection

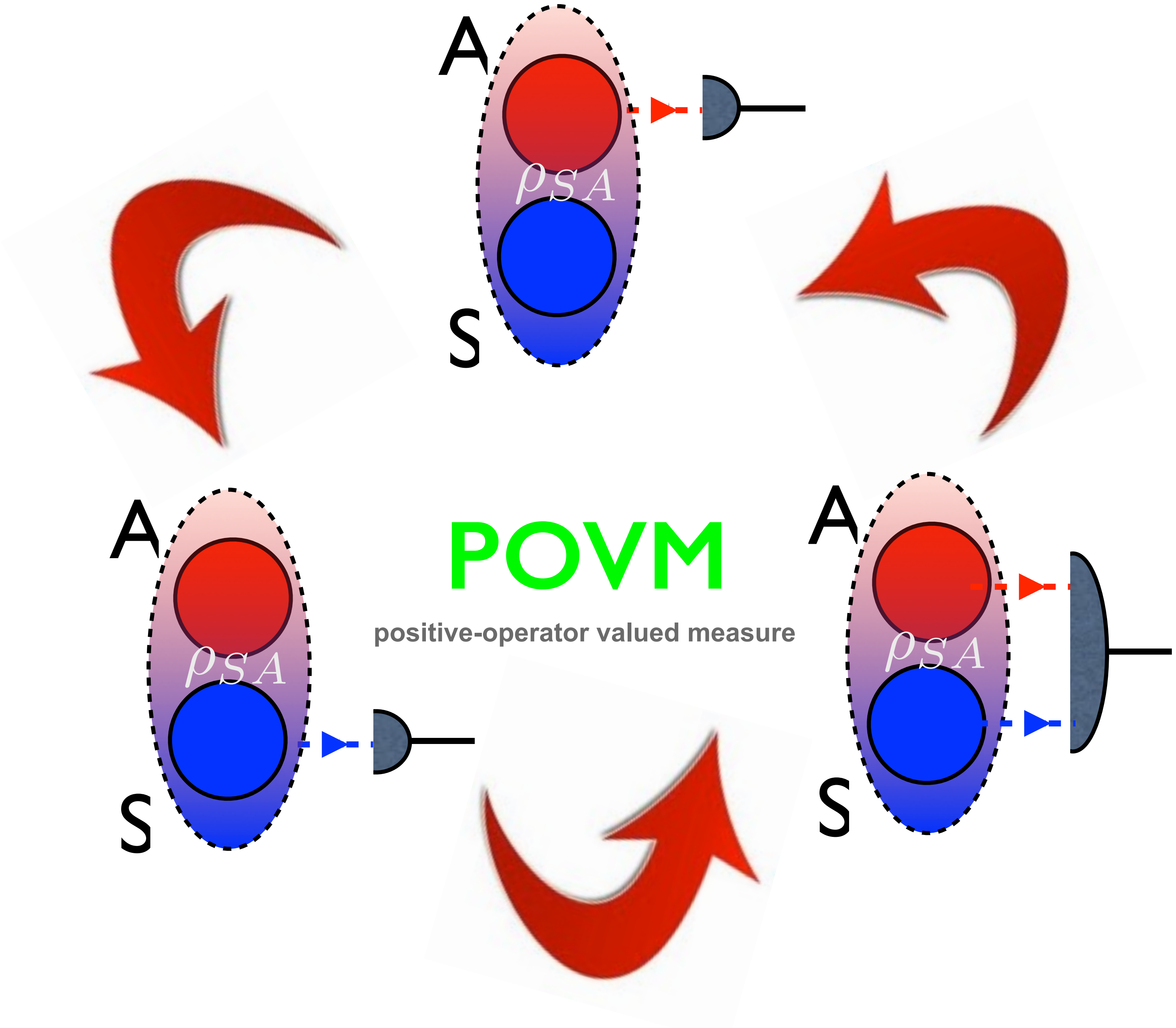

A
S
σ
ρ
ρ_{SA}
ρ_{SA}



SCENARIO III
Difference photo current
Photo diode
Local Oscillator
for optical amplification
Data acquisition
("Signal")
Joint detection



A
S
σ
ρ
ρ_{SA}



$$E_j \geqslant 0$$

$$\sum_{j=1}^{n} E_j = I$$

$$p(j|\rho) = \mathrm{Tr}[E_j \rho]$$

Limitations

TECHNOLOGICAL LIMITS

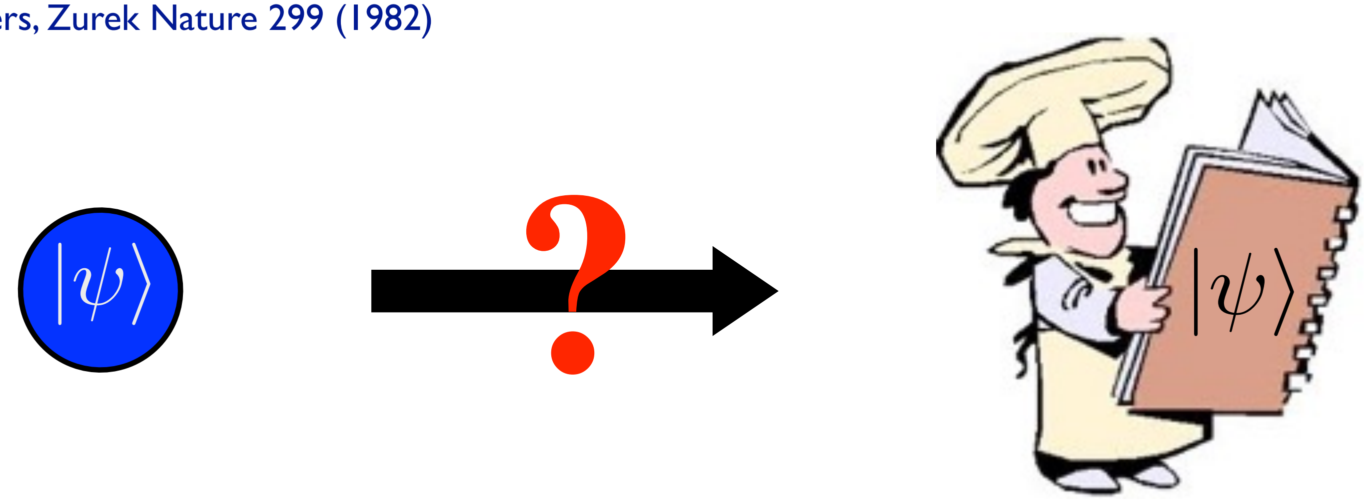
UNCERTAINTY RELATIONS

$$\Omega\Theta \neq \Theta\Omega \quad \Rightarrow \quad \Delta\Omega\Delta\Theta \geq \frac{|\langle[\Omega,\Theta]\rangle|}{2}$$

incompatible observables — Robertson inequality

NO CLONING

Wootters, Zurek Nature 299 (1982)



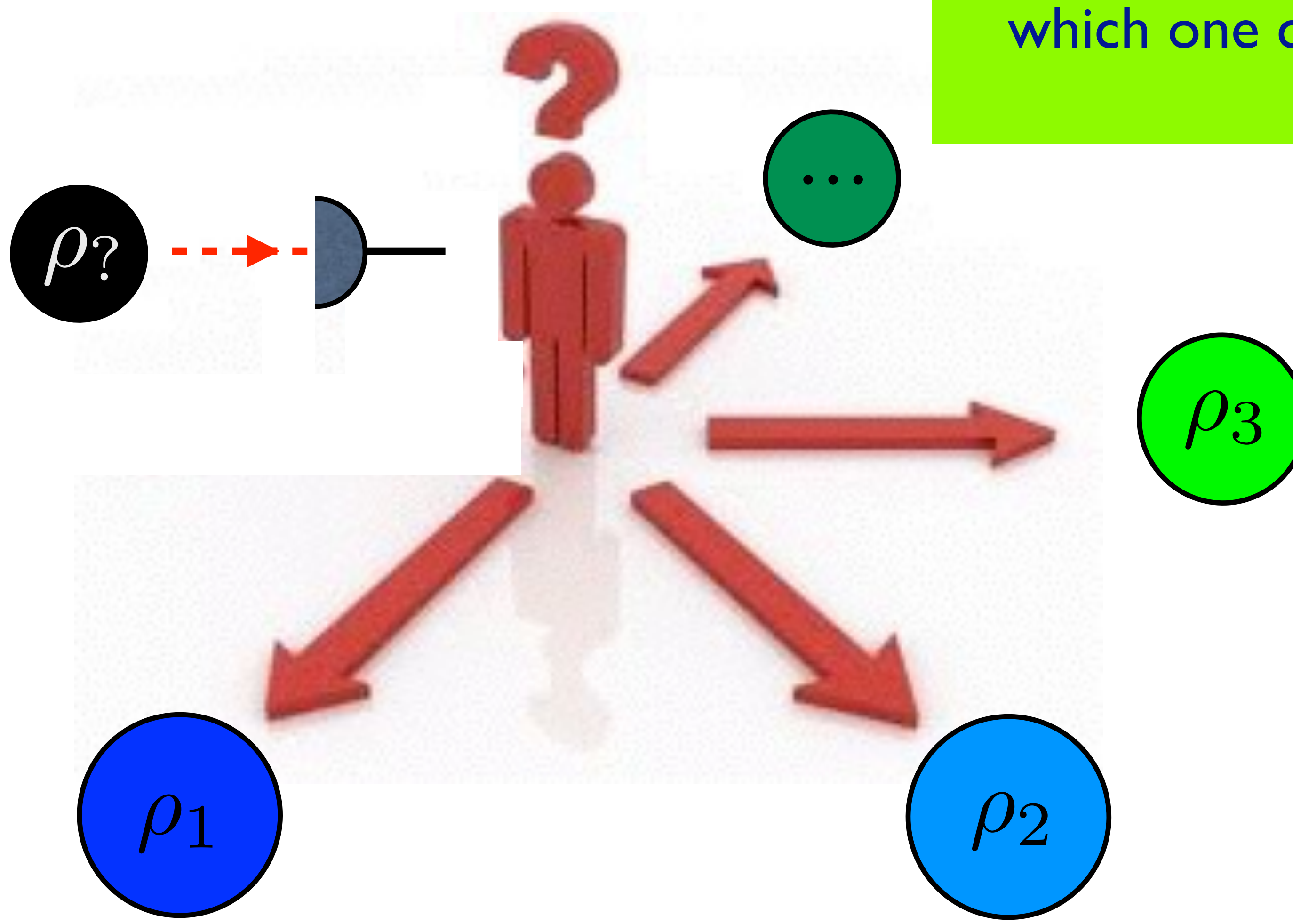
THE MOTHER OF ALL PROBLEMS in Q-METROLOGY

State Discrimination

Given a finite collection of possible states,

$$\rho_1, \rho_2, \cdots, \rho_n$$

and a *single copy* of a state $\rho_?$ extracted randomly from the set of possible states, determine which one correspond to $\rho_?$.

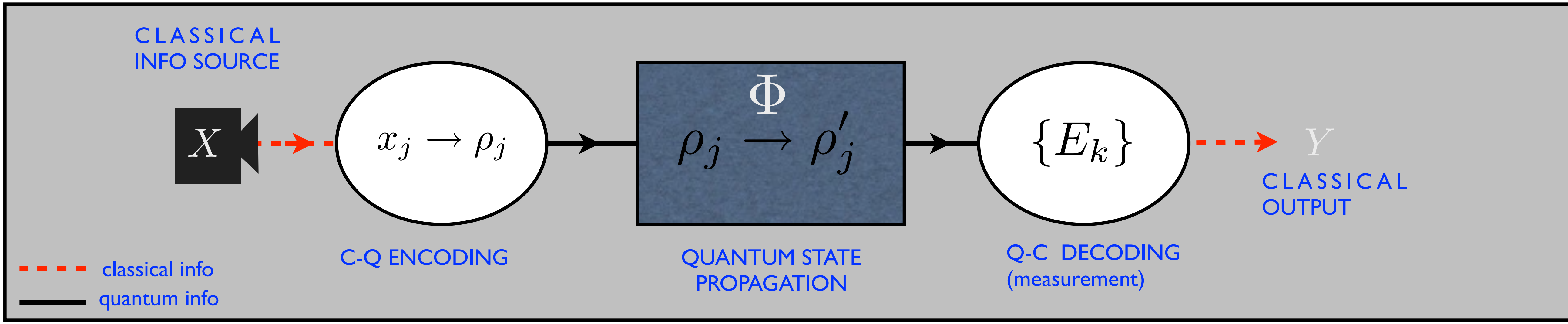


Find the (optimal) POVM which gives the best chance of success [e.g. the lowest error probability]

NB: $\rho_?$ is one of the selected states $\rho_1, \rho_2, \cdots, \rho_n$ but, a priori, we don't know which one.

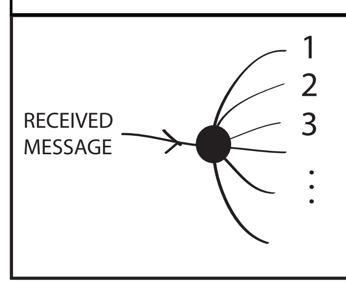
Quantum Communication

Transferring Classical Info over a quantum channel

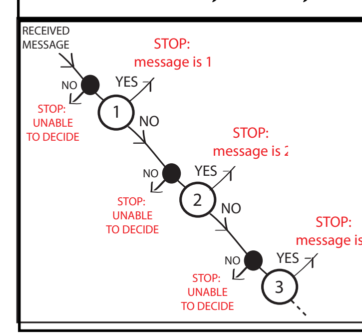


capacity achieved by
POVMs
which are asymptotically
optimal ...
(built upon the notion of
typical spaces)

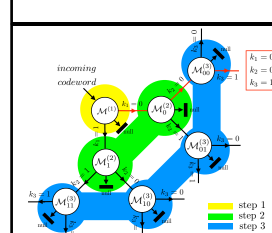
"pretty good measurement"
HOLEVO, SCHUMACHER, WESTMORELAND (1998)



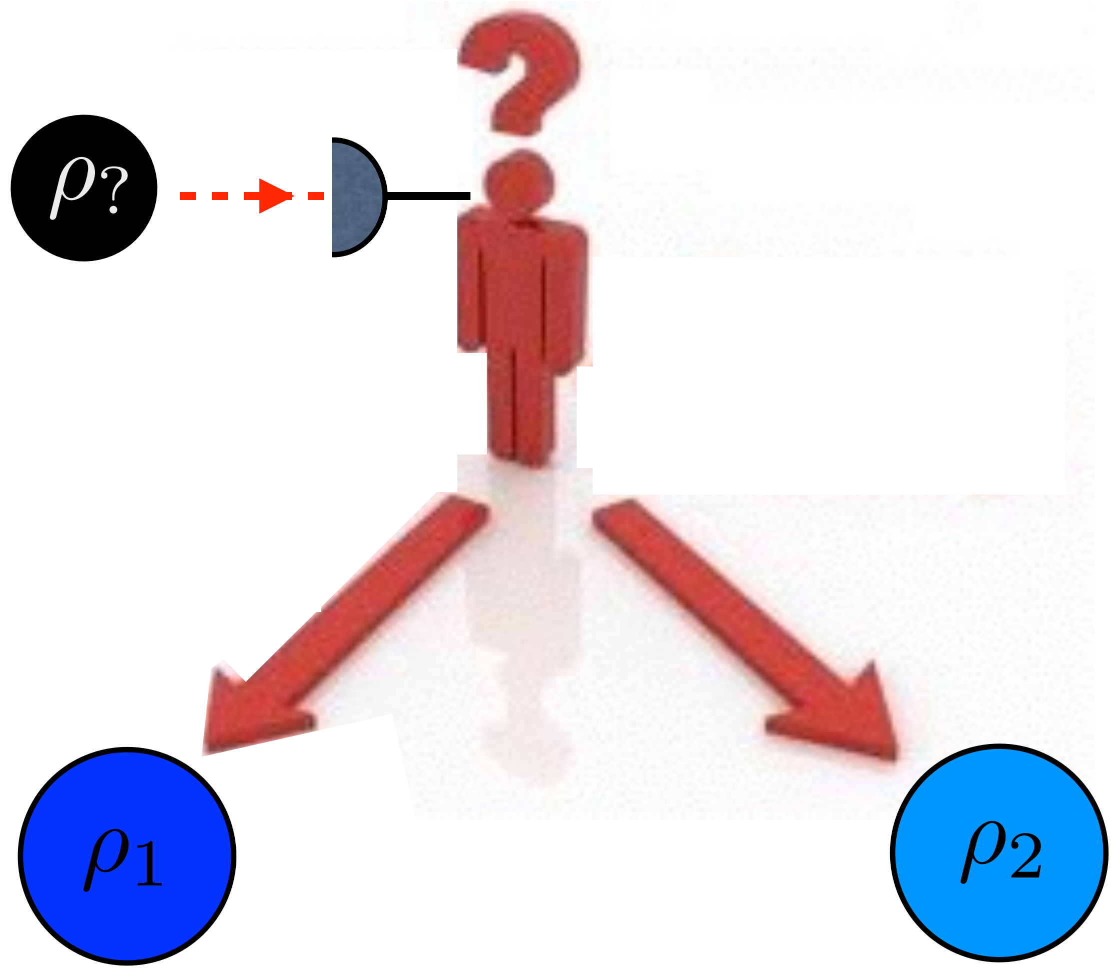
sequential decoding schemes
LLOYD, VG, MACCONE (2011)



bisection decoding schemes
ROSATI, VG (2015)



Two-state discrimination problem with a single copy of the unknown state



Hellstrom Bound

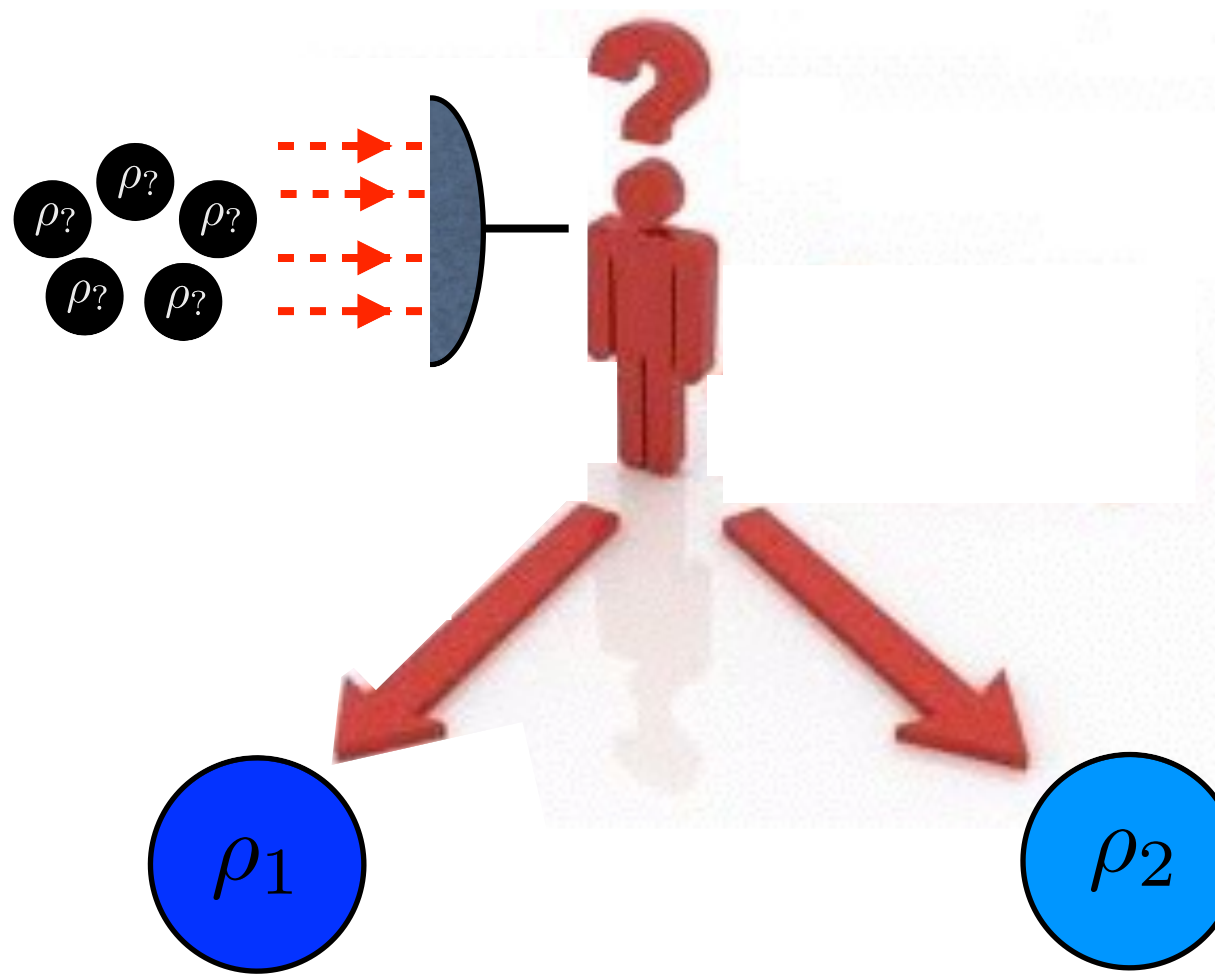
the optimal measurement is a (non trivial) von Neumann projection

$$P_E \geq P_E^{(min)} = \frac{1 - \|\rho_1 - \rho_2\|_1/2}{2}$$

$$\|\Theta\|_1 = \mathrm{Tr}[\sqrt{\Theta^\dagger \Theta}]$$

THE QUANTUM CHERNOFF BOUND

Two-state discrimination problem with N copies of the unknown state

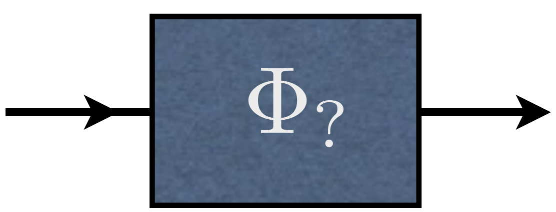


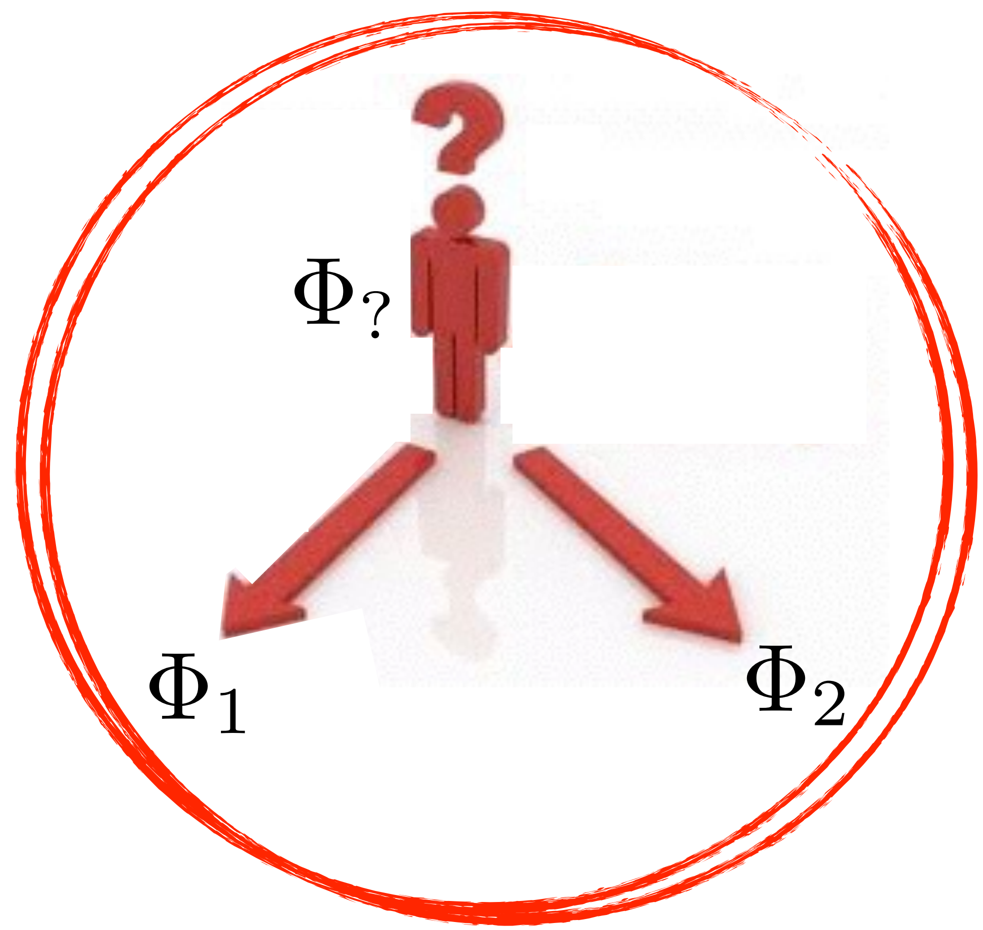
$$P_E^{(N)} = \frac{1 - \|\rho_1^{\otimes N} - \rho_2^{\otimes N}\|_1/2}{2} \lesssim \exp[-N\,\xi_{QCB}]$$

$$\xi_{QCB} = -\log\left\{\min_{0\leq s\leq 1} \mathrm{Tr}[\rho_1^s \rho_2^{(1-s)}]\right\}$$

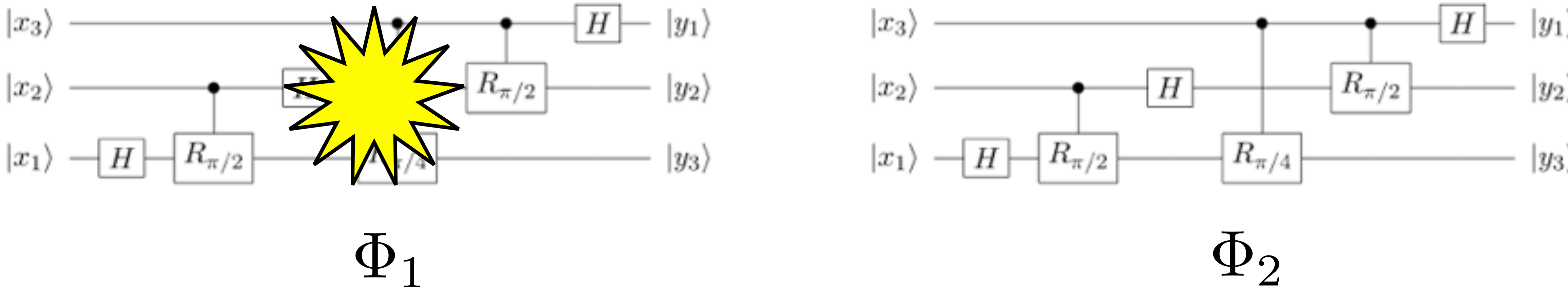
Audenaert et al. Phys. Rev. Lett. (2007)

Process Discrimination



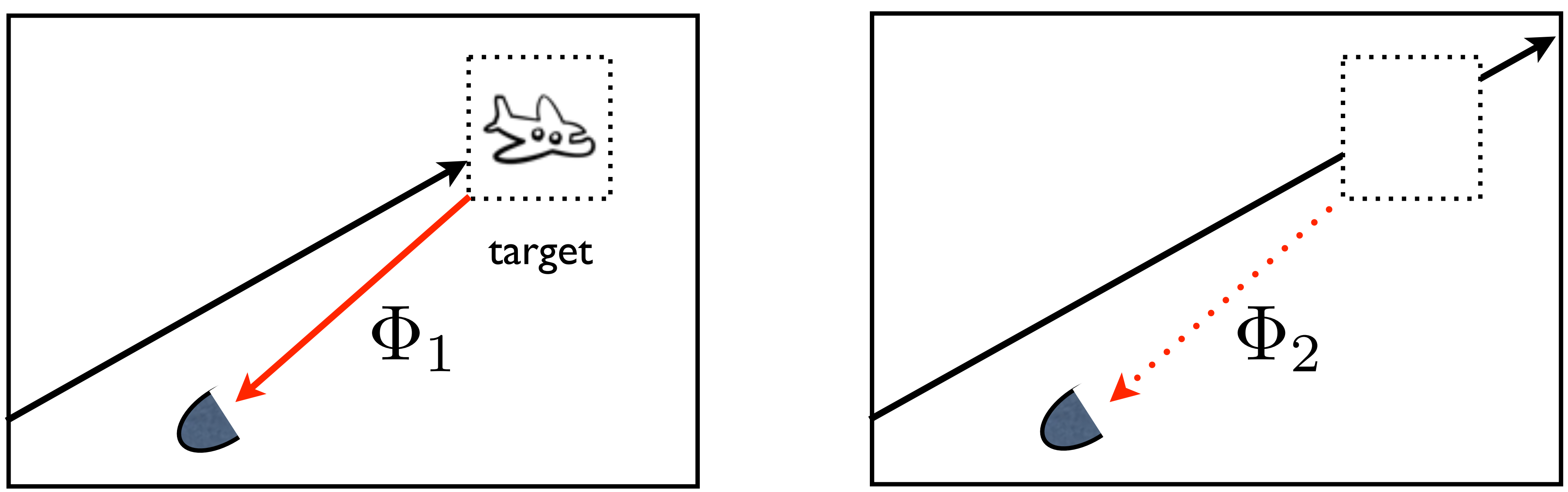


Noise detection



Quantum Illumination

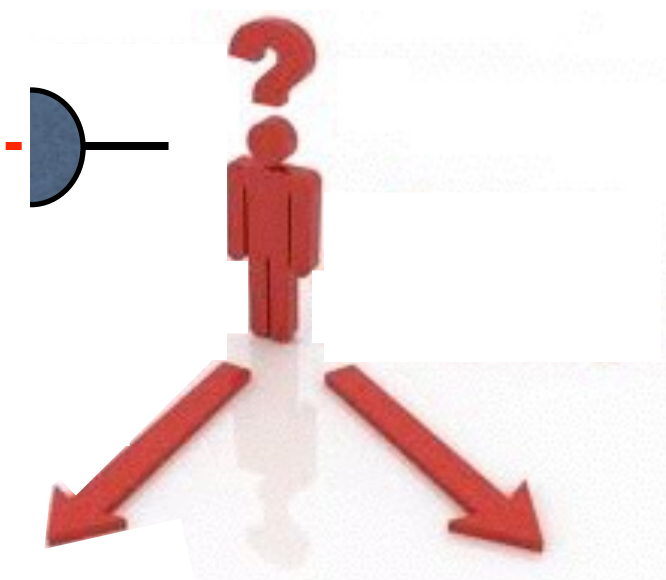
Tan, et al. Phys. Rev. Lett. (2008)



Process Discrimination → State discrimination

Use input state as a probe....

$$\rho \longrightarrow \boxed{\Phi_?} \longrightarrow \rho' = \Phi_?(\rho)$$

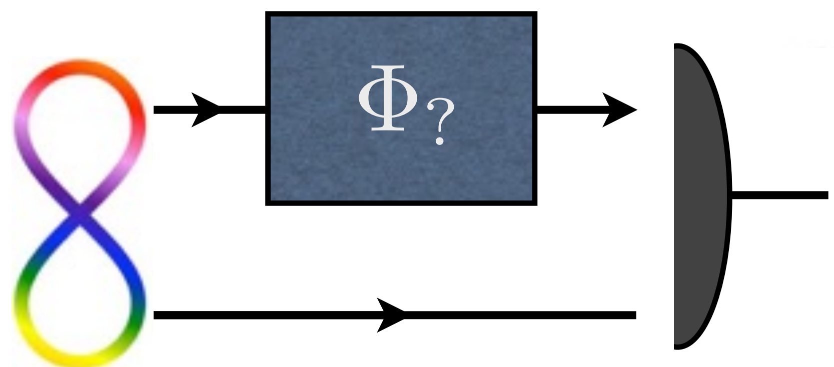


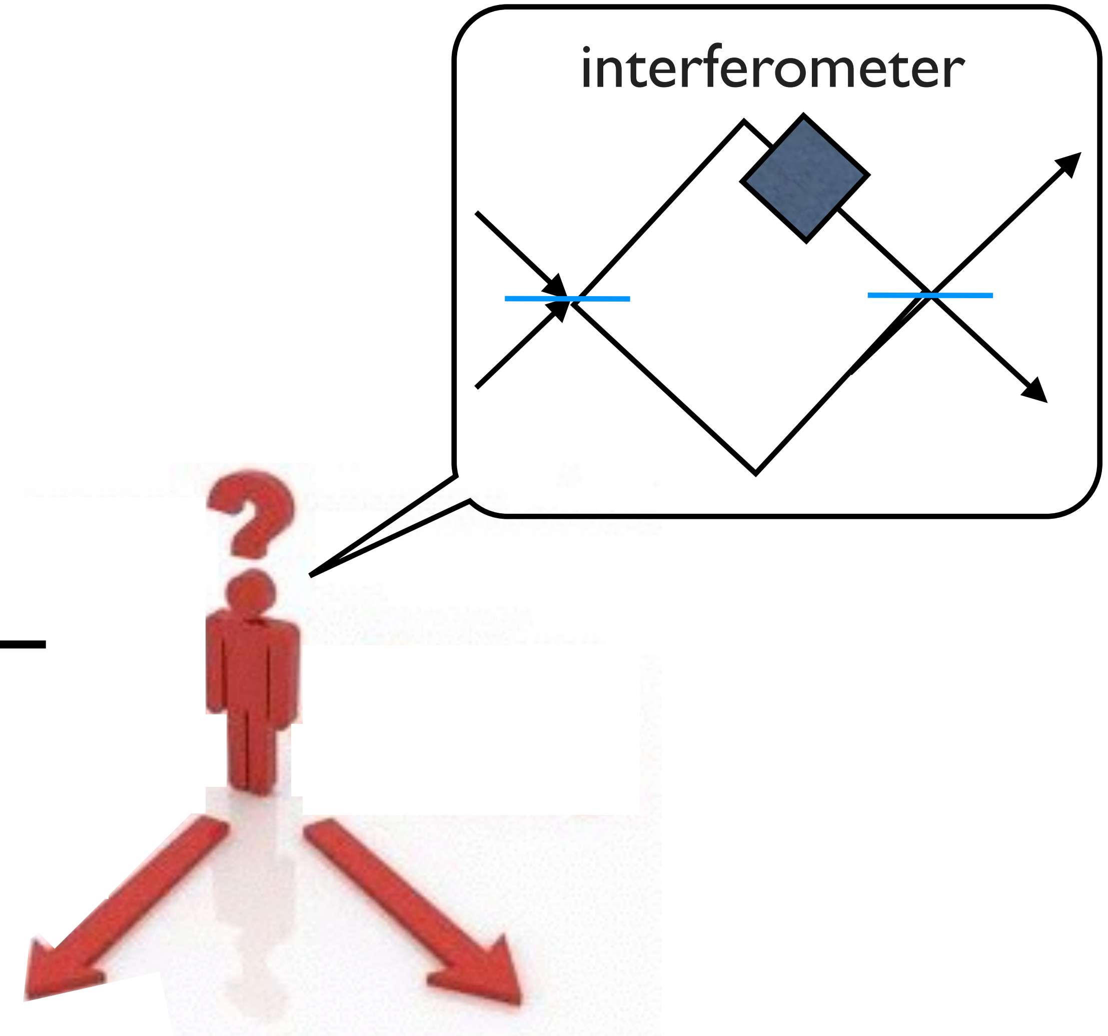
$$\rho_1 = \Phi_1(\rho) \qquad \rho_2 = \Phi_2(\rho)$$

Find optimal input state of the probe

Process Discrimination ➡ State discrimination

side channels + entangled probes





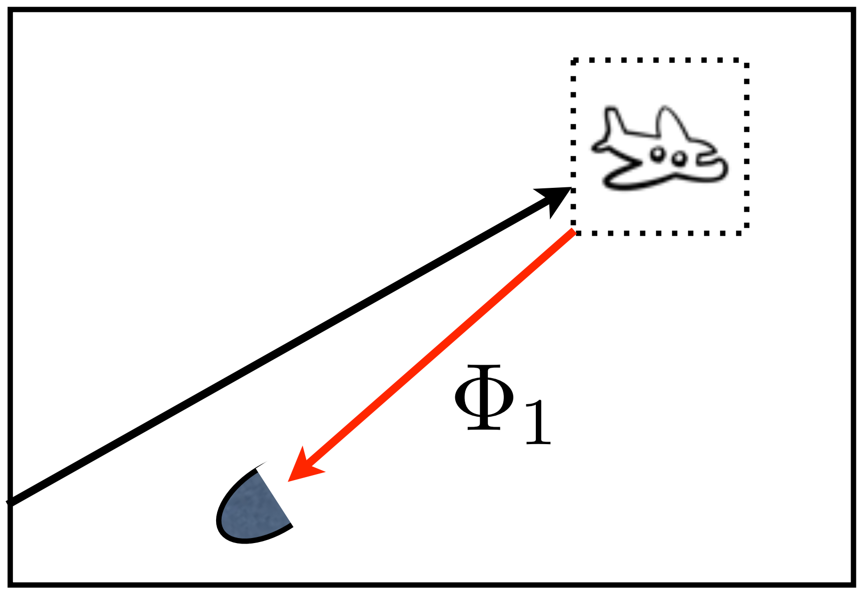
use ancillary system as a reference....

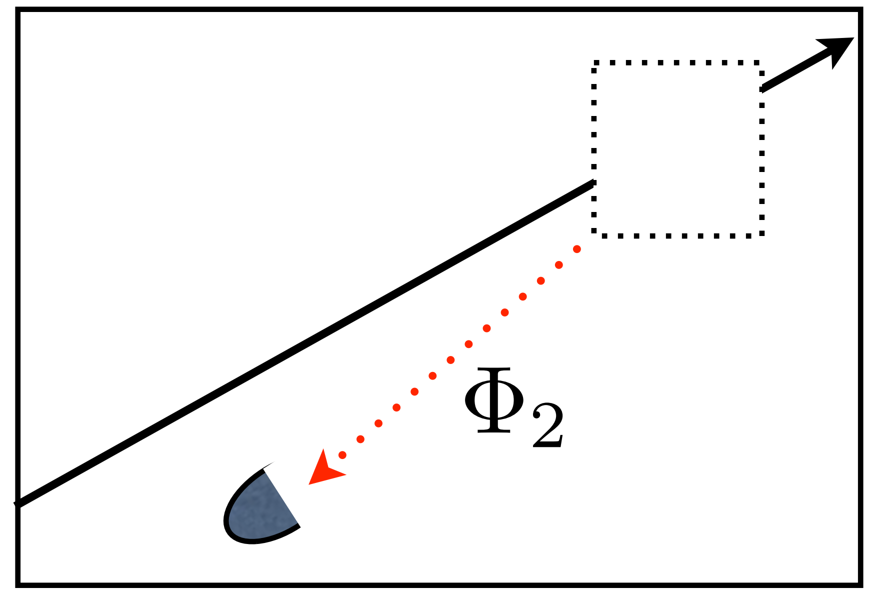
$$\rho_1 = (\Phi_1 \otimes I)(\rho) \qquad \rho_2 = (\Phi_2 \otimes I)(\rho)$$

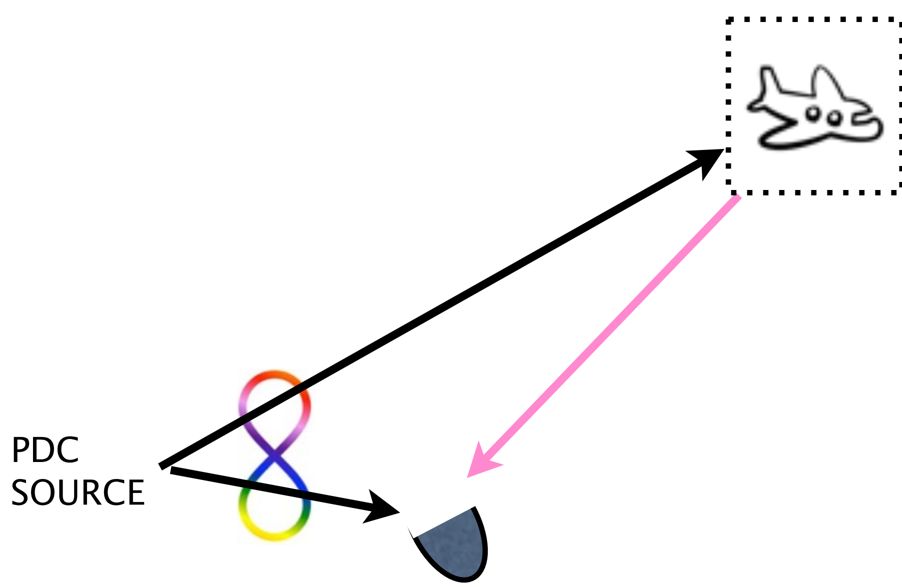
[*Choi-Jamiolkowski isomorphism*]()

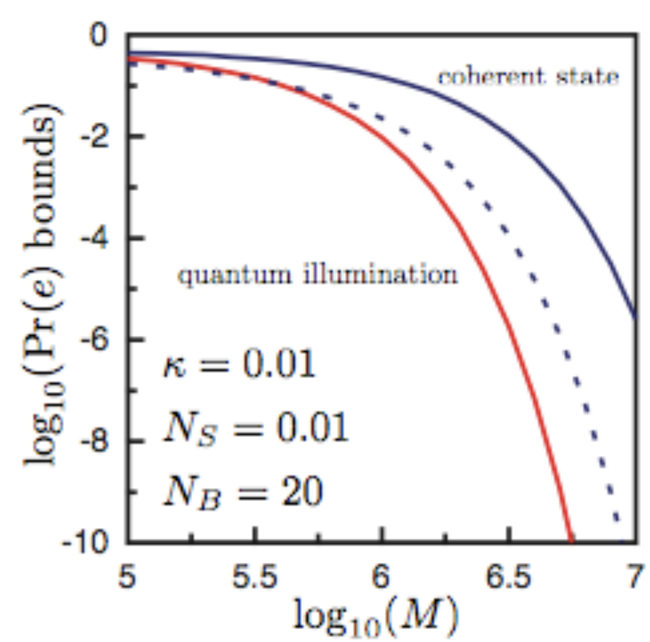
Quantum Illumination

Tan, et al. Phys. Rev. Lett. (2008)

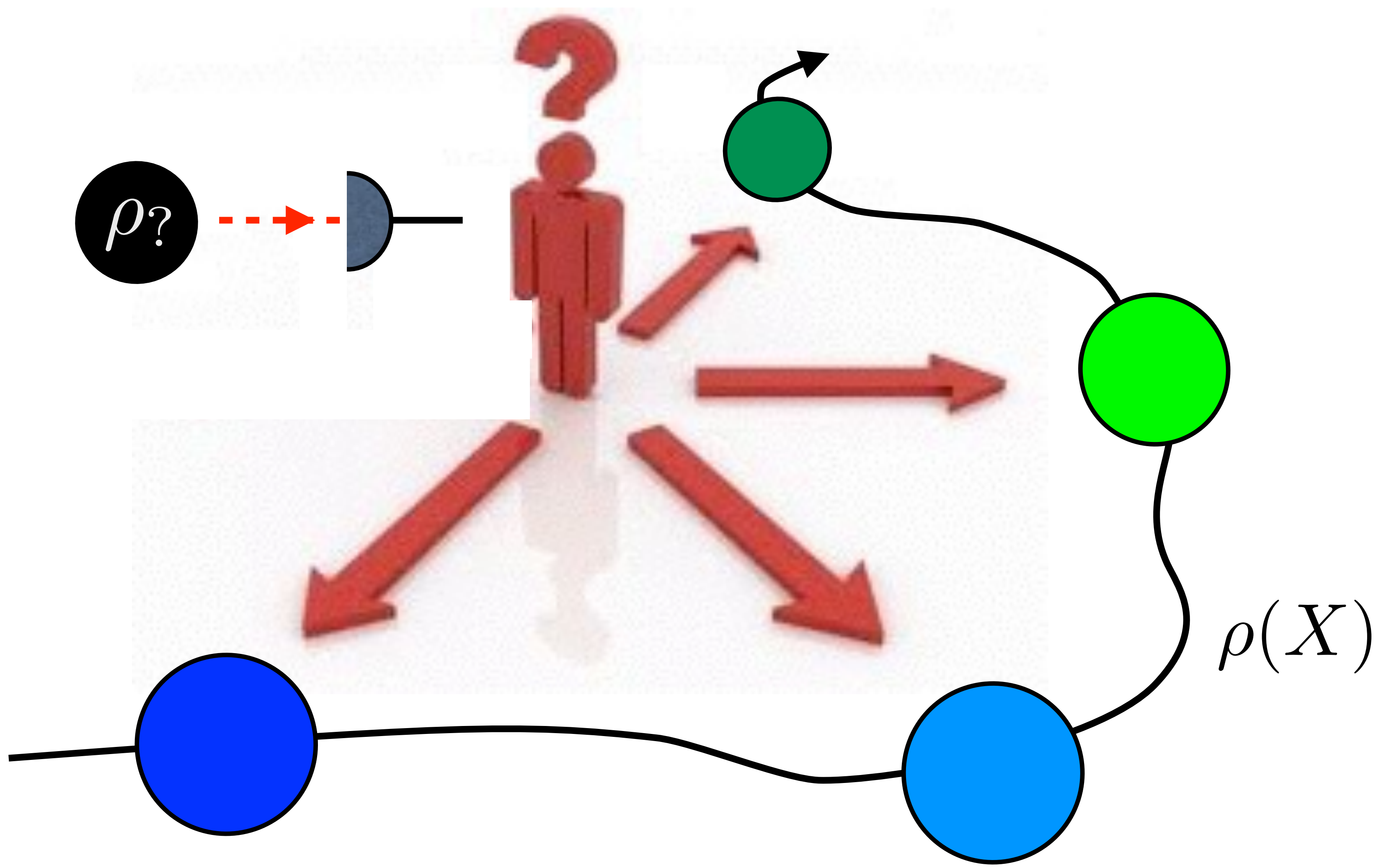








from State Discrimination to Parameter estimation....

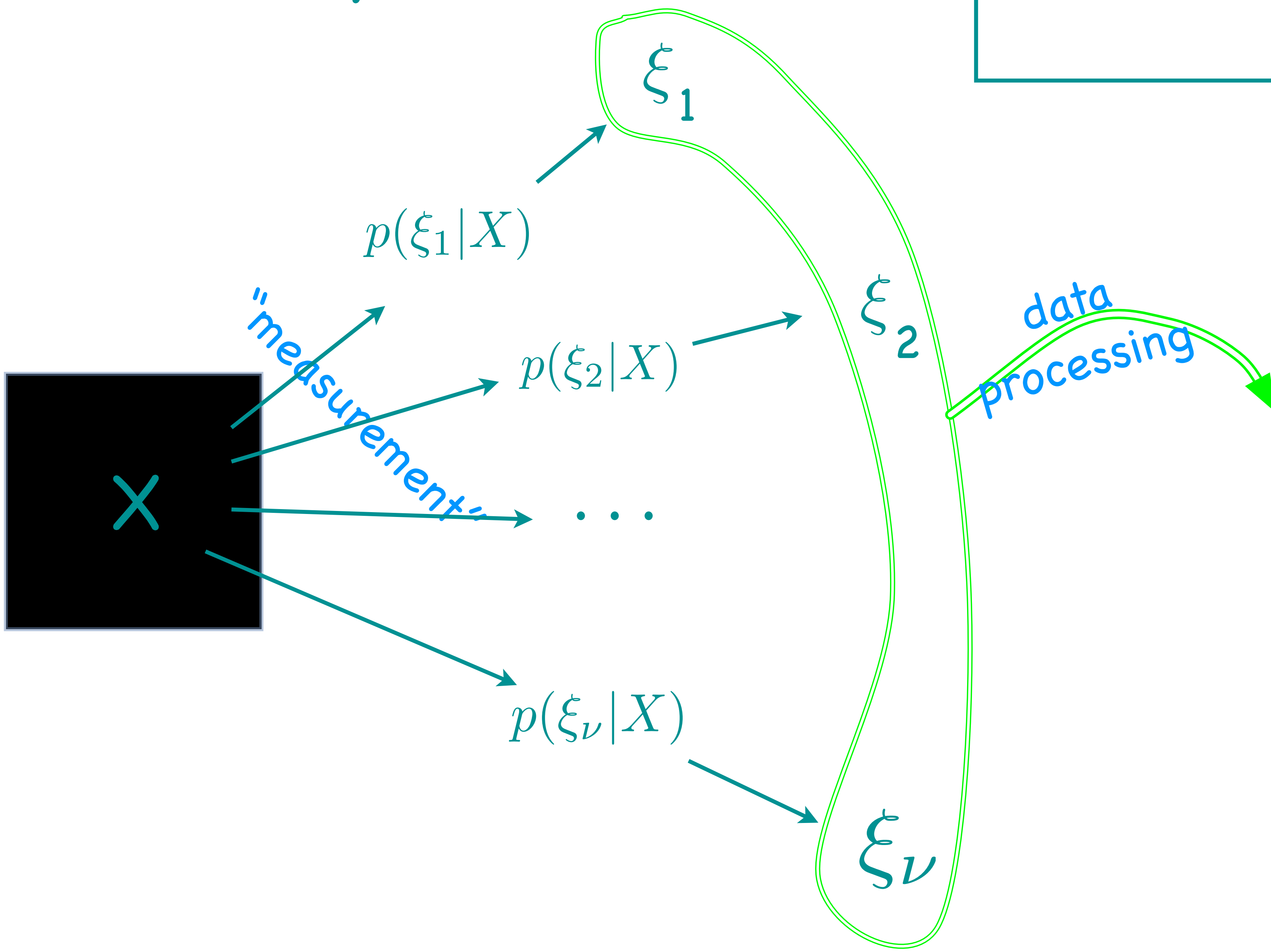


the possible candidates form now a (say) 1-parameter continuous family

determining the state is equivalent to determine the value of the parameter....

Classical theory

$p(\xi|X)$ conditional probability of getting ξ given the value X



$$\vec{\xi} \equiv (\xi_1, \xi_2, \cdots, \xi_\nu)$$

$$X_{est}(\vec{\xi})$$

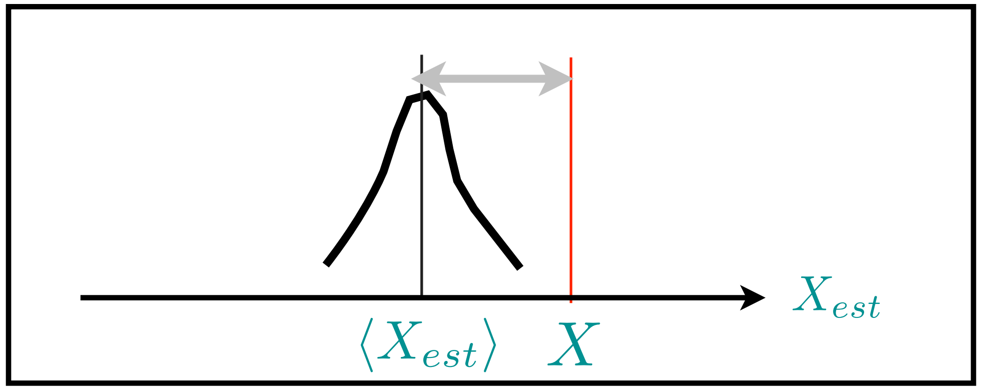
Estimation of X after ν measurements

i.i.d. process $$P(\vec{\xi}|X) = \prod_{j=1}^{\nu} p(\xi_j|X)$$

Figure of merit of the achievable accuracy:

Root Mean Square Error (RMSE)

$$\delta X \equiv \sqrt{\sum_{\vec{\xi}} P(\vec{\xi}|X)\left[X_{\mathrm{est}}(\vec{\xi}) - X\right]^2} = \sqrt{\Delta X_{est}^2 + (X - \langle X_{est}\rangle)^2}$$



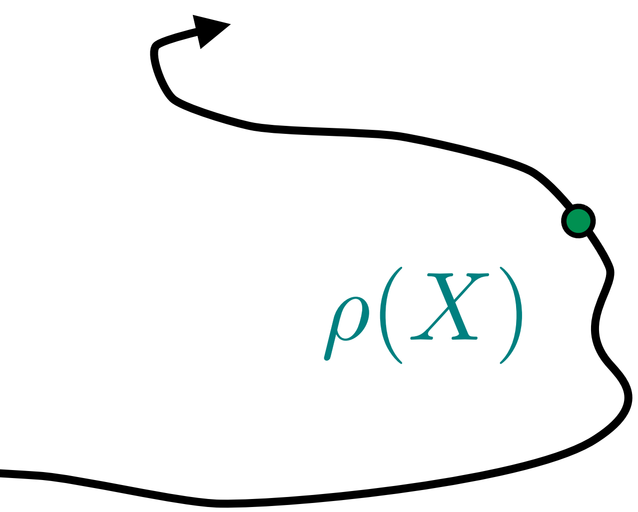
CRAMER-RAO bound

$$\delta X \geqslant \frac{1}{\sqrt{\nu F(X)}}$$

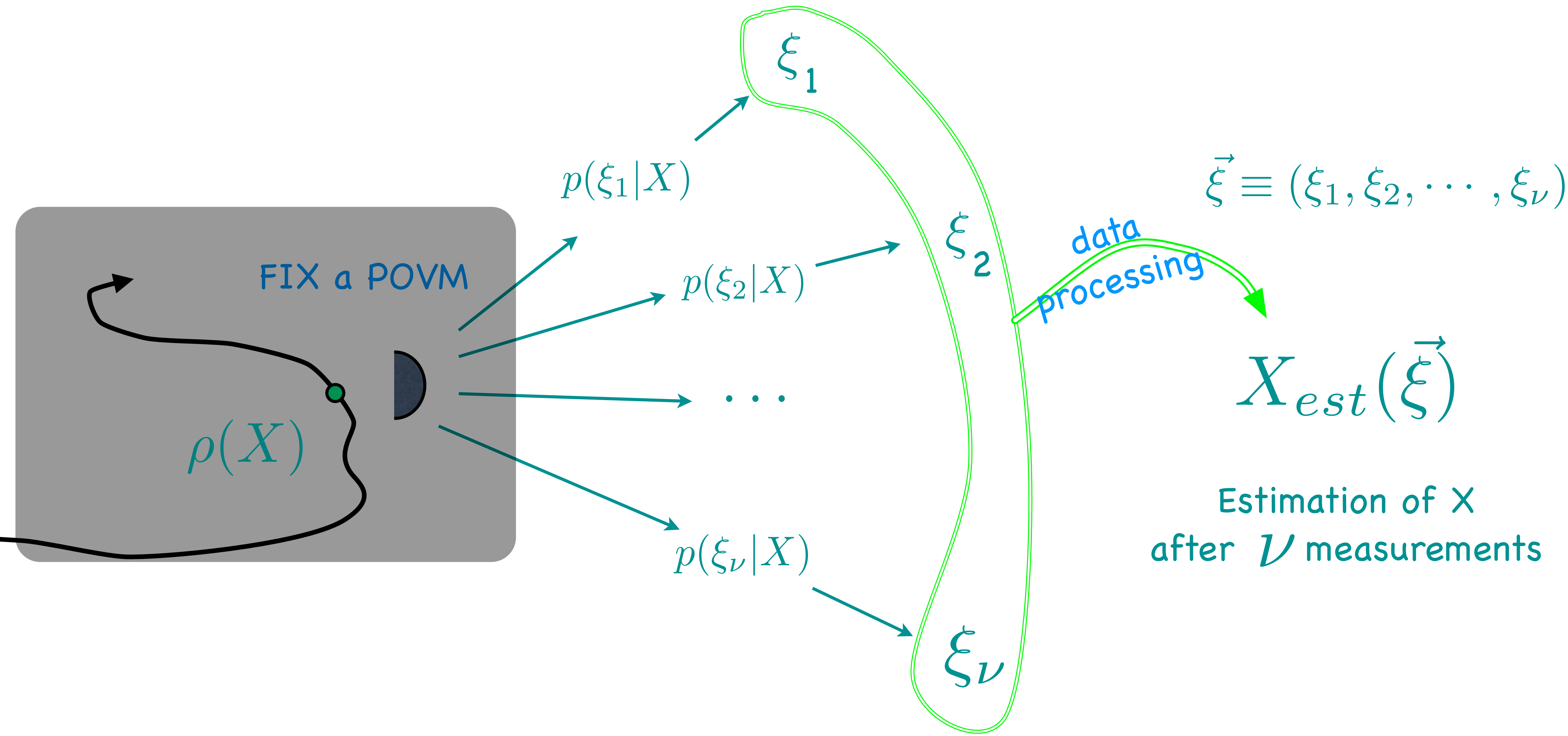
ACHIEVABLE FOR LARGE ENOUGH ν

$$F(X) \equiv \langle \Big[\frac{\partial}{\partial X} \ln p(\xi | X) \Big]^2 \rangle$$ FISHER information

$1/\sqrt{\nu}$ scaling with respect to the number times we repeat the measurement let us go back to the quantum scenario....



let us go back to the quantum scenario....



i.i.d. process $$P(\vec{\xi}|X) = \prod_{j=1}^{\nu} p(\xi_j|X)$$

Given $\rho(X)$, which is the best estimation of X we can get?

For each POVM we can write $$\delta X_{POVM} \geqslant \frac{1}{\sqrt{\nu F_{POVM}(X)}}$$

Therefore the optimal estimation error is given by

$$\boxed{\delta X \geqslant \frac{1}{\sqrt{\nu F_0(X)}}}$$

quantum CRAMER-RAO bound

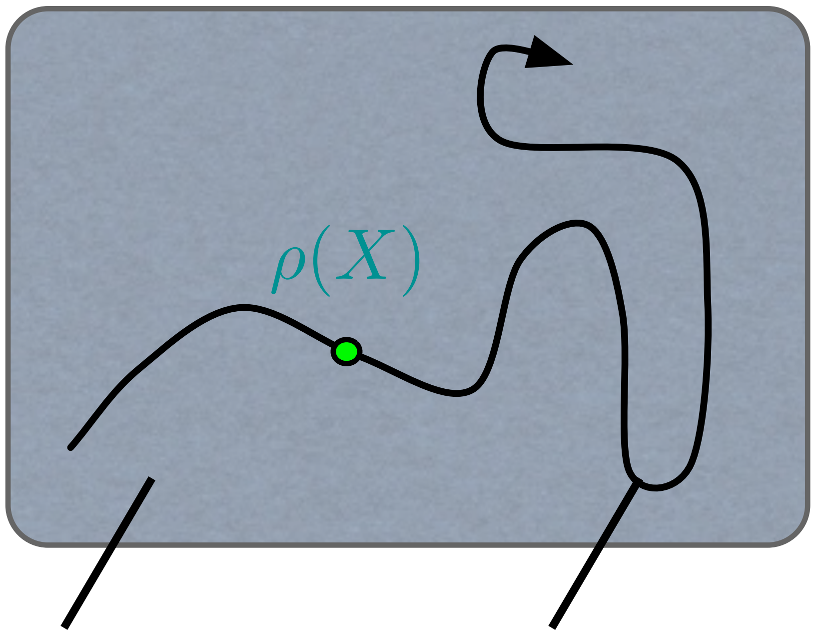
where

$$F_0(X) \equiv \max_{POVM} F_{POVM}(X)$$

maximum with respect to all POVM

Evaluating $F_0(X)$ is difficult.

However, it can be done if the "X-trajectory" of the system is Hamiltonian



$$\frac{d\rho(X)}{dX} = -i[h(X), \rho(X)]$$

Hermitian generator of the trajectory: we include also the case in which it depends on X.

In this case one has

$$\boxed{F_0(X) \leqslant 4\mathrm{Tr}\left[\rho(X)\,\Delta^2 h(X)\right] = 4\langle\Delta^2 h(X)\rangle_X}$$

variance of the Hamiltonian

with

$$\Delta h(X) = h(X) - \langle h(X)\rangle_X$$

and

$$\langle\ldots\rangle_X \equiv \mathrm{Tr}[\ldots\ \rho(X)]$$

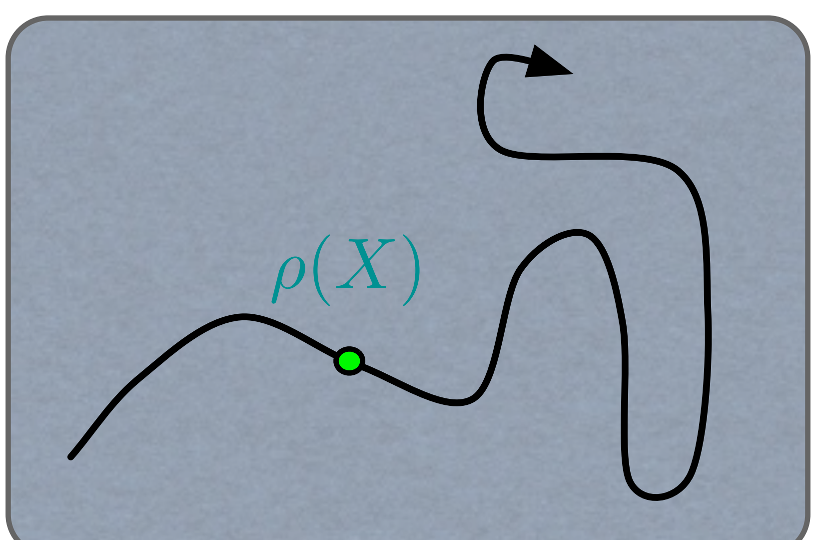
Braunstein and Caves, (1994)

Helstrom (1976) , Holevo (1982)

$$\delta X \geqslant \frac{1}{\sqrt{\nu F_0(X)}} \geqslant \frac{1}{2\sqrt{\nu \langle \Delta^2 h(X) \rangle_X}}$$

$$\delta X \; \Delta h \; \geqslant \frac{1}{2\sqrt{\nu}}$$

Generalized "Energy-Time" uncertainty relation

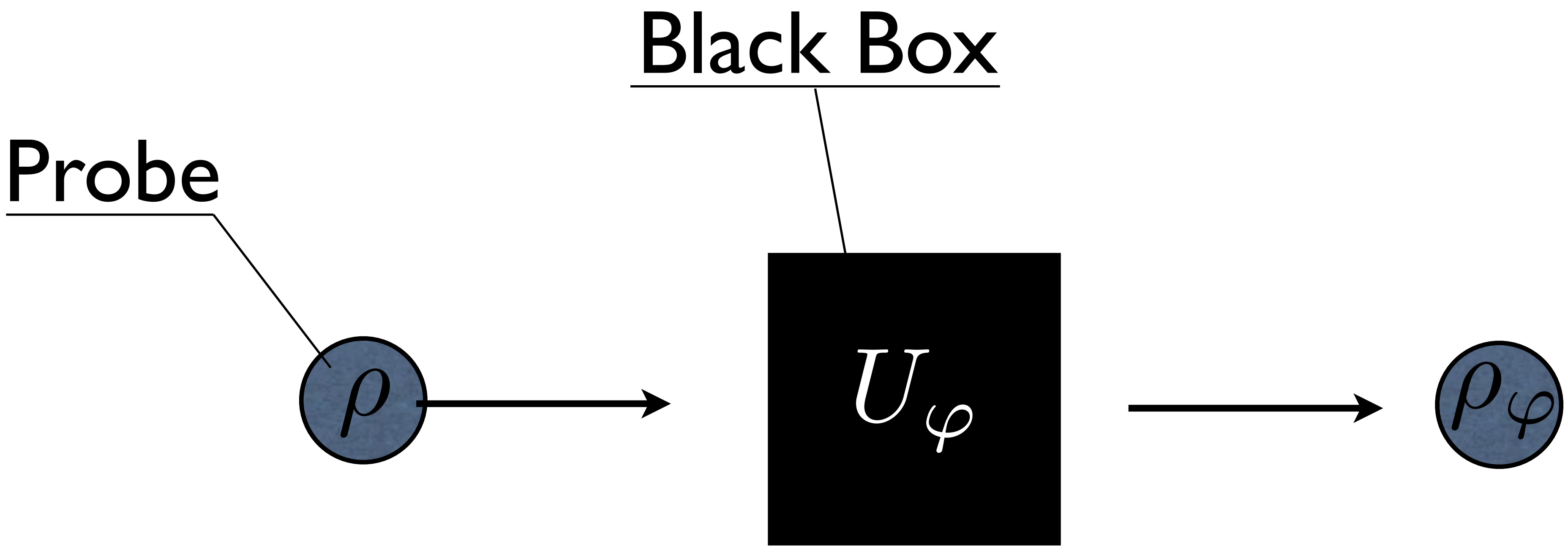


It relate the precision on the estimation of the "time" X with the spread of the Hamiltonian h, i.e.

$$\Delta h \equiv \sqrt{\langle \Delta^2 h(X) \rangle_X}$$

Shot Noise vs Heisenberg

(entanglement as a resource)



it transforms the incoming probes by applying the unitary U_φ

$$\rho_\varphi = U_\varphi \, \rho \, U_\varphi^\dagger$$

The physical mechanism which is responsible for the process is known. What we do NOT know is the value of the phase φ .

$$U(\varphi) \equiv \exp[-iH\varphi]$$

known Hamiltonian

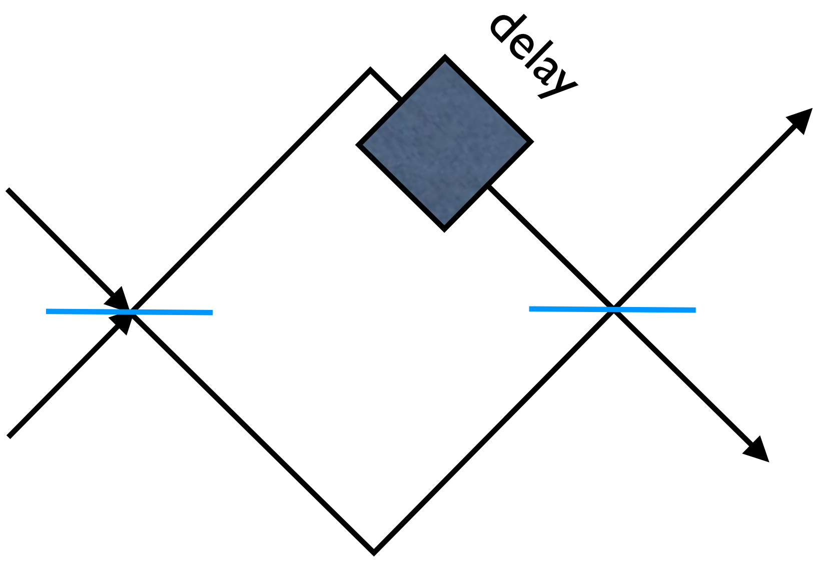
unknown parameter

GOAL: determine φ

Which precision can be achieved with N probes?

Examples

- Interferometry
- Positioning
- Clock Sync
- Transfer of reference frame
- ...



Examples

- Interferometry
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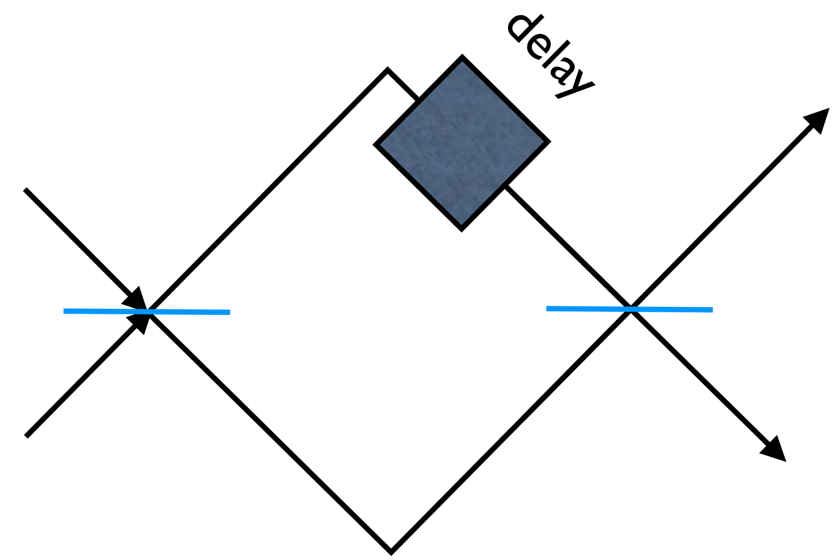
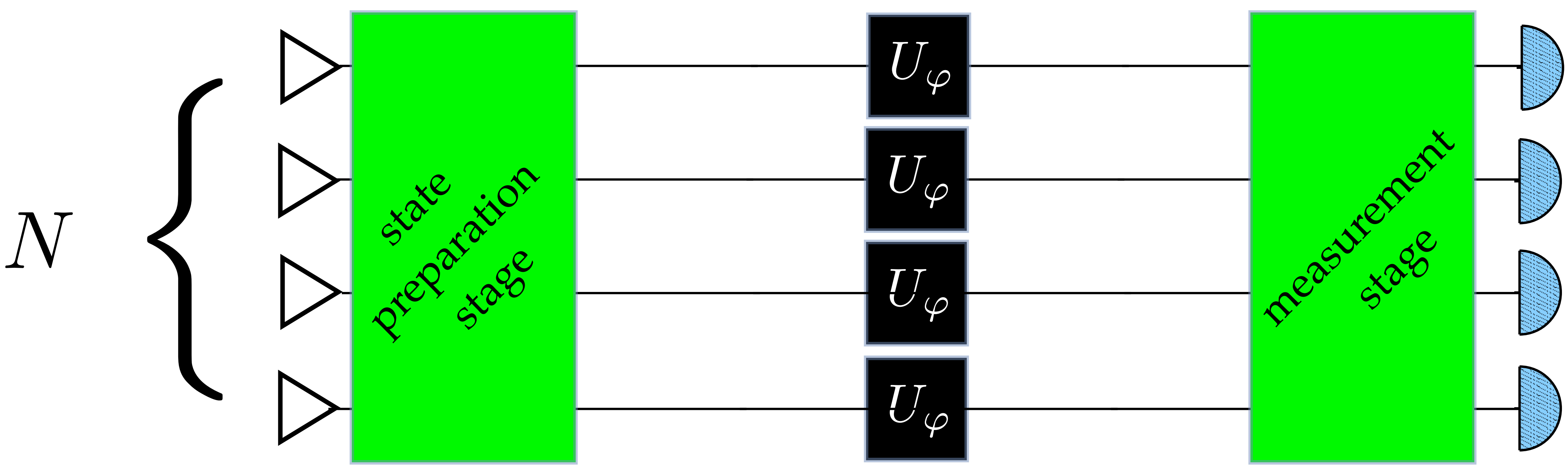
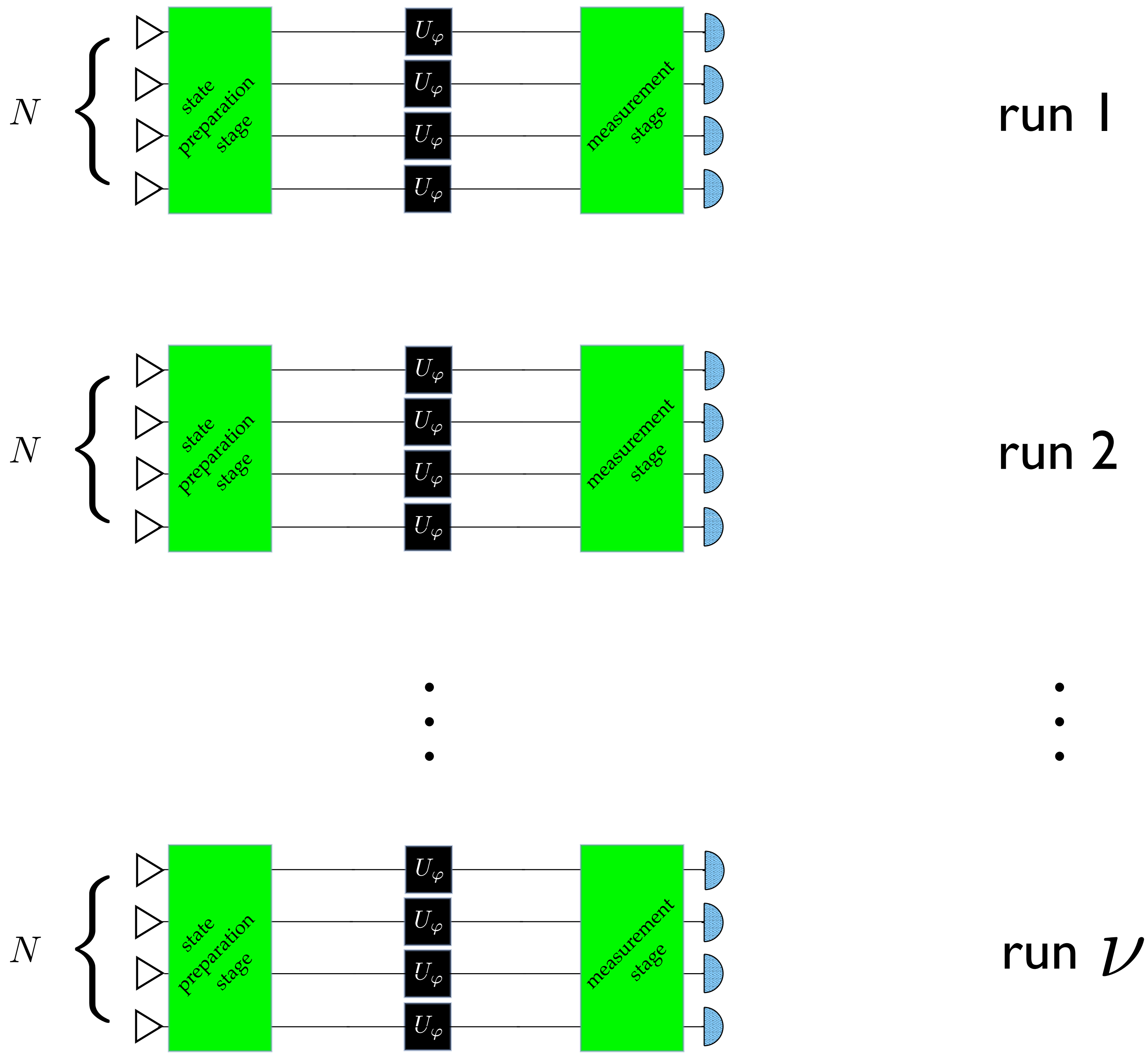




In each experimental run
we have N probes.



Four scenarios

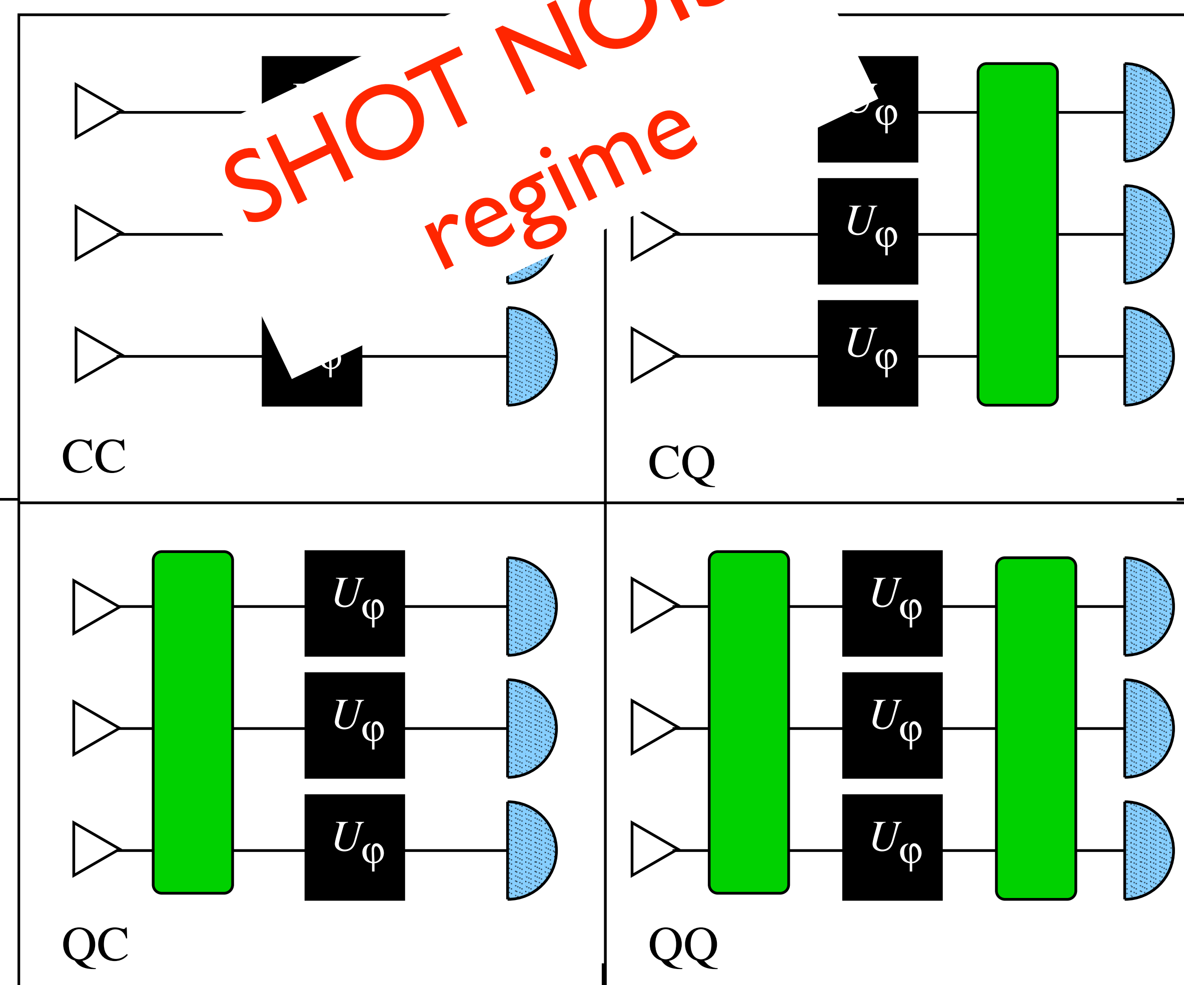
the parameter φ is encoded into a separable state of the N probes.
Each probe is measured individually.

$$\delta\varphi \geqslant 1/(\sqrt{\nu N})$$

the parameter φ is encoded into a separable state of the N probes.
The probes are jointly measured.

$$\delta\varphi \geqslant 1/(\sqrt{\nu N})$$

SHOT NOISE regime



HEISENBERG regime

the parameter φ is encoded into a joint state of the N probes.
Each probe is measured individually.

$$\delta\varphi \geqslant 1/(\sqrt{\nu}N)$$

the parameter φ is encoded into a joint state of the N probes.
The probes are jointly measured.

$$\delta\varphi \geqslant 1/(\sqrt{\nu}N)$$

VG,Lloyd,Maccone PRL 2006

SUMMARY:

- ✦ state discrimination (Helstrom and Chernoff bounds)
- ✦ parameter estimation (Cramer-Rao Bound)
- ✦ Heisenberg scaling
- ✦ Thermal Suscetibility
- ✦ Quantum Correlations.

OPEN PROBLEMS:

- noise????

Thank you!