Leibniz Universität Hannover

# Quantum Aspects of Optomechanics

Advanced School and Workshop on Foundations and Applications of Nanomechanics

Klemens Hammerer

Institute for Theoretical Physics

Institute for Gravitational Physics
Albert Einstein Institute

QUEST Ring Lecture

Trieste Sep 2017

# References

## Reviews on Optomechanics


Cavity Optomechanics
Markus Aspelmeyer, Tobias J. Kippenberg, Florian Marquardt
arXiv:1303.0733, Rev. Mod. Phys. 86, 1391 (2014)

Quantum Control in Optomechanical Systems
Sebastian G. Hofer, Klemens Hammerer
Advances in Atomic, Molecular, and Optical Physics, Volume 66, Chapter 5, Page 263 (2015)
https://doi.org/10.1016/bs.aamop.2017.03.003


## PhD thesis by Sebastian G. Hofer


Quantum Control of Optomechanical Systems
Sebastian G. Hofer
Universität Wien (2015)
http://ubdata.univie.ac.at/AC12696766

First **Quantum** Optomechanics: Physikalische Zeitschrift **22** 817 (1909)



**A. Einstein (Zürich), Über die Entwicklung unserer Anschauungen über das Wesen und die Konstitution der Strahlung.**

Als man erkannt hatte, daß das Licht die Erscheinungen der Interferenz und Beugung zeige, da erschien es kaum mehr bezweifelbar, daß das Licht als eine Wellenbewegung aufzufassen sei. Da das Licht sich auch durch das Vakuum fortzupflanzen vermag, so mußte man sich vorstellen, daß auch in diesem eine Art besonderer Materie vorhanden sei, welche die Fortpflanzung der Lichtwellen vermittelt. Für die Auffassung der Gesetze der Ausbreitung des Lichtes in ponderabeln Körpern war es nötig, anzunehmen, daß jene Materie, welche man Lichtäther nannte, auch in diesen vorhanden sei, und daß es auch im Innern der ponderabeln Körper im wesentlichen der Lichtäther sei, welcher die Ausbreitung des Lichtes vermittelt. Die Existenz jenes Lichtäthers schien unbezweifelbar. In dem 1902 erschienenen ersten Bande des vortrefflichen Lehrbuches der Physik von Chwolson findet sich in der Einleitung über den Äther der Satz: „Die Wahrscheinlichkeit der Hypothese von der Existenz dieses einen Agens grenzt außerordentlich nahe an Gewißheit“.

Heute aber müssen wir wohl die Ätherhypothese als einen überwundenen Standpunkt ansehen. Es ist sogar unleugbar, daß es eine ausgedehnte Gruppe von die Strahlung betreffenden Tatsachen gibt, welche zeigen, daß dem Lichte gewisse fundamentale Eigenschaften zukommen, die sich weit eher vom Standpunkte der Newtonschen Emissionstheorie des Lichtes als vom Standpunkte der Undulationstheorie begreifen lassen. Deshalb ist es meine Meinung, daß die nächste Phase der Entwicklung der theoretischen Physik uns eine Theorie des Lichtes bringen wird, welche sich als eine Art Verschmelzung von Undulations- und Emissionstheorie des Lichtes auffassen läßt. Diese Meinung zu begründen, und zu zeigen, daß eine tiefgehende Änderung unserer Anschauungen vom Wesen und von der Konstitution des Lichtes unerläßlich ist, das ist der Zweck der folgenden Ausführungen.

Der größte Fortschritt, welchen die theoretische Optik seit der Einführung der Undulationstheorie gemacht hat, besteht wohl in Maxwells genialer Entdeckung von der Möglichkeit, das Licht als einen elektromagnetischen Vorgang aufzufassen. Diese Theorie führt statt der mechanischen Größen, nämlich Deformation

# Einstein 1909

- Talk at the “81st meeting of the Society of Natural Scientists and Medics” in Salzburg

Physikalische Zeitschrift. 10. Jahrgang. No. 22. 817

**A. Einstein (Zürich), Über die Entwicklung unserer Anschauungen über das Wesen und die Konstitution der Strahlung.**

“About the development of our conception regarding the nature and constitution of radiation.”

- Known at that time: Planck’s formula for blackbody radiation spectrum

$$\varrho = \frac{8\pi h \nu^3}{c^3} \cdot \frac{1}{e^{\frac{h\nu}{kT}} - 1}.$$

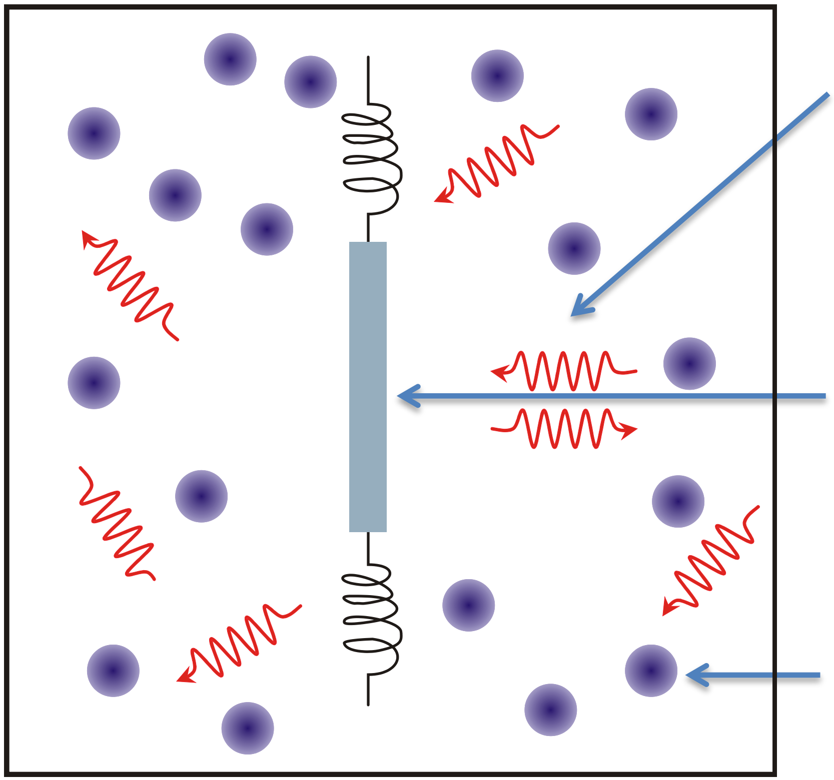
$\rho$ energy density at frequency $\nu$

- “Assuming Planck’s formula to be correct, what can be deduced about the constitution of radiation?”

theorie. Wir sehen die Plancksche Strahlungsformel als richtig an und fragen uns, ob aus ihr etwas gefolgert werden kann bezüglich der Konstitution der Strahlung. Von zwei Betrach-

# Gedankenexperiment

temperature T



blackbody radiation

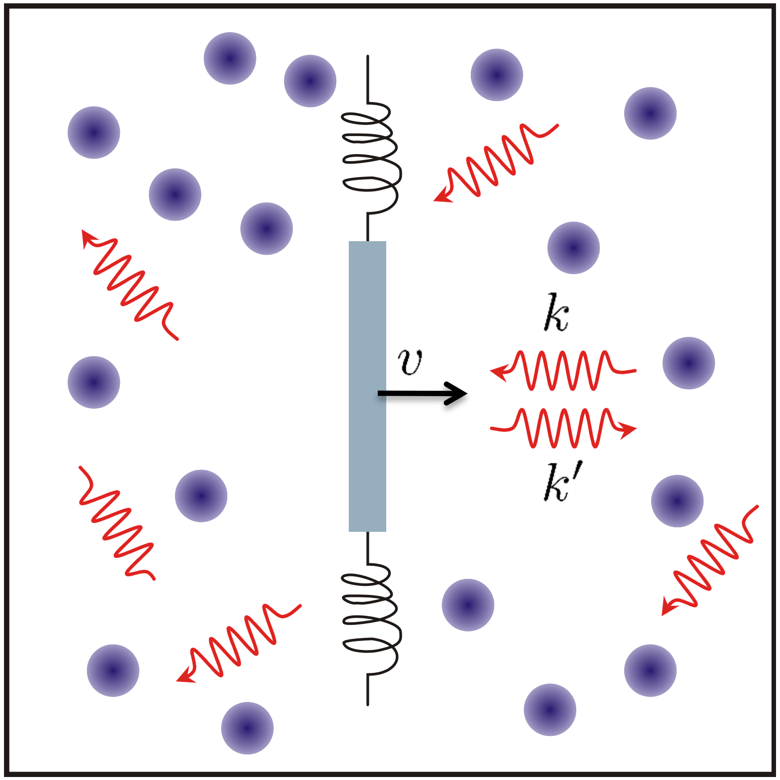
$$\varrho = \frac{8\pi h \nu^3}{c^3} \cdot \frac{1}{e^{\frac{h\nu}{kT}} - 1} .$$

perfectly reflecting,
harmonically bound
mirror

mirror in thermal equilibrium e.g. via
contact to ideal gas, or via suspension

# Gedankenexperiment

temperature T




Doppler shift of reflected wave:

$$k' = -k\left(1 - \frac{2v}{c}\right)$$

Radiation pressure force due to momentum $\sim k - k'$ transfer on mirror:

$$F_{\mathrm{rp}} = -\frac{2P}{c}\left(1 - \frac{v}{c}\right) \qquad \text{power} \quad P$$

Radiation provides friction:

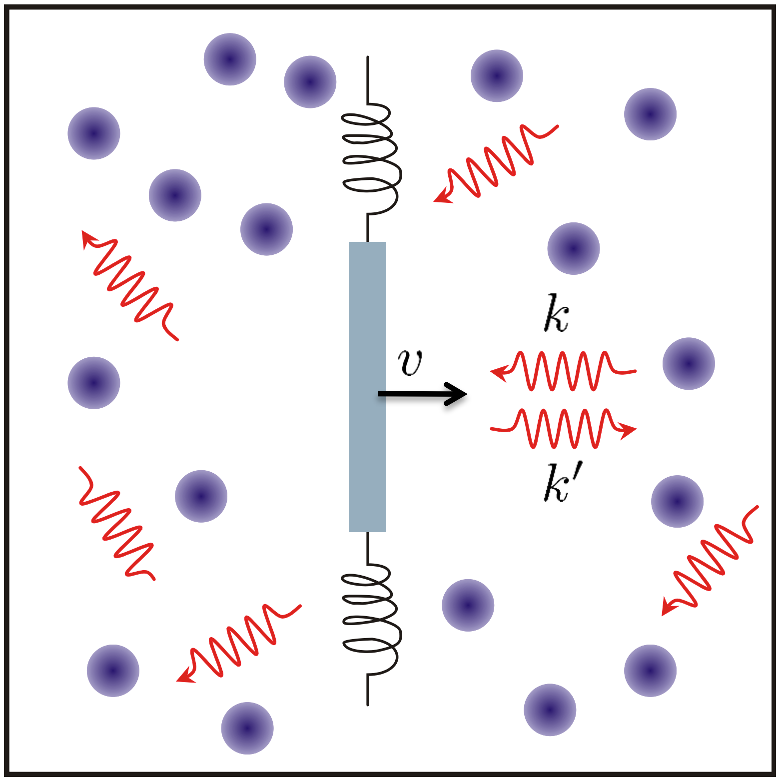
$$\dot{p} = -\gamma p \qquad\qquad \gamma = \frac{2P}{mc^2}$$

"Doppler cooling" of mirror via radiation pressure

$\longrightarrow$ Thermal equlibrium?

# Gedankenexperiment

temperature T




Fluctuation – Dissipation:
Radiation pressure force is fluctuating according to Plancks formula, causing mometum diffusion of the mirror.

Average squared momentum transferred due to radiation pressure fluctuations:

$$\overline{\Delta^2} = \frac{1}{c}\left[h\varrho\nu + \frac{c^3}{8\pi}\frac{\varrho^2}{\nu^2}\right] d\nu f \tau .$$

Looks like localized **particles** moving with energy ***hv***

Allows for an explanation as an interference effect of classical **waves**

→ Radiation has both, particle and wave characteristic

→ In radiation pressure force on mirror **particle** characteristics are dominant for low energy density!

# Standard Quantum Limit in Gravitational Wave Detectors

PHYSICAL REVIEW LETTERS

VOLUME 45 | 14 JULY 1980 | NUMBER 2

## Quantum-Mechanical Radiation-Pressure Fluctuations in an Interferometer

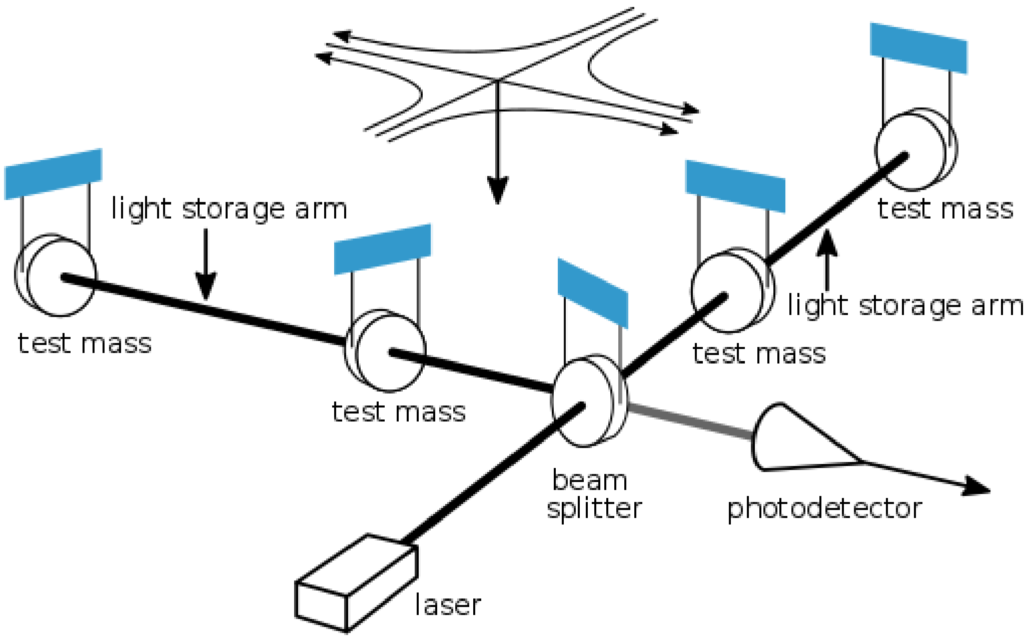
Carlton M. Caves

*W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125*

(Received 29 January 1980)

The interferometers now being developed to detect gravitational vaves work by measuring small changes in the positions of free masses. There has been a controversy whether quantum-mechanical radiation-pressure fluctuations disturb this measurement. This Letter resolves the controversy: They do.

# Standard Quantum Limit in Gravitational Wave Detectors




# Standard Quantum Limit in Gravitational Wave Detectors

*a) Determination of the position of a free particle.*—As a first example of the destruction of the knowledge of a particle's momentum by an apparatus determining its position, we consider the use of a microscope.[1] Let the particle be moving at such a distance from the microscope that the cone of rays scattered from it through the objective has an angular opening $\epsilon$. If $\lambda$ is the wave-length of the light illuminating it, then the uncertainty in the measurement of the $x$-co-ordinate (see Fig. 5) according to the laws of optics governing the resolving power of any instrument is:

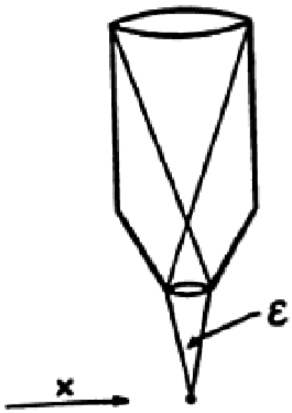
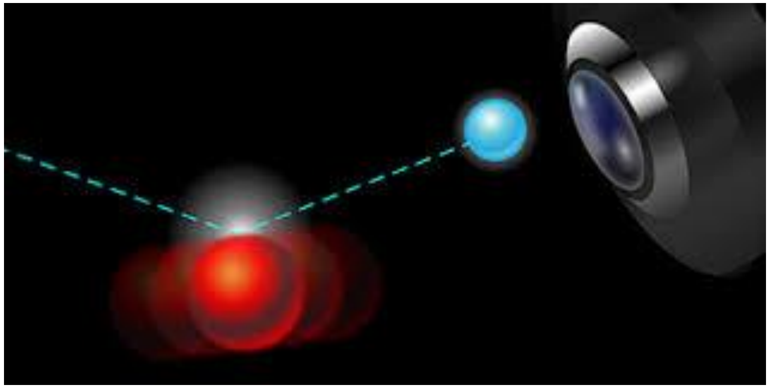



FIG. 5



$$\Delta x = \frac{\lambda}{\sin \varepsilon/2}$$

uncertainty in Compton recoil

$$\Delta p_x = 2\frac{h}{\lambda}\sin \varepsilon/2$$

$$\Delta x \Delta p_x \simeq 2h$$

Heisenberg, The Physical Principles of the Quantum Theory, 1930

# Optomechanical Entanglement & Limits of Quantum Mechanics

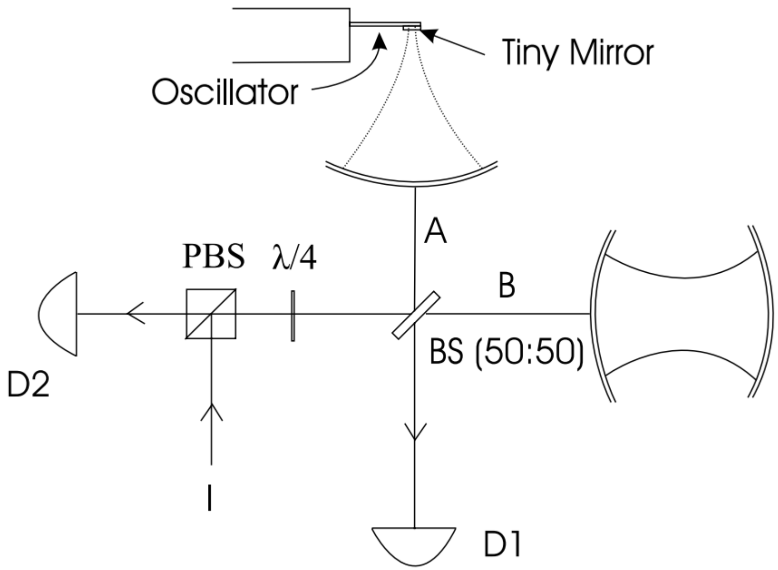
## Towards Quantum Superpositions of a Mirror

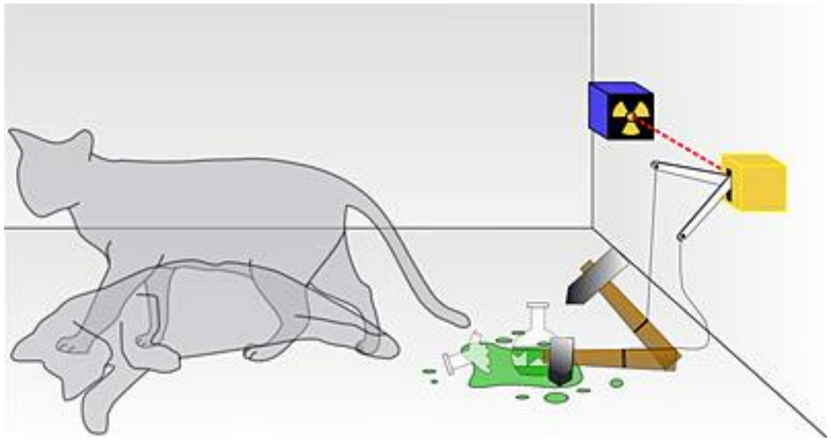
William Marshall,[1,2] Christoph Simon,[1] Roger Penrose,[3,4] and Dik Bouwmeester[1,2]

[1]*Department of Physics, University of Oxford, Oxford OX1 3PU, United Kingdom*
[2]*Department of Physics, University of California, Santa Barbara, California 93106, USA*
[3]*Center for Gravitational Physics and Geometry, The Pennsylvania State University, University Park, Pennsylvania 16802, USA*
[4]*Department of Mathematics, University of Oxford, Oxford OX1 3LB, United Kingdom*

(Received 30 September 2002; published 23 September 2003; publisher error corrected 25 September 2003)

We propose an experiment for creating quantum superposition states involving of the order of $10^{14}$ atoms via the interaction of a single photon with a tiny mirror. This mirror, mounted on a high-quality mechanical oscillator, is part of a high-finesse optical cavity which forms one arm of a Michelson interferometer. By observing the interference of the photon only, one can study the creation and decoherence of superpositions involving the mirror. A detailed analysis of the requirements shows that the experiment is within reach using a combination of state-of-the-art technologies.

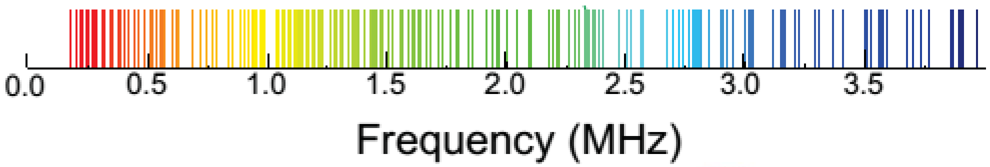


# Mechanical oscillators

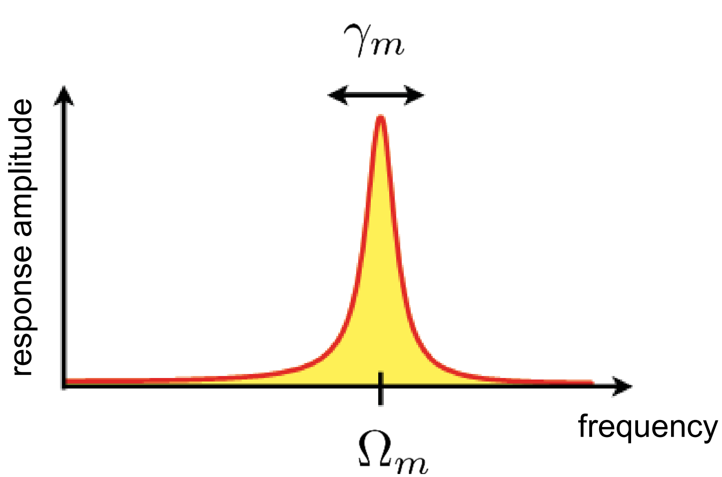
Small vibrations described by:

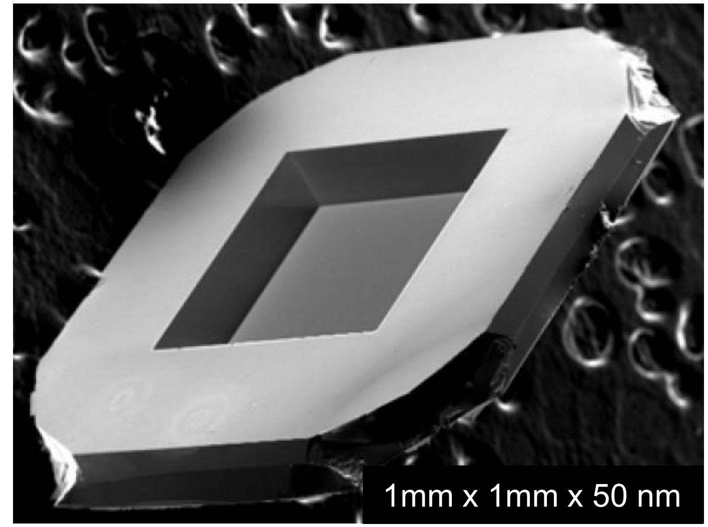
$$\Omega_m$$



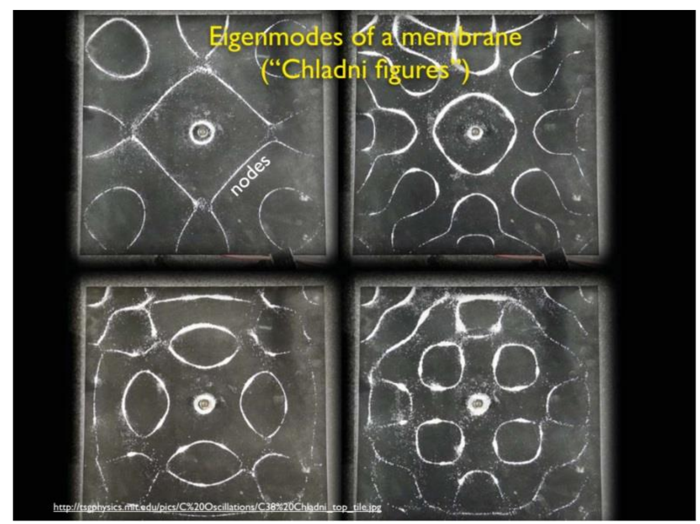

Eigenmodes $\vec{u}_m(\vec{r})$

Displacement field $\vec{u}(\vec{r}) = \sum_m X_m(t)\vec{u}_m(\vec{r})$



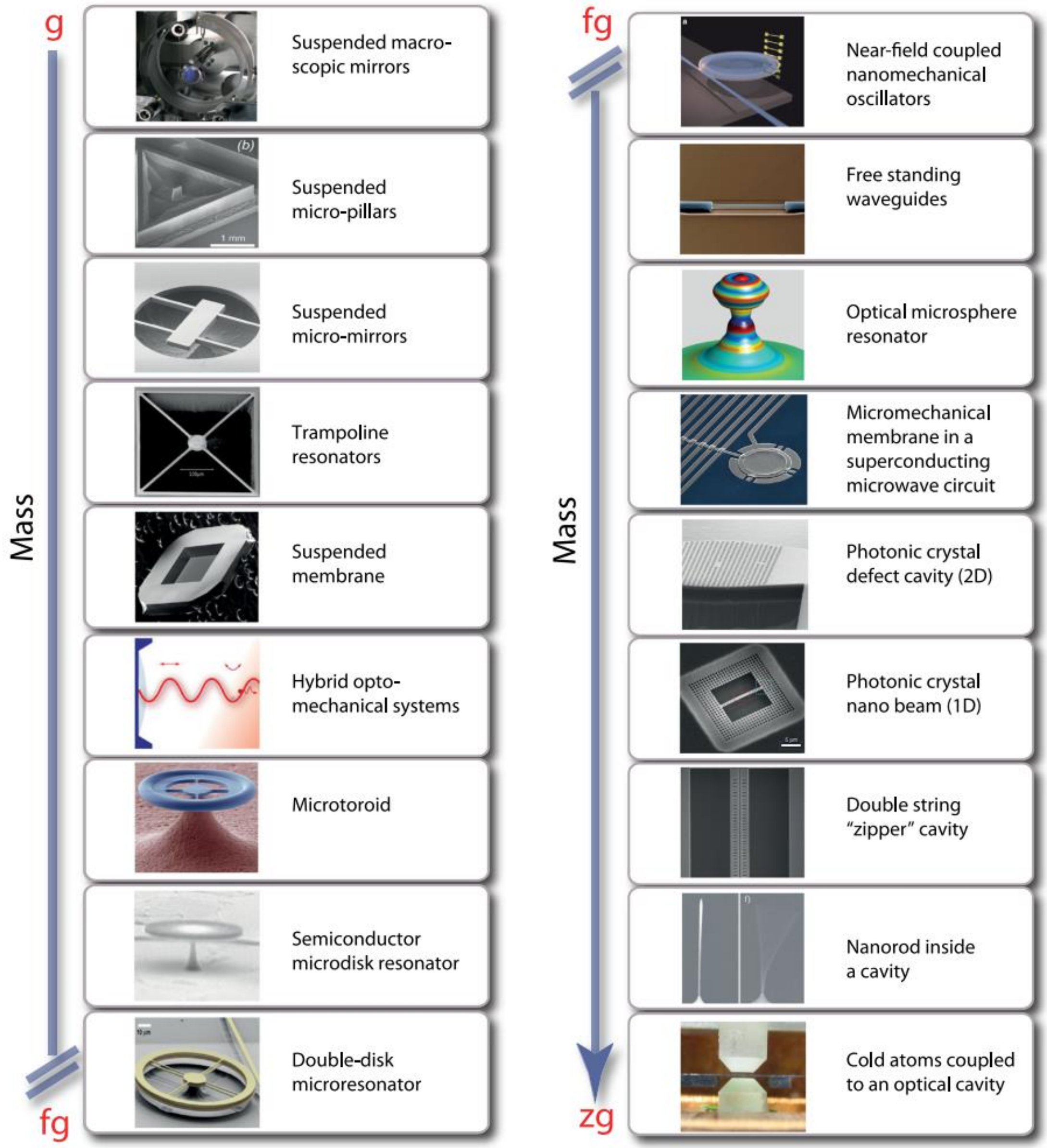




J. Harris



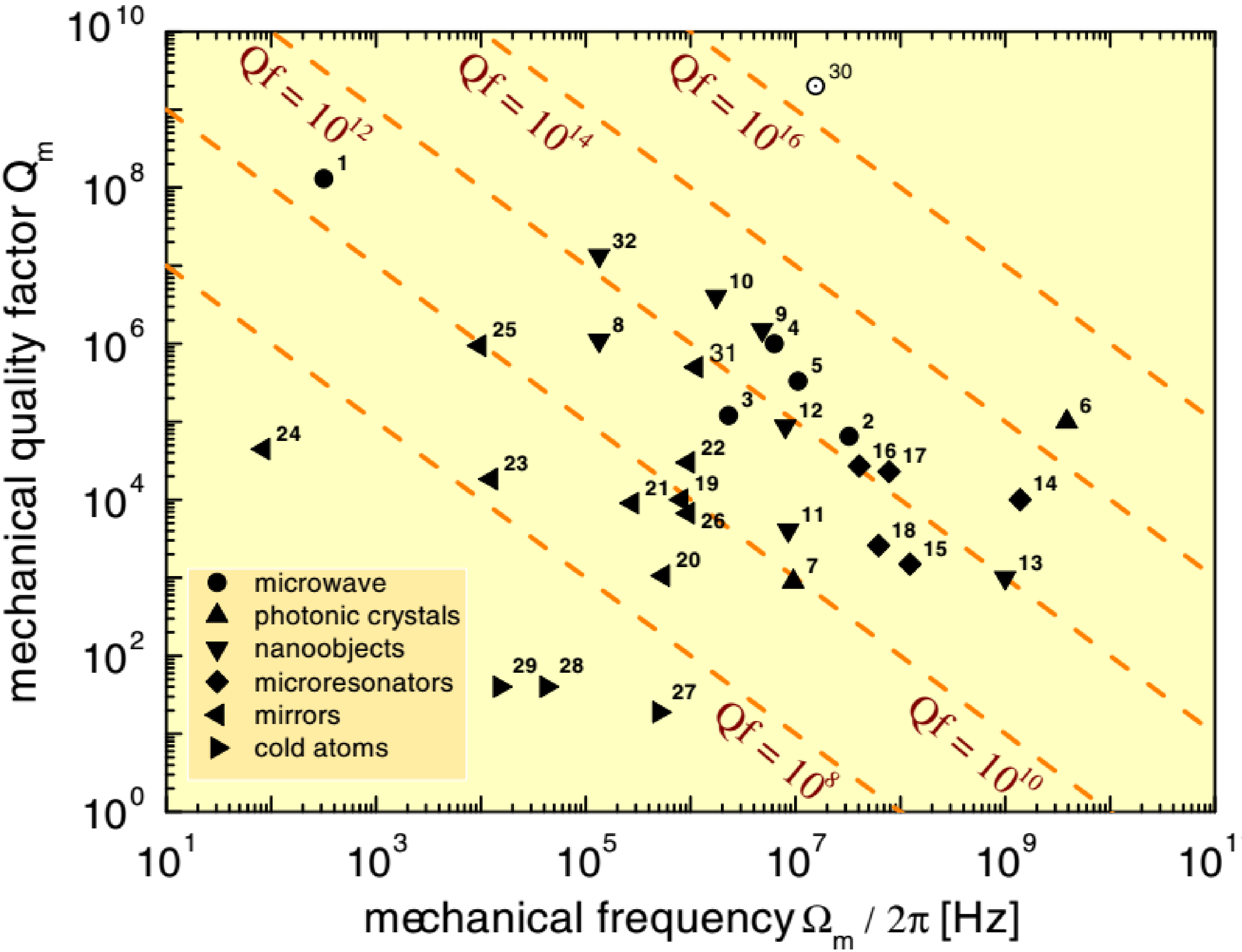

slide: F. Marquardt

# Mechanical Oscillators




Aspelmeyer, Kippenberg, Marquardt, Cavity Optomechanics, Rev. Mod. Phys. 86 1391 (2014)

# Mechanical Oscillators

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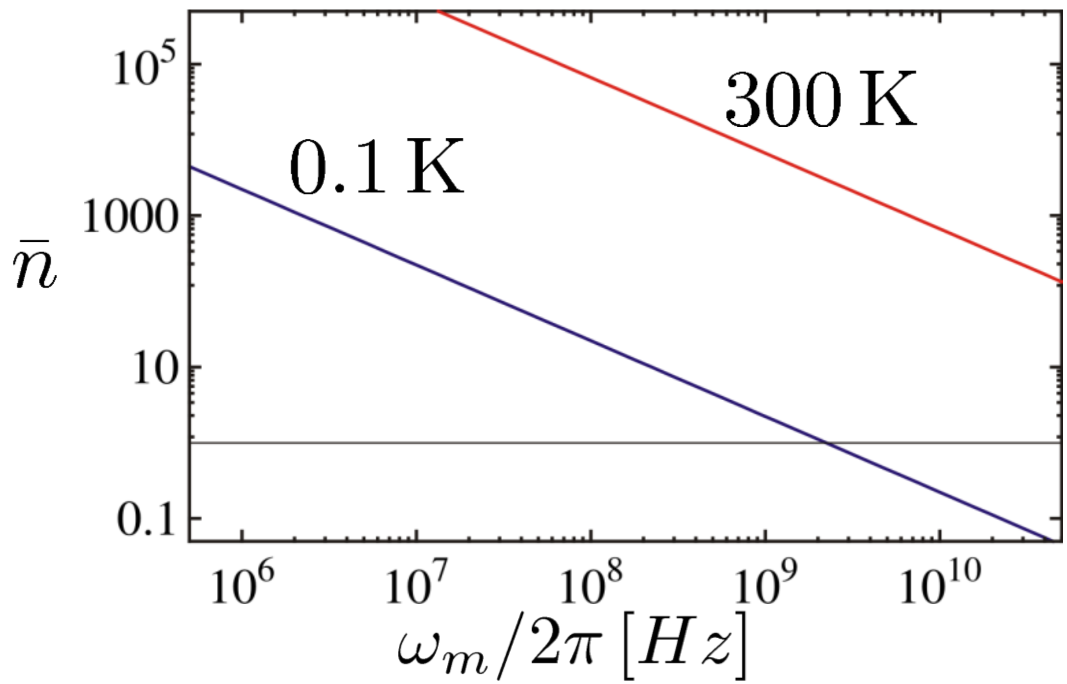
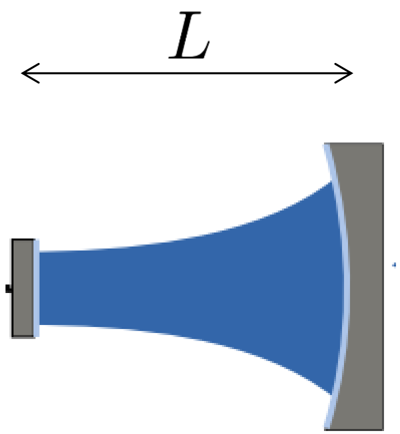
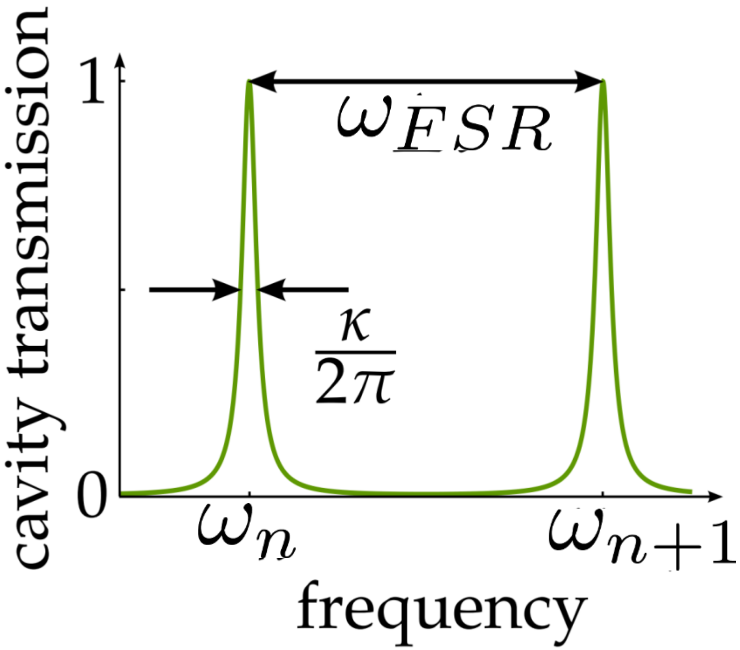
# Mechanical Oscillators

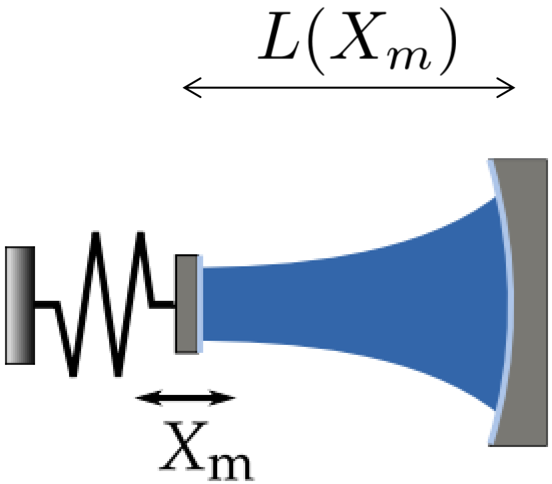


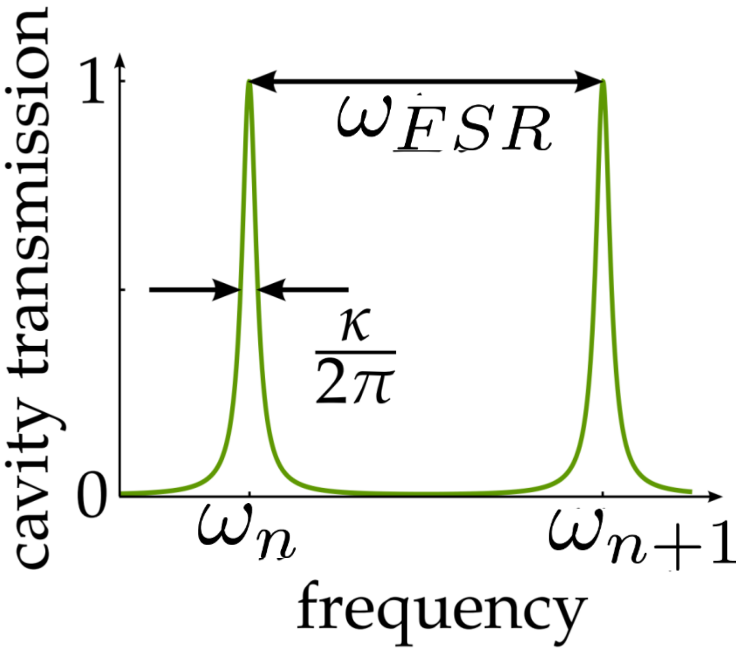
# Optical Resonators



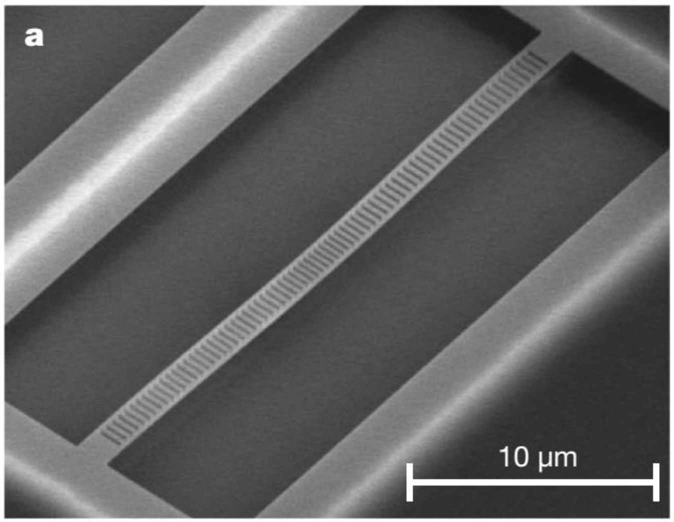
# Optomechanical Interaction

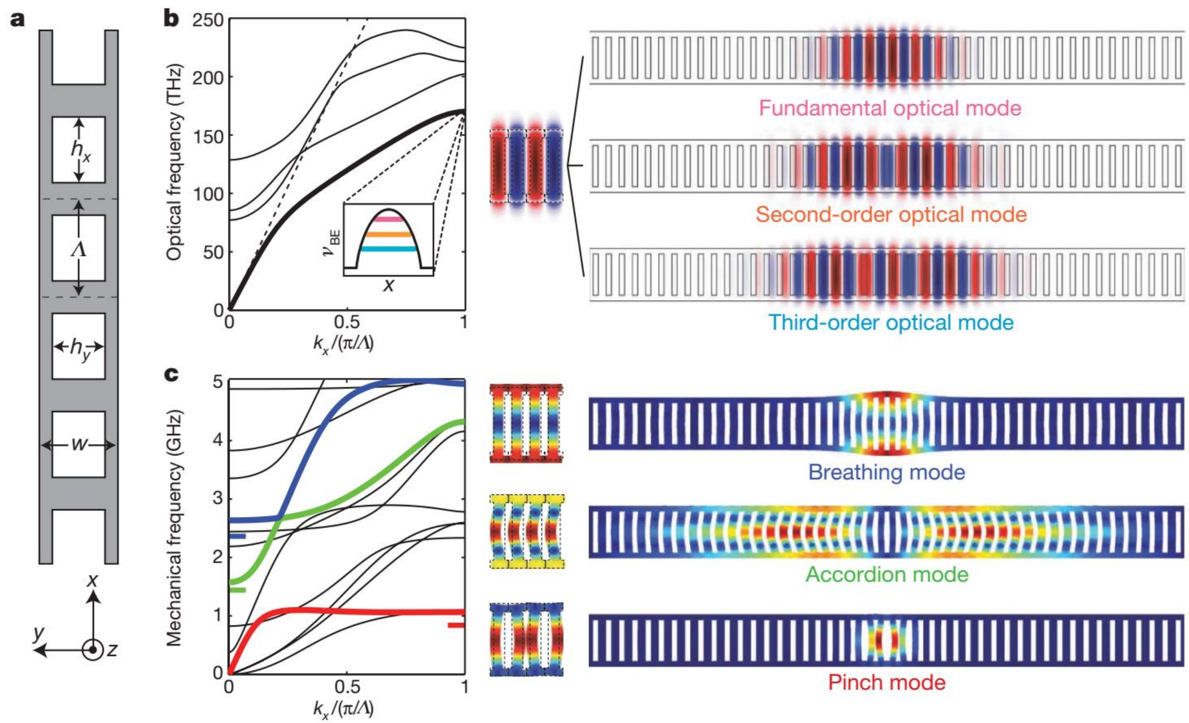


# Optomechanical Interaction

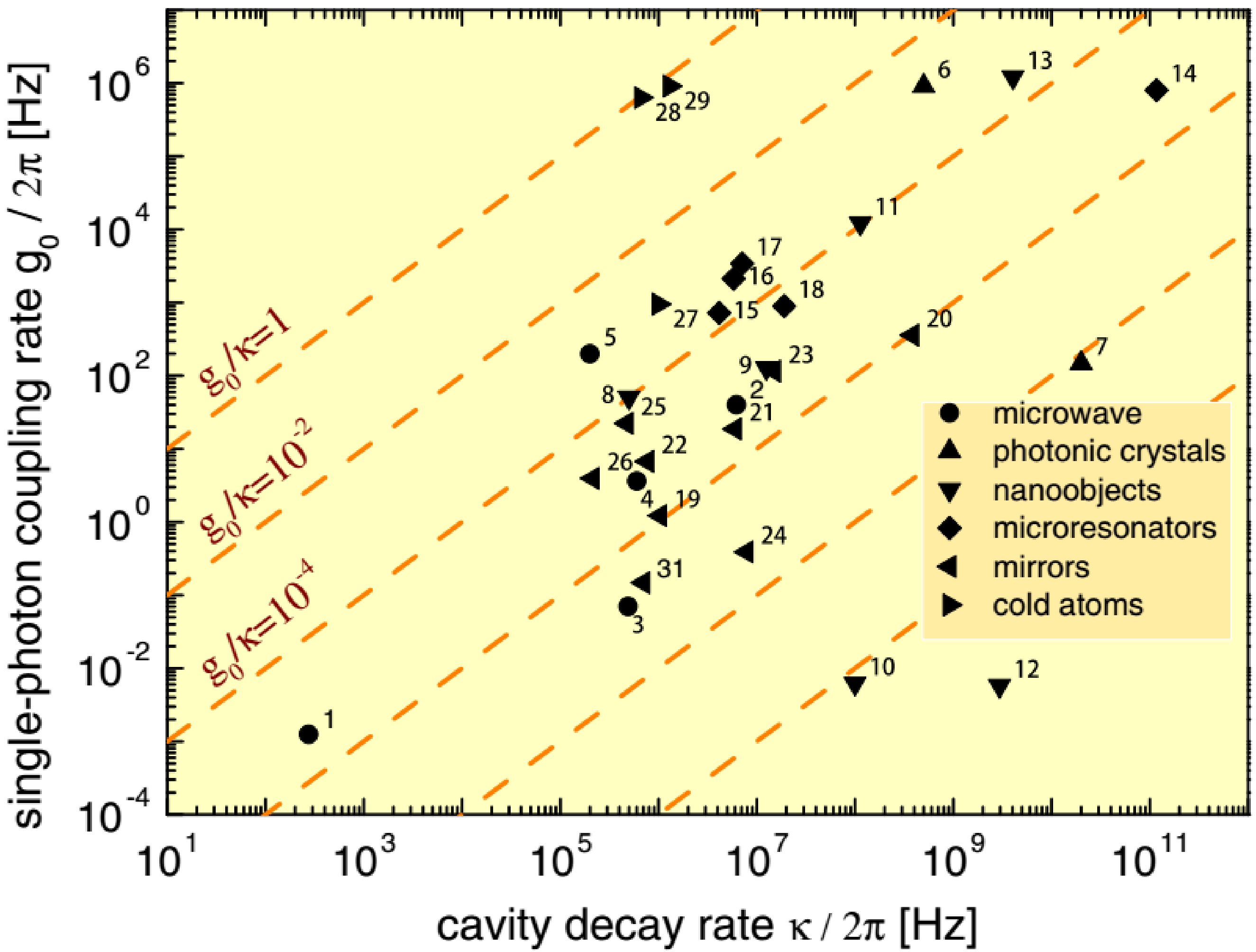
## Optomechanical Crystals



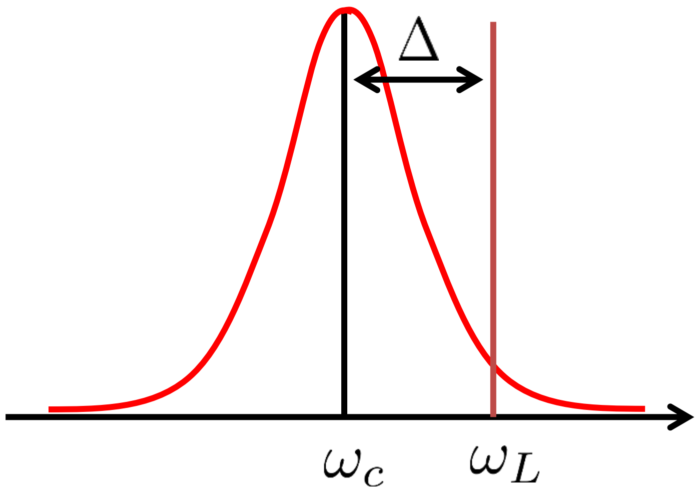



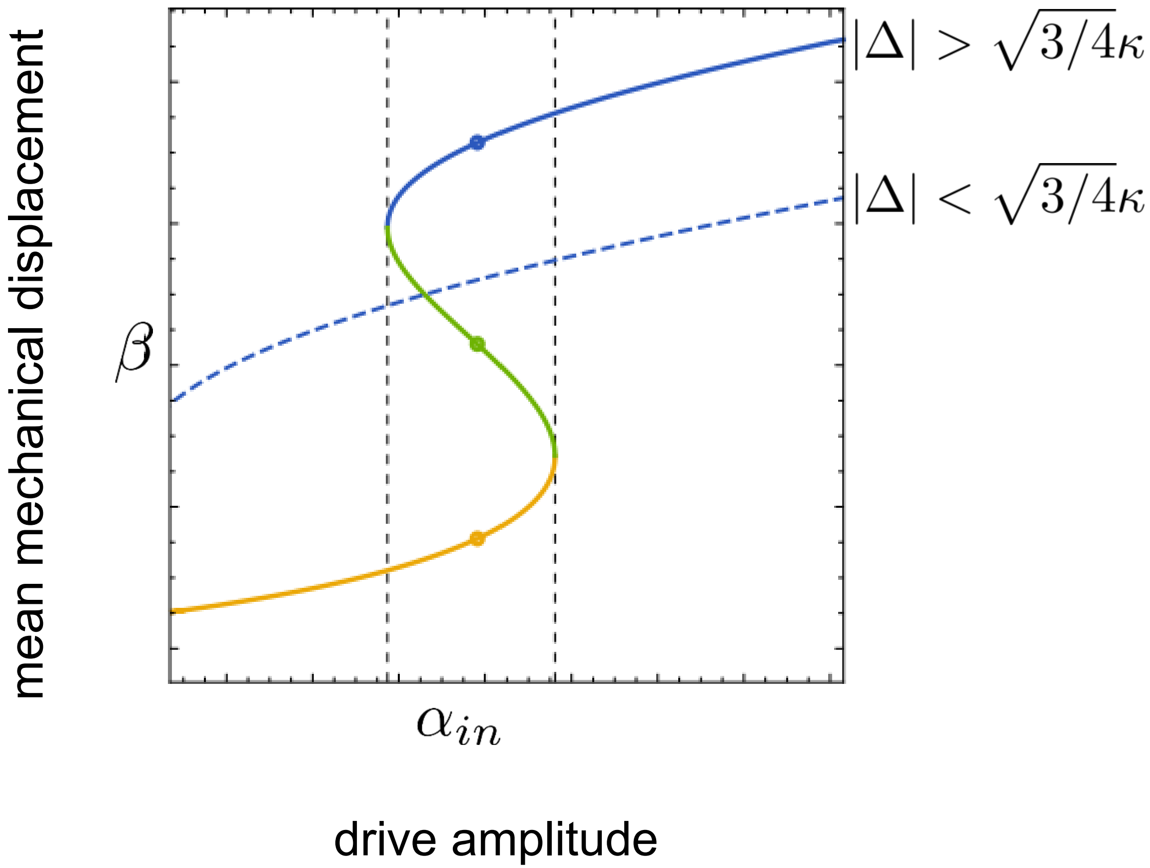

Eichenfield et al. Nature 462 78 (2009)

# Optomechanical Interaction

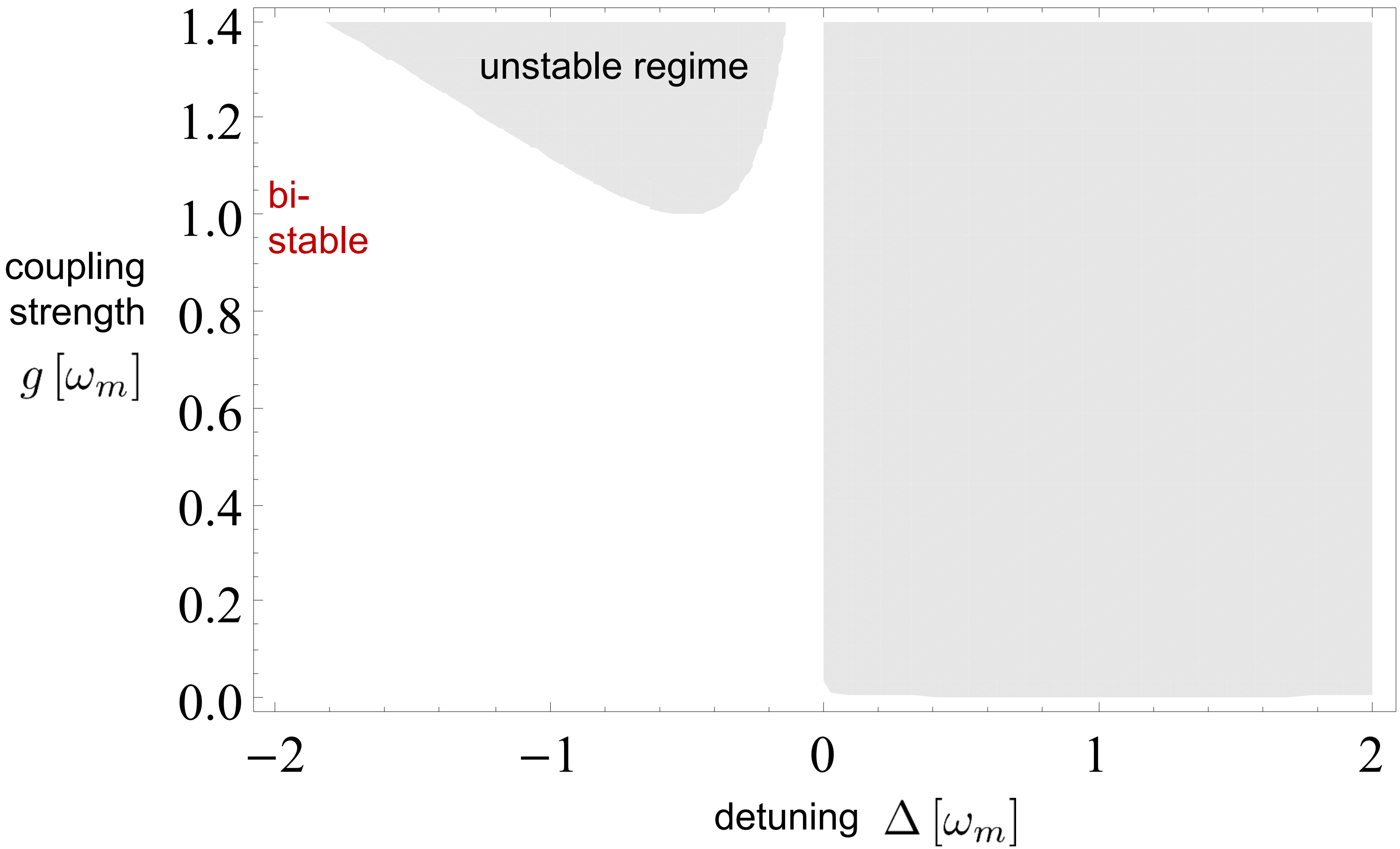


# Optomechanical Equations of Motion



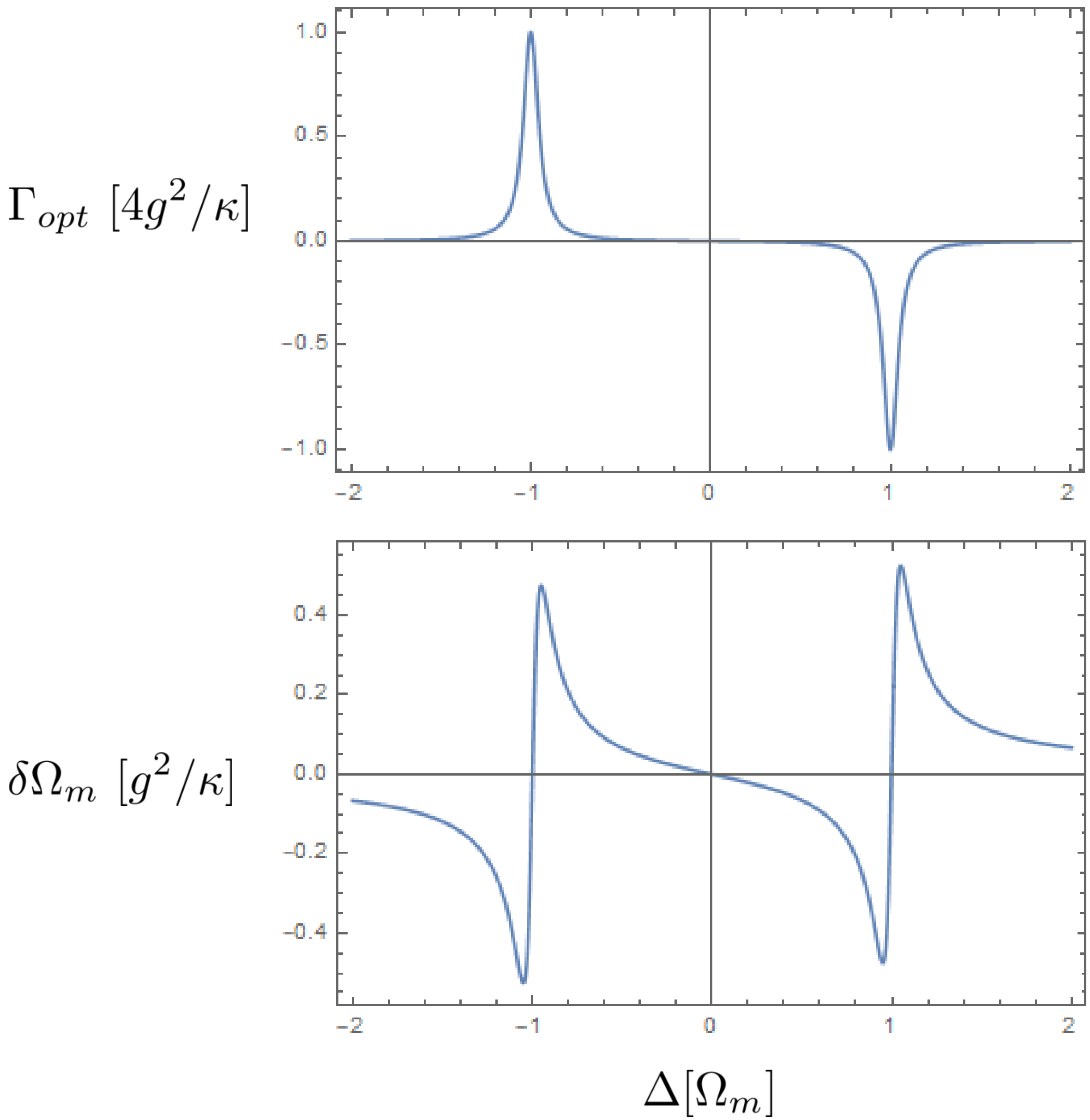
# Optomechanical Equations of Motion



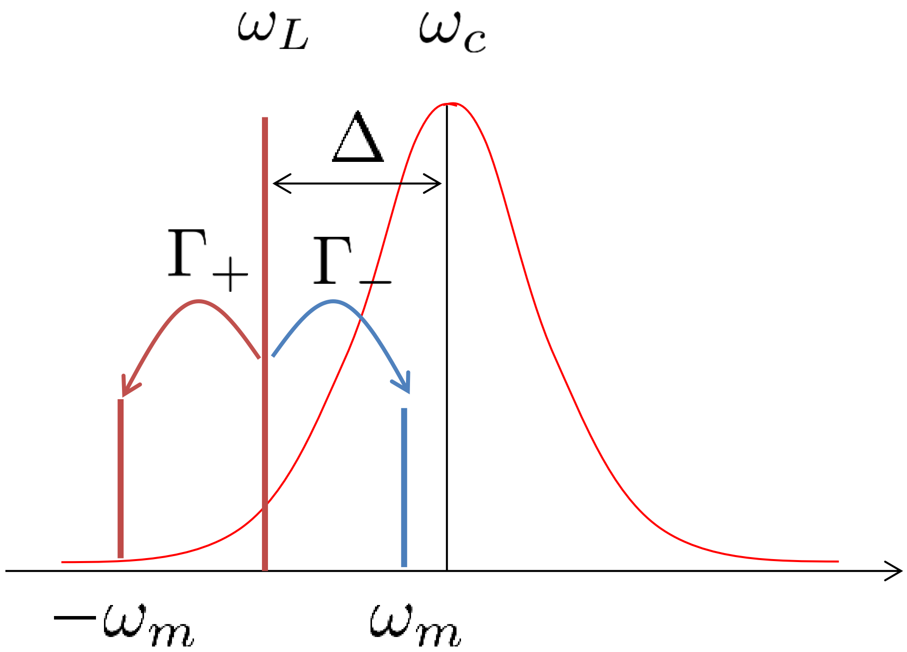

$Q = 10^6 \quad T = 5\mathrm{K} \quad \kappa/\omega_m = 0.5$

# Weak Coupling Optomechanics

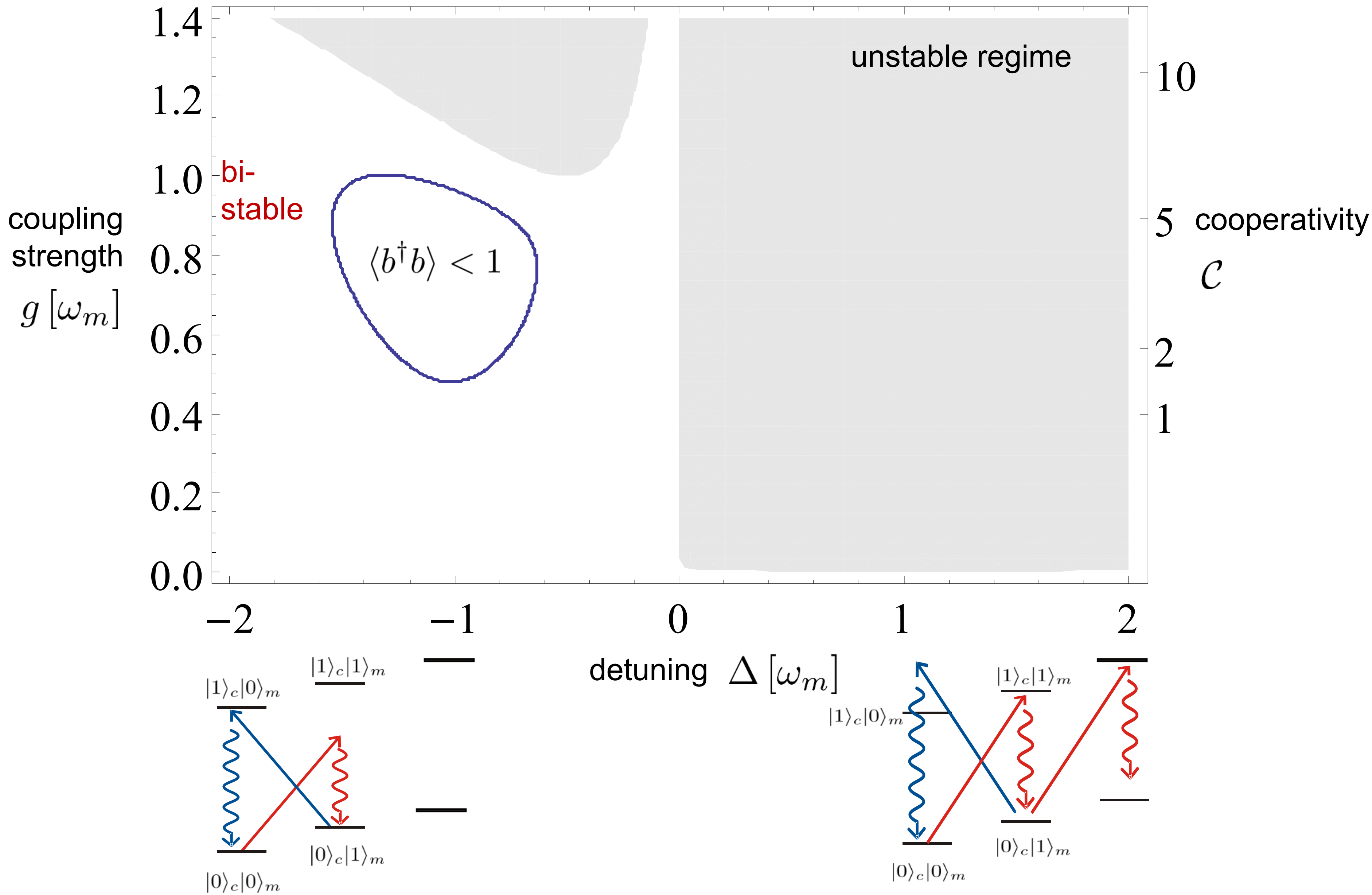
$\kappa/\Omega_m = 0.1$



# Weak Coupling Optomechanics



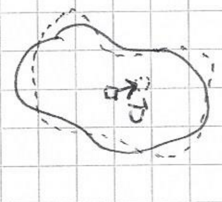
# Optomechanical Instabilities



# I Introduction to Optomechanics

## I.A Mechanical Oscillators

Deformation of a solid body described by ~~displacement~~ <u>displacement field</u>



$$\vec{u}(\vec{r},t) = \sum_m X_m(t)\, u_m(\vec{r})$$

eigenmodes $u_m(\vec{r})$ with amplitude $X_m(t)$

For small displacements harmonic oscillators

$$\ddot{X}_m(t) + \gamma_m \dot{X}_m(t) + \Omega_m^2 X_m(t) = \frac{F_{ext}(t)}{m_{eff}} \qquad (1)$$

resonance frequency $\Omega_m$ ~~[illegible]~~

pictures / values

damping constant $\gamma_m$

effective mass $m_{eff}$ (depends on mode volume of $\omega_m(\vec{r})$)

external forces $F_{ext}$

[ Damping is due to the coupling of eigenmodes to other ~~modes of~~ degrees of freedom of solid (phonons, ~~[illegible]~~ ) ]

Quality factor $Q_m = \frac{\Omega_m}{\gamma_m}$

$Q_m \gg 1 \rightarrow$ underdamped oscillations

Restrict to one eigenmode in the following.

These dofs will act as a bath on the mechanical eigenmodes and provide a fluctuating ~~force~~ (thermal) force $F_T(t)$ in $F_{ext}(t)$. In the following we assume $F_{ext}(t) = F_T(t)$

Quantum mechanical treatment

$$[X_m, P_m] = i\hbar$$

~~Dimensionless position and momentum~~

Intrinsic length scale = zero point fluctuation of ground state

Values
$\Omega_m \simeq 1\,\text{MHz}$
$m_{eff} \simeq 1\,\text{ng}$
$x_{ZPF} \simeq 10^{-14}\,\text{m}$
$= 10\,\text{fm}$

$$x_{ZPF}^2 = \langle 0 | X_m^2 | 0 \rangle = \sqrt{\frac{\hbar}{2 m_{eff} \Omega_m}}$$

Dimensionless position and momentum operators

$$x_m = \frac{X_m}{\sqrt{2}\, x_{ZPF}} \qquad p_m = \frac{P_m}{\sqrt{2}\, m_{eff} \Omega_m x_{ZPF}} = \frac{\sqrt{2}\, x_{ZPF}}{\hbar} P_m$$

which fullfill

$$[x_m, p_m] = i \, .$$

Creation + annihilation operators

$$b = \frac{1}{\sqrt{2}}(x_m + i p_m)$$

$$b^+ = \frac{1}{\sqrt{2}}(x_m - i p_m)$$

Harmonic oscillator Hamiltonian

$$H_m = \frac{P_m^2}{2 m_{eff}} + \frac{m_{eff}\Omega_m^2}{2} X_m^2 = \frac{\hbar\Omega_m}{2}(x_m^2 + p_m^2)$$

$$= \hbar\Omega_m \left( b^+ b + \frac{1}{2} \right)$$

equations of motion

$$\dot{x}_m = \tfrac{i}{\hbar}[H, x_m] = i\frac{\Omega_m}{2}[p_m^2, x_m] = i\frac{\Omega_m}{2}(-2ip_m)$$

$$= \Omega_m p_m \; ; \quad \dot{p}_m = \tfrac{i}{\hbar}[H, p_m] = -\Omega_m x_m$$

Quantum mechanical equations of motion equivalent to (1)

$$\dot{x}_m = \Omega_m p_m \qquad (2)$$
$$\dot{p}_m = -\Omega_m x_m - \gamma_m p_m + \sqrt{2\gamma_m}\, f_{\cancel{ext}\,T}(t)$$

where we use a scaled ~~th~~ thermal force

$$f_T(t) = \frac{F_T(t)\, x_{ZPF}}{\hbar \sqrt{\gamma_m}}$$

with dimension $[f_T(t)] = s^{-1/2}$.

For high-T ($\hbar\Omega_m \gg k_B T$) and high-$Q_m$ it is justified to adopt a zero mean, white noise model for ~~the fluctua~~ $f_T(t)$, ~~that is~~ assuming

$$\langle f_T(t) \rangle = 0 \qquad \text{(no average force)}$$

$$\langle f_T(t) f_T(t') + f_T(t') f_T(t) \rangle = (2\bar{n}+1)\, \delta(t-t') \qquad (3)$$

where

values

$\frac{k_B}{\hbar} = 10^{11}\ K^{-1} s^{-1}$

$\Omega_m = 1\ MHz$

$\bar{n} = 10^5\ T$

~~$\langle \ldots \rangle$~~ $\bar{n} = \frac{k_B T}{\hbar \Omega_m} \gg 1$

| T | $\bar{n}$ | |
|---|---|---|
| 4 K | $4 \cdot 10^5$ | $\Omega_m = 1\ MHz$ |
| 20 mK | 200 | |
| 300 K | $3 \cdot 10^7$ | |
| 20 mK | 1 | $\Omega_m = 200\ MHz$ |

is the average occupation number of the mechanical oscillator in thermal equilibrium.

~~Exercise: Solve (2) Show this Prove this!~~

Hint: ~~Solve~~ Use (2) and (3) in order to derive an equation of motion for the average occupation $\langle b^\dagger b \rangle$, Solve this ~~in the limit $t \to \infty$~~

~~and show~~ $\lim_{t\to\infty} \langle b^\dagger b \rangle_t = \bar{n}$

~~For~~ the creation operators (2) implies for high - Q - oscillators

$$\dot{b} = \frac{1}{\sqrt{2}}(\dot{x}_m + i\dot{p}_m)$$

$$= -i\Omega_m b - \frac{\gamma_m}{2}(b + b^+) + i\sqrt{\gamma_m}\, f_T(t)$$

$$\simeq -\left(i\Omega_m + \frac{\gamma_m}{2}\right) b + \sqrt{\gamma_m}\, b_{in}(t)$$

$$\langle b_{in}^+(t)\, b_{in}(t')\rangle = (\bar{n}_m + \tfrac{1}{2})\,\delta(t-t')$$

~~Example~~:

Approximation valid for $Q \gg 1$
implies equal damping of $x_m$ and $p_m$ at rate $\gamma_m/2$.

Example : Solution of EoM

$$b(t) = e^{-(i\tilde{\nu}_m + \frac{\gamma_m}{2})t} \left\{ b(0) + \sqrt{\gamma_m} \int_0^t dt'\, e^{(i\tilde{\nu}_m + \frac{\gamma_m}{2})t'} b_{in}(t') \right\}$$

$$\langle b^\dagger(t) b(t)\rangle = e^{-(-i\tilde{\nu}_m + \frac{\gamma_m}{2})t}\, e^{-(i\tilde{\nu}_m + \frac{\gamma_m}{2})t} \cdot \Big\{ \langle b^\dagger(0) b(0)\rangle$$

$$+ \gamma_m \int_0^t dt' \int_0^t dt''\, e^{(-i\tilde{\nu}_m + \frac{\gamma_m}{2})t'}\, e^{(i\tilde{\nu}_m + \frac{\gamma_m}{2})t''} \langle b_{in}^\dagger(t') b_{in}(t'')\rangle$$

$$+ \text{terms of order } \langle b_{in}^\dagger(t') b(0)\rangle + h.c. \Big\}$$

$0$

no initial correlations

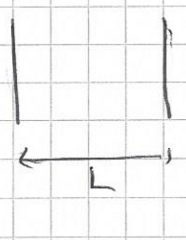
$$= e^{-\gamma_m t} \left\{ \langle b^\dagger(0) b(0)\rangle + \gamma_m \bar{n} \int_0^t dt'\, e^{\gamma_m t'} \right\}$$

$$n_m(t) = e^{-\gamma_m t} n_m(0) + (1 - e^{-\gamma_m t})\, \bar{n}_m$$

Phonon number decays at rate $\gamma_m$.

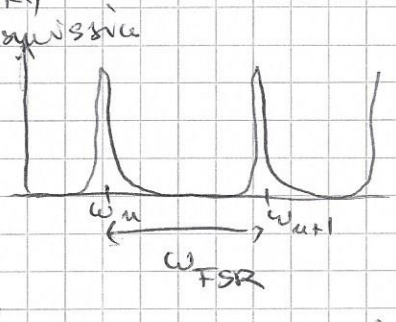
# I.3 Optical Resonators

Example: Fabry Perot resonator




exhibits resonances at

$$\omega_n = n\,\omega_{FSR} \qquad n \in \mathbb{N}$$




with free spectral range

$$\omega_{FSR} = 2\pi \frac{c}{2L}$$

Finite reflectivity of mirrors (and other losses) lead to a power decay rate $\kappa$ for the cavity resonance

The Finesse of the cavity

$$\mathcal{F} = \frac{\omega_{FSR}}{\kappa}$$

For $\mathcal{F} \gg 1$ resonances are resolved.
We will restrict to one cavity resonance, i.e. one cavity mode, with frequency $\omega_c$

Quantized description of cavity mode

$$[a, a^\dagger] = \mathbb{1} \qquad (4)$$

Electric field

$$\vec{E}(\vec{r}) = \vec{A}_c(\vec{r})\left(\frac{a + a^\dagger}{\sqrt{2}}\right)$$

where $\vec{A}_c(\vec{r})$ is the amplitude for a single photon in the mode corresponding to frequency $\omega_c$. We assume $\vec{A}_c \in \mathbb{R}$ for simplicity

Amplitude and phase quadrature

$$X_c = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad p_c = -\frac{i}{\sqrt{2}}(a - a^\dagger)$$

Obey (follows from (4))

$$[x_c, p_c] = i \qquad (5)$$

Hamiltonian

$$H_c = \hbar\omega_c(a^\dagger a + \tfrac{1}{2})$$

Equations of motion (neglecting damping + noise)

$$\dot{a}_c = i/\hbar\,[H_c, a] = -i\omega_c a$$

$$a_c(t) = a(0)\, e^{-i\omega_c t}$$

Time evolved field

$$\vec{E}(\vec{r},t) = \vec{A}_c \frac{1}{\sqrt{2}}(a e^{-i\omega_c t} + a^\dagger e^{i\omega_c t})$$

$$= \vec{A}_c(\cos(\omega_c t)\, x_c + \sin(\omega_c t)\, p_c)$$

$x_c$ ~ amplitude of field oscillating ~ $\cos\omega_c t$

$p_c$ - amplitude of field oscillating $\pi/2$ - phase shifted ~ $\sin\omega_c t$

Heisenberg uncertainty implied by (5)

$$\Delta x\, \Delta p \geq \frac{1}{2}|\langle [x,p]\rangle| = \frac{1}{2}$$

Amplitude and phase quadrature represent conjugate components of field

~~equation of motion~~

$\cancel{\dot{a} = i/\hbar}$

~~Hamiltonian~~

$$\cancel{H_c = \hbar\omega_c\left(a^\dagger a + \tfrac{1}{2}\right)}$$

Equations of motion

$$\cancel{\dot{a} = i/\hbar\,[H, a] = -}$$

$$\cancel{= i\omega_c\,[a^\dagger a, a] = -i\omega_c a}$$

Equations of motion
including decay and noise

$$\dot{a} = -\left(i\omega_c + \frac{\kappa}{2}\right) a + \sqrt{\kappa}\, a_{in}(t)$$

Noise is due to thermal or vacuum fluctuations of the field outside of cavity leaking into the cavity

~~For optical cavities~~

Zero Mean
White noise model $\quad [a_{in}(t), a^\dagger_{in}(t')] = \delta(t-t')$

$$\langle a_{in}(t) \rangle = 0 \qquad \langle a^\dagger_{in}(t)\, a_{in}(t') \rangle = \bar{n}_c\, \delta(t-t')$$

$$\langle a_{in}(t)\, a^\dagger_{in}(t') \rangle = (\bar{n}_c + 1)\, \delta(t-t')$$

[all other averages vanish]

where $\bar{n}_c$ is the thermal average photon number in thermal equilibrium. For optical fields at room temperature $\bar{n}_c \approx 0$.

# I. C. Optomechanical Interaction

Consider a cavity mode whose frequency depends parametrically on the displacement of a mechanical oscillator,

$$\omega_c(X_m) \,.$$

The Hamiltonian for cavity + mec. osc.

$$H = H_m + H_c = \hbar\Omega_m b^\dagger b + \hbar\omega_c(X_m)\, a^\dagger a$$

For small displacements

$$\omega_c(X_m) \simeq \omega_c(0) + \frac{\partial\omega_c(0)}{\partial x} X_m$$

$$= \omega_c(0) + \frac{\partial\omega_c(0)}{\partial x} x_{ZPF}\sqrt{2}\, X_m$$

$$= \omega_c - g_0 (b + b^\dagger)$$

Single photon optomechanical coupling

$$g_0 = -\frac{\partial\omega_c}{\partial x} x_{ZPF}$$

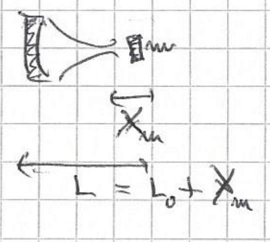
Hamiltonian

$$H = \hbar\Omega_m b^\dagger b + \hbar\omega_c a^\dagger a - \hbar g_0 a^\dagger a (b + b^\dagger) \qquad (6)$$

$g_0 \sim$ how much does the ~~energy~~ frequency of the cavity mode shift per ZPF-displacement

Example 1: Fabry-Perot with movable end mirror

$$\omega_c = n\,\omega_{FSR} = \frac{n\pi c}{L} \qquad n \in \mathbb{N}$$




$$\omega_c(X_m) = \frac{n\pi c}{L_0 + X_m}$$

$$g_0 = -\frac{\partial \omega}{\partial X_m} x_{ZPF} = \omega_c \frac{x_{ZPF}}{L}$$

values $\cancel{L \sim \lambda = \frac{2\pi c}{\omega_c}}$

$\cancel{g_0 = \frac{\omega_c^2 x_{ZPF}}{2\pi c} = \frac{10^{28} \cdot 10^{-14}}{10^{9}} \sim 10^{6}}$

Example 2: ~~Photonic~~ Optomechanical crystal

dielectric material (lossless, non-dispersive) with scalar permittivity

$$\epsilon(\vec{r}) = \begin{cases} \epsilon_1 & \vec{r} \in V \\ \epsilon_2 & \vec{r} \notin V \end{cases} = \epsilon_2 + (\epsilon_1 - \epsilon_2)\,\Theta(\vec{r}) \qquad \Theta(\vec{r}) = \begin{cases} 1 & \vec{r} \in V \\ 0 & \vec{r} \notin V \end{cases}$$

Can generate localized phonon and photon modes via band shaping

Energy of EM-field

$$H_{EM} = \frac{1}{2}\int dV \left\{ \frac{1}{\epsilon(\vec{r})} |\vec{D}(\vec{r})|^2 + \frac{1}{\mu_0} |\vec{B}(\vec{r})|^2 \right\}$$

electric displacement field $\vec{D}(\vec{r}) = \epsilon(\vec{r})\, E(\vec{r})$

Mechanical displacement $\vec{u}(\vec{r})$ affects permittivity by

i) electrostriction: change of permittivity in medium $\epsilon_1(\vec{u}(\vec{r}), \vec{r})$

ii) radiation pressure: change of boundary of medium $\Theta(\vec{u}(\vec{r}), \vec{r})$

For small structures radiation pressure is dominant.

First order correction of $\epsilon(\vec{r})$ w.r.t $\vec{u}(\vec{r})$

$$d\epsilon(\vec{r}) = (\epsilon_1 - \epsilon_2)\ \vec{u}(\vec{r}) \cdot \underline{\vec{\nabla}\theta(\vec{r})}$$

act. ~ Dirac-$\delta$ on surface $\partial V$ of $V$

~~Correction in Hamiltonian~~

In $H_{EM}$ we need the correction to the inverse permittivity

$$\epsilon^{-1}(\vec{r}) = \frac{1}{\epsilon_2} + \left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right)\theta(\vec{r})$$

$$\rightarrow \quad d\epsilon^{-1}(\vec{r}) = \left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right)\vec{u}(\vec{r}) \cdot \vec{\nabla}\theta(\vec{r})$$

Correction to Hamiltonian

$$dH_{EM} = \frac{1}{2}\int dV\ d\epsilon^{-1}(\vec{r})\ |\vec{D}(\vec{r})|^2$$

$$= \frac{1}{2}\int_{\partial V} d\vec{A} \cdot \vec{u}(\vec{r}) \left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right)|\vec{D}(\vec{r})|^2$$

Ill-defined because ~~$D_\parallel(\vec{r})$~~ parallel components $\vec{D}_\parallel(\vec{r})$ on $\partial V$ are discontinuous!

Solution: Decompose $dH$ on $\partial V$ in continuous components $\vec{D}_\perp(\vec{r})$ and $\vec{E}(\vec{r}) = \vec{D}(\vec{r})/\epsilon(\vec{r})$

$$dH_{EM} = \frac{1}{2}\int dV \left\{ - d\epsilon(\vec{r})\,|\vec{E}_\parallel|^2 + d\epsilon^{-1}(\vec{r})\,|\vec{D}_\perp|\right\}$$

$$= -\frac{1}{2}\int_{\partial V} d\vec{A} \cdot \vec{u}(\vec{r}) \left\{(\epsilon_1 - \epsilon_2)\,|\vec{E}_\parallel|^2 + \left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right)|\vec{D}_\perp|^2\right\}$$

For structure with single relevant phonon and photon mode ~~$\mathcal{H} = g_0 a^\dagger a (b+b^\dagger)$~~

$$|E_q|^2 \sim |D_\perp|^2 \sim a^\dagger a$$

$$\vec{U} \sim \cancel{(b+b^\dagger)}\, x_m \sim (b+b^\dagger)$$

$$\mathcal{H} = g_0 a^\dagger a (b+b^\dagger)$$

## I.D ~~Nonlinear~~ Optomechanical equations of motion

$(\hbar = 1)$

(6) simplifies, including loss + noise

~~Hamiltonian $H = \hbar\Omega_m b^\dagger b + \hbar\omega_c a^\dagger a + \hbar g_0 a^\dagger a (b+b^\dagger)$~~

$$\dot{a} = -(i\omega_c + \tfrac{\kappa}{2})a + i g_0 (b+b^\dagger) a + \sqrt{\kappa}\, a_{in}(t)$$

$$\dot{b} = -(i\Omega_m + \tfrac{\gamma_m}{2}) b + i g_0 a^\dagger a + \sqrt{\gamma_m}\, b_{in}(t)$$

Interpretation

$$\dot{a} = -\left\{ i \left[\omega_c - g_0 \underbrace{(b+b^\dagger)}_{\sim x_m}\right] + \tfrac{\kappa}{2} \right\} a + \sqrt{\kappa}\, a_{in}(t)$$

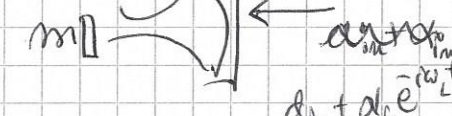
shift of frequency $\sim x_m$

$$\dot{x}_m = \Omega_m p_m - \tfrac{\gamma_m}{2} x_m + \sqrt{\gamma_m/2}\,(b_{in} + b_{in}^\dagger)$$

$$\dot{p}_m = -\Omega_m \cancel{p_m}\, x_m - \tfrac{\gamma_m}{2} p_m + \underbrace{\sqrt{2} g_0\, a^\dagger a}_{} + \sqrt{\gamma_m/2}\,(b_{in} + b_{in}^\dagger)$$

force on mechanical oscillator ~ photon number

Including a laser drive



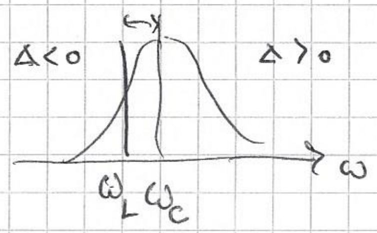

$$a_{in} \longrightarrow a_{in} + \alpha_{in} e^{-i\omega_L t}$$

where $\alpha_{in} = \sqrt{\frac{P}{\hbar\omega_L}} = \sqrt{\frac{\text{photons}}{\text{sec}}} \in \cancel{\mathbb{C}}\,\mathbb{R}$

Refer cavity operators to frame rotating at the drive frequency

$$\tilde{a} = a e^{+i\omega_L t}$$

$$\dot{\tilde{a}} = \left[i(\omega_L-\omega_c) - \frac{\kappa}{2}\right]\tilde{a} + i g_0 (b+b^\dagger)\tilde{a} + \sqrt{\kappa}\,\tilde{a}_{in} + \sqrt{\kappa}\,\alpha_{in}$$




detuning $\Delta_0 = \omega_L - \omega_c$

Drop tilde in the following; remember we are working in a frame rotating at $\omega_L$.

We expect that $a$ and $b$ will acquire a ~~mean~~ non-zero mean component due to driving field $\alpha_{in}$.

Let $\alpha = \langle a \rangle$ and $\beta = \langle b \rangle$

and fluctuations

$$\delta a = a - \langle a \rangle \qquad \delta b = b - \langle \beta \rangle$$

$$\begin{aligned}\delta\dot{a} = {} & \left(i\Delta_0 - \frac{\kappa}{2}\right)\alpha + i g_0 (\beta+\beta^*)\alpha + \sqrt{\kappa}\,\alpha_{in} \\ & + \left(i\Delta_0 - \frac{\kappa}{2}\right)\delta a + i g_0 (\beta+\beta^*)\delta a + i g_0 \alpha(\delta b + \delta b^\dagger) + \sqrt{\kappa}\, a_{in} \\ & + i g_0 (\delta b + \delta b^\dagger)\delta a\end{aligned}$$

$$\dot{\delta b} = -\left(i\Omega_m + \frac{\gamma_m}{2}\right)\beta + i g_0 |\alpha|^2$$

$$- \left(i\Omega_m + \frac{\gamma_m}{2}\right)\delta b + i g_0\left(\alpha\,\delta a^\dagger + \alpha^* \delta a\right) + \sqrt{\gamma_m}\,(b_{in}$$

$$+ i g_0\, \delta a^\dagger \delta a$$

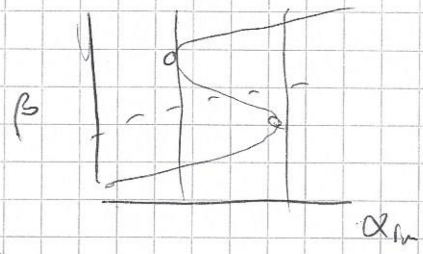
¢-number component: set to zero

$$(i\Delta_0 - \tfrac{\kappa}{2})\alpha + i g_0(\beta + \beta^*)\alpha + \sqrt{\kappa}\,\alpha_{in} = 0$$

$$-(i\Omega_m + \tfrac{\gamma_m}{2})\beta + i g_0 |\alpha|^2$$

Equivalent to cubic equation for $\alpha$ or $\beta$.
Three solutions possible for $|\Delta_0| \geq \sqrt{3/4}\,\kappa$

Assume stable solution for $\alpha$ and $\beta$.




Absorb mean shift in effective detuning

$$\Delta = \Delta_0 + g_0(\beta + \beta^*)$$

Effective optomechanical coupling

$$g = g_0\, \alpha$$

Assumption!

Note: $|\alpha|^2 = \#$ photons in cavity $\gg 1$

↳ $g \gg g_0$

Neglect quadratic terms in dynamics.
Linearized equations of motion (drop $\delta$!)

$$\dot{a} = (i\Delta - \tfrac{\kappa}{2})\, a + i g\,(b + b^\dagger) + \sqrt{\kappa}\, a_{in}$$

$$\dot{b} = -(i\Omega_m + \tfrac{\gamma_m}{2})\, b + i g\,(a + a^\dagger) + \sqrt{\gamma_m}\, b_{in}$$

Effective equations of motion follow from effective Hamiltonian

$$H = -\Delta a^\dagger a + \Omega_m b^\dagger b - g(b+b^\dagger)(a+a^\dagger)$$

dynamics of

EoMs describe fluctuations around mean amplitudes $\alpha$ and $\beta$.

~~Stability of dynamics~~

~~Dynamics are described~~

EoMs are linear in operators, Hamiltonian is quadratic: ~~Equa~~ EoMs of average values are the same as classical EoMs (Ehrenfest theorem).

Stability of dynamics:

Define $\vec{A} = (a, a^\dagger, b, b^\dagger)^T$

$$\dot{\vec{A}} = M\vec{A} + \vec{A}_{in}$$

Stability $\longmapsto$ $\mathrm{Re}(\text{eigenvalues}(M)) < 0$

$\Delta$

# Weak Coupling Optomechanics



$$\dot{a} = (i\Delta - \tfrac{\kappa}{2}) a + ig(b + b^+) + \sqrt{\kappa}\, a_{in}$$

$$\dot{b} = -(i\Omega_m + \tfrac{\gamma_m}{2}) b + ig(a + a^+) + \sqrt{\gamma_m}\, b_{in}$$

solve for $b(t)$ assuming

- ~~red detuning~~
- weak coupling $\frac{g}{\kappa} \ll 1$
- no restriction on $\frac{\Omega_m}{\kappa}, \frac{\Delta}{\kappa}$

Move to rotating frame

$$\tilde{a} = a\, e^{-i\Delta t} \qquad \tilde{b} = b\, e^{i\Omega_m t}$$

$$\dot{\tilde{a}} = -\frac{\kappa}{2}\tilde{a} + ig\left(\tilde{b}\, e^{-i(\Delta+\Omega_m)t} + \tilde{b}^+ e^{-i(\Delta-\Omega_m)t}\right)$$

$$+ \sqrt{\kappa}\, e^{-i\Delta t} a_{in}$$

adiabatic solution correct in $O(\frac{g}{\kappa})$

$$\tilde{a}(t) = ig \int_0^t d\tau\, e^{-\kappa/2\,\tau} \left\{ \overset{\tilde{b}(t)}{\overset{\text{is}}{\tilde{b}_{(t-\tau)}}}\, e^{-i(\Delta+\Omega_m)(t-\tau)} + \tilde{b}^+_{(t-\tau)}\, e^{-i(\Delta-\Omega_m)(t-\tau)} \right\}$$

$$+ \sqrt{\kappa}\, \int_0^t dt'\, e^{+\kappa/2(t'-t)}\, e^{-i\Delta t'}\, a_{in}(t')$$

$$\simeq ig\left\{ \tilde{b}\, \eta_+\, e^{-i(\Delta+\Omega_m)t} + \tilde{b}^+ \eta_-\, e^{-i(\Delta-\Omega_m)t} \right\}$$

$$+ \frac{\cancel{\sqrt{\kappa}}}{\kappa/2 - i\Delta}\frac{2}{\sqrt{\kappa}}\, \tilde{a}_{in}(t)\, \cancel{e^{-i\Delta t}}$$

← coarse grained white noise

$$\eta_\pm = \frac{1}{\kappa/2 - i(\Delta \pm \Omega_m)}$$

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$$\dot{\tilde{b}} = -\frac{\gamma_m}{2}\tilde{b} + ig\left(\tilde{a}\, e^{i(\Omega_m+\Delta)t} + \tilde{a}^\dagger e^{i(\Omega_m-\Delta)t}\right) + \sqrt{\gamma_m}\, \tilde{b}_{in}\, e^{i\Omega_m t}$$

insert

$$\dot{\tilde{b}} = -\frac{\gamma_m}{2}\tilde{b} + \sqrt{\gamma_m}\, b_{in}\, e^{i\Omega_m t}$$

$$- g^2(\eta_- - \eta_+^*)\tilde{b} + \frac{i\sqrt{2}g}{\sqrt{2}} e^{i\Omega_m t}\left(\frac{\sqrt{\kappa}}{\kappa/2 - i\Delta}\tilde{a}_{in} + \frac{\sqrt{\kappa}}{\kappa/2 + i\Delta}\tilde{a}_{in}^\dagger\right)$$

$$+ O(e^{2i\Omega_m t})$$

effect of OM-coupling

- shift in frequency $\delta\Omega_m = g^2 \operatorname{Im}(\eta_- - \eta_+^*)$
- shift in damping $\Gamma_{opt} = 2g^2 \operatorname{Re}(\eta_- - \eta_+^*)$
- additional radiation pressure noise " $\Gamma_- - \Gamma_+$

$$\dot{b} = -\left[i(\Omega_m + \delta\Omega_m) + \frac{\gamma_m + \Gamma_{opt}}{2}\right] b$$

$$+ \sqrt{\gamma_m}\, b_{in}(t) + \sqrt{2}g\left(\frac{\sqrt{\kappa}}{\kappa/2 - i\Delta} a_{in} + h.c.\right)$$

$\Rightarrow$ Red Detuning: Cooling + State swap $\Rightarrow i\sqrt{\Gamma_-}\, a_{in,-} - i\sqrt{\Gamma_+}\, a_{in,+}^\dagger$

Mean occupation number in stationary state (if exists)

$\langle a_{in,\pm}\, a_{in,\pm}^\dagger \rangle = 0$

$[a_{in,\pm}, a_{in,\pm}^\dagger] = \delta(t-t')$

$\langle a_{in,+}\, a_{in,+}^\dagger \rangle$

$$n_{eff} = \langle b^\dagger b\rangle_{t\to\infty}$$

$$= \frac{\gamma_m}{\gamma_m + \Gamma_{opt}}\,\underbrace{\bar{n}}_{\langle b_{in}^\dagger b_{in}\rangle} + \frac{\Gamma_+}{\gamma_m + \Gamma_{opt}}$$

Resolved sideband cooling

$$\Delta = -\Omega_m$$

$$\Gamma_- \simeq \Gamma_{opt} = \frac{4g^2}{\kappa} \gg \gamma_m \qquad \Gamma_+ \simeq \frac{4g^2}{\kappa}\left(\frac{\kappa}{4\Omega_m}\right)^2$$

$$n_{eff} = \frac{\kappa\gamma_m \bar{n}}{4g^2} + \left(\frac{\kappa}{4\Omega_m}\right)^2$$

$$\bar{n} \longrightarrow \bar{n}_{eff} \approx \frac{1}{C} + \left(\frac{\kappa}{4\Omega_m}\right)^2$$

for $\Omega_m \gg \kappa$, $\Delta = -\Omega_m$,

$$C = \frac{4g^2}{\kappa\gamma\bar{n}} \approx \frac{\Gamma_{opt}}{\gamma\bar{n}} \gg 1 \text{ thermal cooperativity}$$

$$\hookrightarrow \quad \Gamma_{opt} \approx \frac{4g^2}{\kappa} > \gamma\bar{n} \text{ .. heating rate}$$

remaining occupation due to Antistokes scattering vacuum fluctuations of light

cooling is due to broadening of mechanical line

$$\gamma_m \longrightarrow \gamma_m^{eff} = \gamma_m + \Gamma_{opt} \approx \Gamma_{opt}$$

$$Q_m \longrightarrow Q_m^{eff} = \frac{\Omega_m}{\gamma_m^{eff}} \ll Q_m$$

thermal decoherence

$$\gamma_{th} = \gamma_m \bar{n} \longrightarrow \gamma_m^{eff}\bar{n}_{eff} = \Gamma_{opt}\frac{\gamma\bar{n}}{\Gamma_{opt}} = \underset{!}{\gamma\bar{n}}$$

~~What happens for~~

Time dependent dynamics for $\Delta = -\Omega_m$, $\Omega_m \gg \kappa$, $\Gamma_{opt} \gg \gamma\bar{n}$ in rotating frame

$$\dot{b} = -\left[\frac{\Gamma_{opt}}{2}\right] b + i\sqrt{\Gamma_{opt}}\, a_{in}(t)$$

$$+ \text{ thermal noise } + \text{ Antistokes}$$

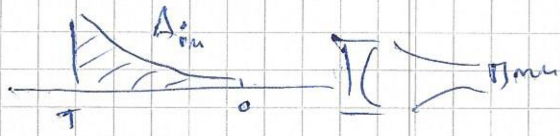
$$b_{out} = b(t) = e^{-\Gamma_{opt}t/2}\left(\underset{b_{in}}{\overset{\shortparallel}{b(0)}} - i\sqrt{\Gamma_{opt}}\int_0^t dt'\, e^{t'\Gamma_{opt}/2}\, a_{in}(t')\right)$$

$$A_{in} = -i z^{-1}\int_0^T dt'\, e^{t'\Gamma/2}\, a_{in}(t') \qquad [a_{in}(t), a_{in}^\dagger(t')] = \delta(t-t')$$

$$[A_{in}, A_{in}^\dagger] = z^{-2}\int_0^T dt\, e^{\Gamma t} = z^{-2}\,\frac{e^{\Gamma T}-1}{\Gamma} = 1$$

$$z = \sqrt{\frac{e^{\Gamma T}-1}{\Gamma}}$$

$$b_{out}(T) = b_{in}(0)\, e^{-\Gamma_{opt}T/2} + \frac{g}{|g|}\sqrt{1-e^{-\Gamma_{opt}T/2}}\; A_{in}$$




$\Gamma_{opt} T \gg 1$

$$= \frac{g}{|g|} A_{in}$$

$$b^\dagger(T) = A_{in}^\dagger \qquad \text{State swap!}$$

$$|\psi\rangle_L |0\rangle_m = \sum_n c_n (A_{in}^\dagger)^n |0\rangle_L |0\rangle_m$$

$$\to \sum_n c_n (b^\dagger)^n |0\rangle_L |0\rangle_m = |0\rangle_L |\psi\rangle_m$$

$$\frac{1}{\Gamma_{opt}} \lesssim T \ll \frac{1}{\gamma_m \bar n_{th}} = \frac{1}{\gamma \bar n} \to 1 \ll \frac{\Gamma_{opt}}{\gamma \bar n} = C$$

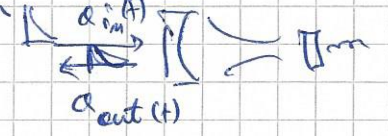
$$\Gamma_{opt} T \gg 1$$

What happens to state of mechanics?

$$A_{out} \overset{?}{=} b_{in}$$

Cavity input-output relation

$$a_{out}(t) = a_{in}(t) - \sqrt{\kappa}\, a(t)$$




insert adiabatic solution for $a(t)$, integrate, define mode function

$$\cancel{a_{out}(t) = a_{in}(t) - \sqrt{\Gamma_{opt}}\, b(t)}$$

$$A_{out} = e^{-T\Gamma_{opt}/2} A_{in} + \sqrt{1-e^{-T\Gamma_{opt}/2}}\, b_{in}$$

$$\overset{T\Gamma_{opt}\gg 1}{=} b(0)$$

## Blue Detuning: Instability and Entanglement

Instability in $\Delta > 0$ is due to $\qquad |\Gamma_+| \gg \Gamma_-$

$$\gamma_m^{eff} = \gamma_m + \Gamma_{opt} < 0 \qquad \Gamma_{opt} \simeq -\frac{4g^2}{\kappa} = -\Gamma$$

No steady state described by linear model

→ Limit cycles stabilized by nonlinearity of optomechanical interaction

Time dependent dynamics:

$$\dot{b} = -\left(-\frac{\Gamma}{2}\right) b - i\sqrt{\Gamma}\, a_{in}^{\dagger}(t)$$

$$B_{out} = b(T) = e^{\Gamma T/2}\, b_{in} + \sqrt{e^{\Gamma T} - 1}\, A_{in}^{\dagger}$$

using input output relations

$$A_{out} = e^{\Gamma T/2} A_{in} + \sqrt{e^{\Gamma T} - 1}\, b_{in}$$

Corresponds to a two-mode squeezing dynamics: Unitary

$$S = \exp[(Ab - A^{\dagger}b^{\dagger})r]$$

$$A_{out} = S A_{in} S^{\dagger} = \cosh(r) A_{in} + \sinh(r) b^{\dagger}$$

$$b_{out} = \cosh(r) b + \sinh(r) A^{\dagger}$$

$$r = \operatorname{arccosh}(e^{\Gamma T/2})$$

Schrödinger picture

$$S|0\,0\rangle = \frac{1}{\cosh(r)} \sum_n [-\tanh(r)]^n e^{in\varphi} |n, n\rangle$$

H Optomechanics on resonance: Position/Force Sensing + Standard Quantum Limit

$$g < \kappa \;;\; \Delta = 0 \;\longrightarrow\; \delta\Omega_m = 0 \;,\; \Gamma_{opt} = 0$$

$$\dot{b} = -\left[i\Omega_m + \frac{\gamma_m}{2}\right] b + \sqrt{\gamma_m}\, b_{in}(t) - i\sqrt{\Gamma}\,(a_{in} + a_{in}^{\dagger})$$

$$\Gamma = \frac{2g^2}{\sqrt{(\kappa/2)^2 + \Omega_m^2}} \overset{\kappa \gg \Omega_m}{\approx} \frac{4g^2}{\kappa}$$

radiation pressure noise due to amplitude vacuum fluctuations of incoming field

$$X_{in}^{L}(t) = \frac{1}{\sqrt{2}}(a_{in} + a_{in}^{\dagger}) \qquad P_{in}^{L} = -\frac{i}{\sqrt{2}}(a_{in} - a_{in}^{\dagger})$$

$$[X_{in}^{L}(t), P_{in}^{L}(t')] = i\,\delta(t - t')$$

$$\langle x_{in}^L(t)\, x_{in}^L(t') + x_{in}(t')\, x_{in}(t) \rangle = \delta(t-t')$$

$$\langle p_{in}^L(t)\, p_{in}^L(t') + p_{in}(t')\, p_{in}(t) \rangle = \delta(t-t')$$

EoM for mechanical quadratures

$$\dot{x}_m(t) = \Omega_m p_m - \frac{\gamma_m}{2} x_m + \sqrt{\gamma_m}\, x_{in}^m(t)$$

$$\dot{p}_m(t) = -\Omega_m x_m - \frac{\gamma_m}{2} p_m + \sqrt{\gamma_m}\, p_{in}^m(t) + \sqrt{\Gamma}\, x_{in}^L(t)$$

adiabatic solution for cavity

$$a \approx i \frac{2g}{\kappa} (b + b^\dagger) + \frac{2}{\sqrt{\kappa}} a_{in}$$

$$a_{out} = a_{in} - \sqrt{\kappa}\, a$$

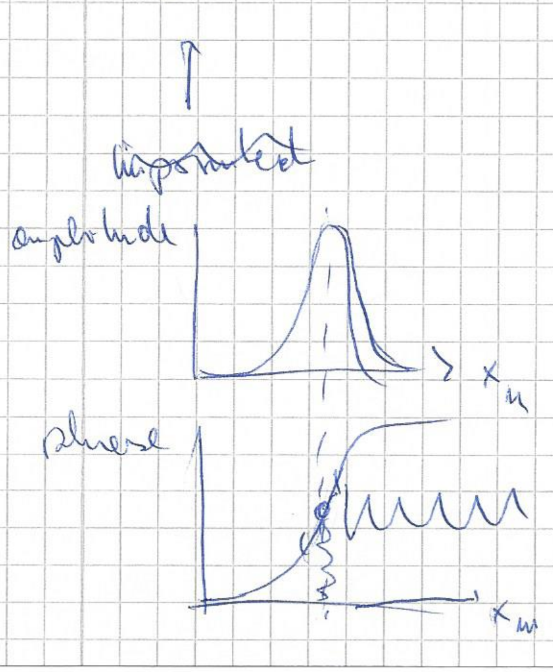
$$= - a_{in} - i \sqrt{\Gamma} (b + b^\dagger)$$

$$x_{out}^L(t) = - x_{in}^L(t)$$

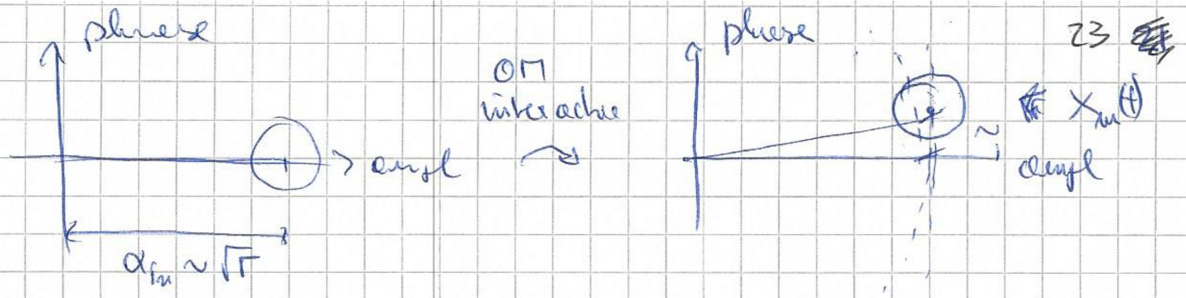
$$p_{out}^L(t) = - p_{in}^L(t) + \sqrt{\Gamma}\, x_m(t)$$

unperturbed

Position of oscillator imprinted on phase of light leaving cavity




slope $\sim \sqrt{\Gamma}$ $\qquad \Gamma = \frac{4g^2}{\kappa}$ "readout rate"




~~Sensitivity of measurement of position at time t~~

Integrate over time T

$$P^{\text{out}}_{\text{out}} \cong \frac{1}{\sqrt{T}}\int dt\, P^{L}_{\text{out}}(t) = P^L_{\text{in}} + \sqrt{\Gamma T}\, x_m(t)$$

$$\rightarrow x_m^{\text{est}}(t) = \frac{1}{\sqrt{\Gamma T}} P^{\text{out}}_L = x_m(t) + \underbrace{\frac{1}{\sqrt{\Gamma T}} P^L_{\text{in}}}_{\text{added noise}}$$

signal strength $\cancel{\sqrt{\Gamma T}}$ $\Gamma T$

Shot noise variance $(\Delta P^L_{\text{in}})^2 = 1/2$

Sensitivity : $\frac{\text{noise variance}}{\text{signal}} = \frac{(\Delta P^L_{\text{in}})^2}{\Gamma T} = \frac{1}{2\Gamma T}$

with units $x^2_{ZPF} \cdot \frac{1}{\Gamma T}$ [m²/sec]

But : At later times $x_m(t+\tau)$ will be impaired by amplitude fluctuations.

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~~Solution of $x_m$~~ $\quad x(\Omega) = \int dt\, x(t)\, e^{i\Omega t}$ $\quad [x(\Omega)] = \frac{1}{Hz}$

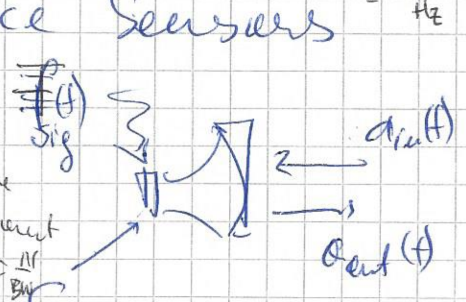
with units $[x(\Omega) \times x_{ZPF}] = \frac{m}{Hz}$

Optomechanical System as Force Sensors

Solution to EOMs in frequency space

estimate Fourier component $f(\Omega)$ $\quad [f(\Omega)] = \frac{N}{Hz}$

~~$[f_{sig}] \sim \frac{N}{sec}$~~




$$x(\Omega) = \chi_m(\Omega)\left(f_{sig}(\Omega) + \sqrt{\gamma_m}\, f_T(\Omega) + \sqrt{\Gamma}\, x^L_{in}(\Omega)\right)$$

$$\chi_m(\Omega) = \frac{\Omega_m}{\Omega_m^2 - \Omega^2 - i\gamma_m \Omega} \overset{Q \gg 1}{\simeq} \frac{1}{(\Omega_m - \Omega) - i\gamma_m}$$

Input - Output for Light

$$P^L_{out}(\Omega) = -P^L_{in}(\Omega) + \sqrt{\Gamma}\, x_m(\Omega)$$

$$= \sqrt{\Gamma}\,\chi_m(\Omega)\, f_{sig}(\Omega)$$

$$+ \underbrace{\left\{ -P^L_{in}(\Omega) + \sqrt{\Gamma}\,\chi_m(\Omega)\, f_T(\Omega) + \xi\,\Gamma\,\chi_m(\Omega)\, x^L_{in}(\Omega) \right\}}_{\text{added noise}}$$

$$f_{est}(\Omega) = \frac{1}{\sqrt{\Gamma}\,\chi_m(\Omega)}\, P^L_{out}$$

Spectral density of noise process $x(\Omega)$

$$S(\Omega)\,\delta(\Omega - \bar{\Omega}) = \langle x(\Omega)\, x^\dagger(\bar{\Omega}) + x(\bar{\Omega})\, x^\dagger(\Omega) \rangle$$

E.g. $x^L_{in}(\Omega)$ $\quad S_{x_{in}}(\Omega) = 1/2$

$P^L_{in}(\Omega)$ $\quad S_{P_{in}}(\Omega) = 1/2$

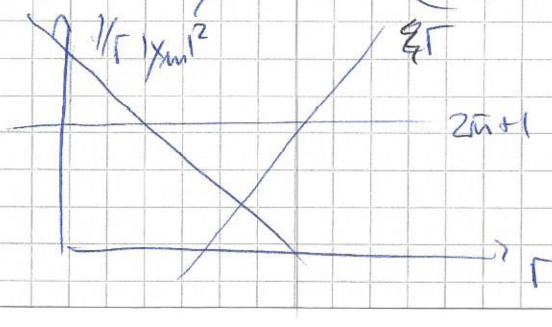
$f_T(\Omega)$ $\quad S_{f_T}(\Omega) = 2\bar{n} + 1/2$

$$S_{\text{added noise}}(\Omega) = \frac{1}{2} + \Gamma\,|\chi_m(\Omega)|^2\,(2\bar{n}+1) + \xi^2\,\Gamma^2\,|\chi_m(\Omega)|^2\,\frac{1}{4}$$

Signal strength $\Gamma\,|\chi_m|^2$

sensitivity at frequency $\Omega$

$$S(\Omega) = \frac{1}{\Gamma|\chi_m|^2} + (2\bar{n}+1) + \xi^2\,\Gamma$$

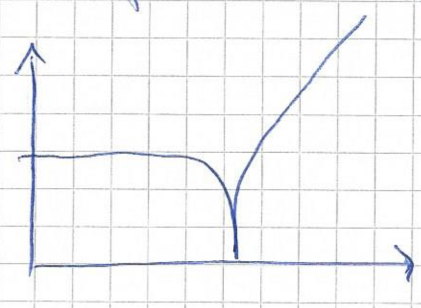


neglect thermal noise

$$S(\Omega) \geq 2\sqrt{\frac{\hbar}{\hbar |\chi_m|}} = \frac{\hbar}{|\chi_m|}$$

$\alpha x + \beta/x \geq 2\sqrt{\alpha\beta}$

Standard Quantum Limit of Continuous Position Sensing



$$SQL(\Omega) \geq 2\hbar\sqrt{(\Omega-\Omega_m)^2 + (\gamma/2)^2}$$

$$= 2\begin{cases} \hbar\,\Omega_m & \Omega < \Omega_m \\ \hbar\,\gamma_m/2 & \Omega = \Omega_m \\ \hbar\,\Omega & \Omega > \Omega_m \end{cases}$$

with units multiply by $\hbar$/meff $\Omega_m$

Back Action

Measurement Noise above Thermal Noise

$$\gamma_m \Gamma\, |\chi_m(\Omega)|^2 (\bar{n}_m + 1/2) < \Gamma^2 |\chi_m|^2$$

$$\gamma_m \bar{n} \simeq \gamma_m(\bar{n} + 1/2) < \Gamma$$

$$\frac{\Gamma}{\gamma_m \bar{n}} = C_{th} > 1$$

$$[S_{SQL}] = \frac{(N/Bw)^2}{s}$$