Outline

- Digital CMOS Design
- Arithmetic Operators
  - Adders
  - Comparators
  - Shifters
  - Multipliers
  - Square rooting

Square root

An real number y using floating point representation

Find a real number x such as

$$(x+\varepsilon)^2 = y$$

- Calculation cannot be implemented in hardware
- Need iterative operation

Square root

- Direct method ⇨ digit-by-digit
- Indirect method ⇨ resolve $x^2 - y = 0$

Square root - direct

Find x such as $x = \sqrt{y}$

$$y = (-1)^0 \times M \times 2^E \quad \textit{With} \quad M \in [1, 2[$$

$$\sqrt{y} = (-1)^0 \times \sqrt{M} \times 2^{\frac{E}{2}}$$

E odd ?

$$y = (-1)^0 \times M' \times 2^{2E'} \quad \textit{With} \quad M' \in [1, 4[$$

$$x = \sqrt{y} = (-1)^0 \times \sqrt{M'} \times 2^{E'}$$

$$x = (-1)^0 \times X \times 2^{E'} \quad \textit{With} \quad X \in [1, 2[$$

Square root - direct

Find x such as $x = \sqrt{y}$

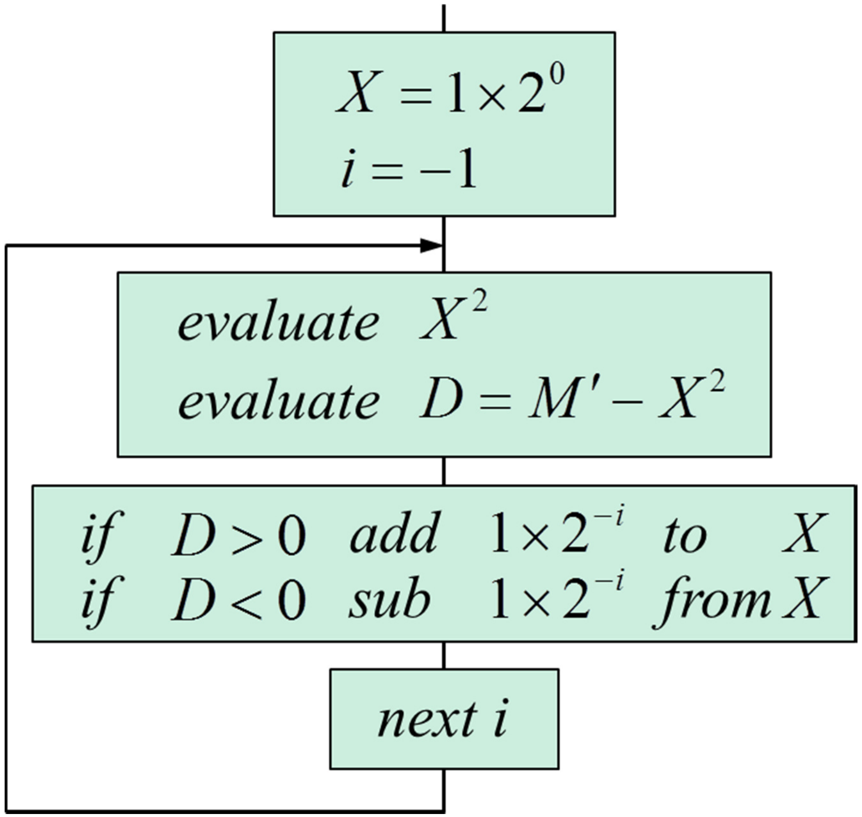
$$y = (-1)^0 \times M' \times 2^{2E'} \quad With \quad M' \in [1, 4[$$
$$x = (-1)^0 \times X \times 2^{E'} \quad With \quad X \in [1, 2[$$

$$X = \sum_{i=0}^{-n} x_i \times 2^i$$

Iterate on i and evaluate x_i

Square root - direct

$$y = (-1)^0 \times M' \times 2^{2E'} \quad \textit{With} \quad M' \in [1, 4[$$
$$x = (-1)^0 \times X \times 2^{E'} \quad \textit{With} \quad X \in [1, 2[$$



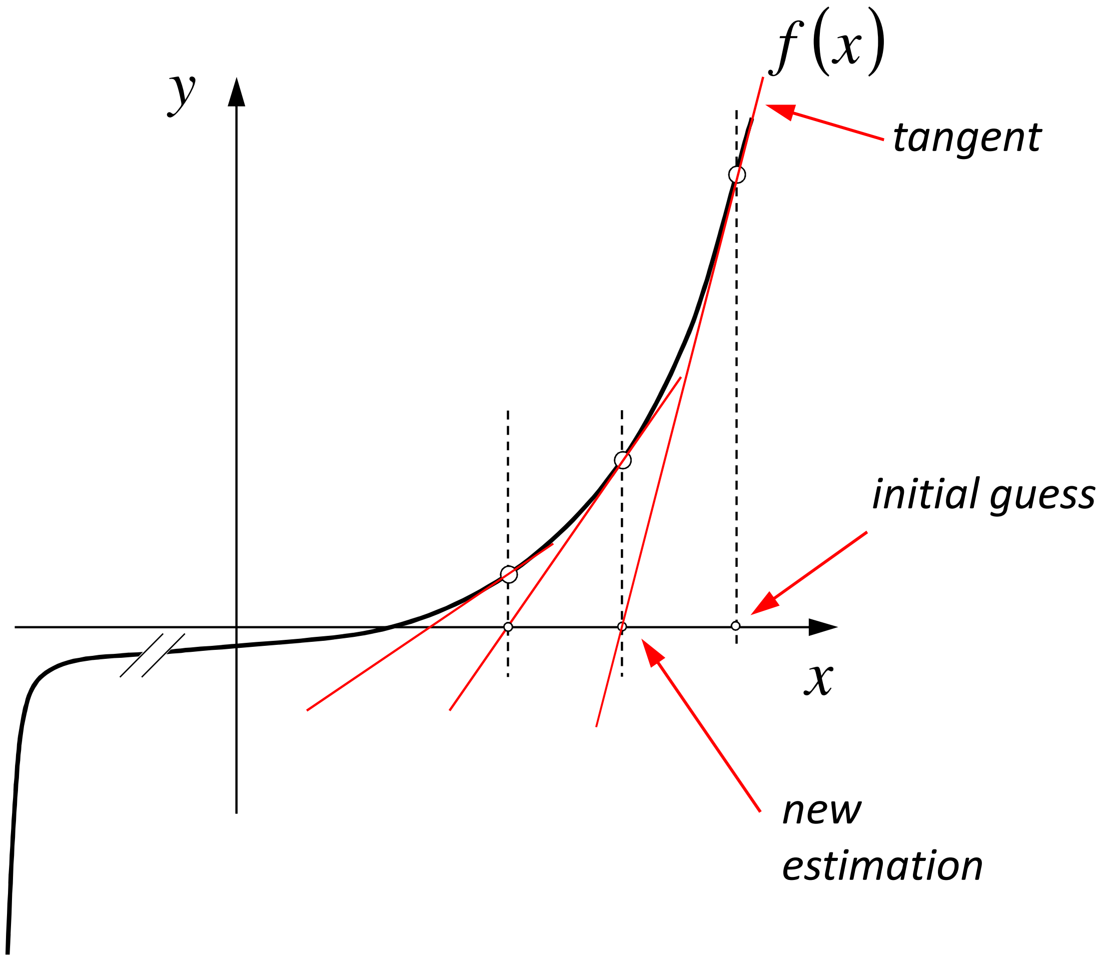
Square root

- Direct method ⇨ digit-by-digit
- Indirect method ⇨ resolve $x^2 - y = 0$

Square root - indirect

Resolving a non linear equation

$$f(x)=0$$



Square root - indirect

Resolving a non linear equation $f(x) = 0$

Taylor series in the neighborhood of x_0

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \cdots$$

1st order : $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

Square root - indirect

Resolving a non linear equation $f(x) = 0$

Iterative resolution starting from an initial guess x_0

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = 0 \qquad f(x_0) + f'(x_0)(x - x_0) = 0$$

$$x = \frac{-f(x_0)}{f'(x_0)} + x_0$$

Newton-Raphson method

Square root - indirect

Resolving $x = \sqrt{y}$

Find a function f *such as* $f(x) = 0$ *for* $x = \sqrt{y}$

$$f(x) = x^2 - y$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) \approx (x_0{}^2 - y) + 2x_0(x - x_0)$$

$f(x) = 0$

$$x = \frac{(y - x_0{}^2)}{2x_0} + x_0$$

$$x = \frac{1}{2}\left(\frac{y}{x_0} + x_0\right)$$

Square root - indirect

Resolving $x = \sqrt{y}$

Each iteration $x_{i+1} = \frac{1}{2}\left(\frac{y}{x_i} + x_i\right)$

division !!

Hard to implement

~~$f(x) = x^2 - y$~~

Square root - indirect

Resolving $x = \sqrt{y}$

$$Find\ a\ function\ \ f\ \ such\ as\ \ f(u) = 0 \quad for \quad u = \frac{1}{\sqrt{y}}$$

$$f(u) = u^{-2} - y$$

$$f(u) \approx f(u_0) + f'(u_0)(u - u_0)$$

$$f(u) \approx (u_0^{-2} - y) - 2u_0^{-3}(u - u_0)$$

$$f(u) = 0 \qquad u = \frac{(u_0^{-2} - y)}{2u_0^{-3}} + u_0 \qquad u = \frac{1}{2}u_0(3 - u_0^{2} y)$$

Square root - indirect

Resolving $x = \sqrt{y}$

Each iteration $u_{i+1} = \frac{1}{2} u_i \left(3 - u_i^{\,2} y\right)$

multiply !!

$$u = \frac{1}{\sqrt{y}}$$

$$x = \sqrt{y} = \frac{y}{\sqrt{y}} = u \cdot y$$

Square root - indirect

Resolving $x = \sqrt{y}$ $u = \dfrac{1}{\sqrt{y}}$

$$y = (-1)^0 \times M \times 2^E \quad with \quad M \in [1, 2[$$

$$\sqrt{y} = (-1)^0 \times \sqrt{M} \times 2^{\frac{E}{2}}$$

E odd ?

$$y = (-1)^0 \times M' \times 2^{2E'} \quad with \quad M' \in [1, 4[$$

$$\sqrt{y} = (-1)^0 \times \sqrt{M'} \times 2^{E'}$$

$$\sqrt{y} = (-1)^0 \times \frac{M'}{\sqrt{M'}} \times 2^{E'}$$

Square root - indirect

Resolving $x = \sqrt{y}$ $u = \frac{1}{\sqrt{y}}$

Initial guess

$$u = \frac{1}{\sqrt{M'}} \quad \textit{with} \quad M' \in [1, 4[$$

$$u \in]0.5, 1]$$