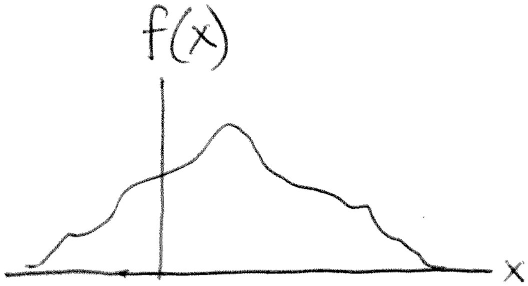
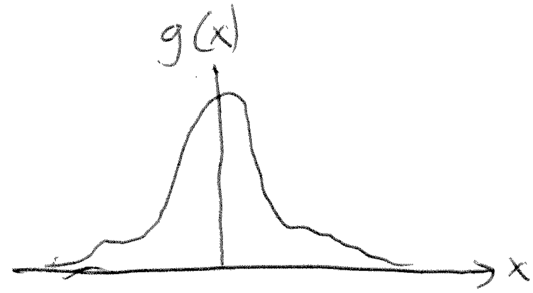
Preliminaries

1) <u>Convolution</u>: consider two functions, f & g.





Their convolution is defined as

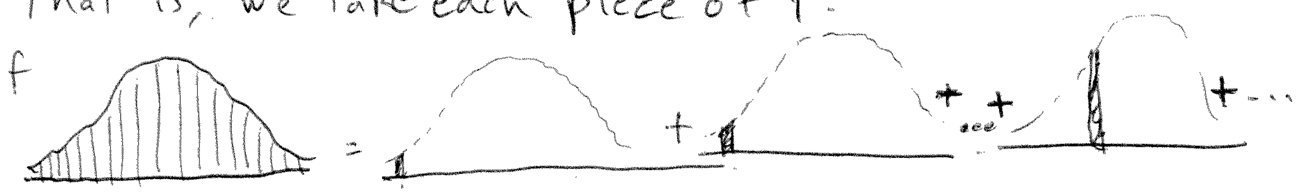
$$f*g\,(x) = \int_{-\infty}^{\infty} f(x')\,g(x-x')\,dx'.$$

The convolution of f with g can be interpreted as a "blurring" of f with g. To see this, use the Riemann sum interpretation of the integral:

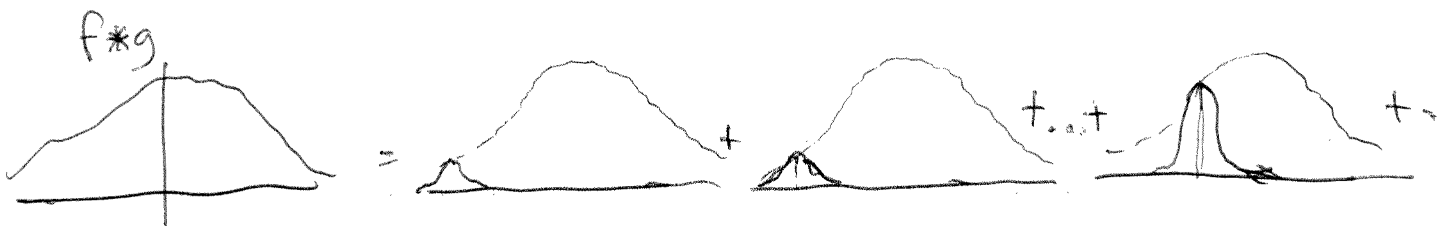
$$x' \to x_m = m\Delta x, \qquad \text{for} \quad \Delta x \to 0.$$

$$f*g = \lim_{\Delta x \to 0} \sum_m \frac{f(x_m)}{\cancel{\Delta x}}\, g(x - x_m)\,\Delta x$$

That is, we take each piece of f:



and "blur" each piece with a displaced version of g:

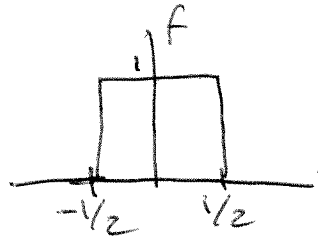


Notice that the convolution is commutative, i.e.

$$f*g(x) = \int_{-\infty}^{\infty} f(x')\,\underbrace{g(x-x')}\,dx' = -\int_{\infty}^{-\infty} g(x'')\,f(x-x'')\,dx'' = \int_{-\infty}^{\infty} g(x'')\,f(x-x'')\,dx''$$

$$x'' = x - x', \quad dx' = -dx'' \qquad = g*f(x).$$

Exercise:

1) Let $f_1(x) = \text{rect}(x) =$ 

$= \begin{cases} 1, & |x| \le \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$

find $f_1 * f_1$

2) Let $f_2(x) = \dfrac{e^{-\pi\left(\frac{x}{a}\right)^2}}{a}$

find $f_2 * f_2$

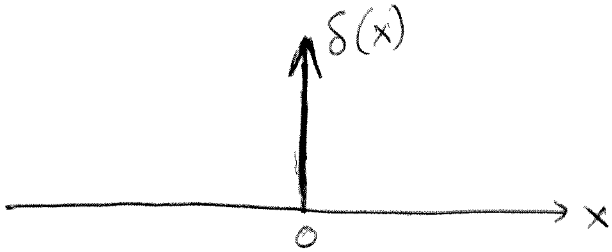
3) (Only for those who like maths!)

find $f_1 * f_2$

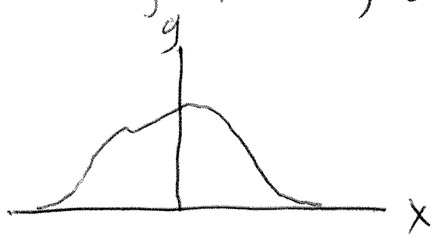
Hint: $\text{erf}(\tau) = \dfrac{2}{\sqrt{\pi}} \int_0^{\tau} e^{-t^2}\,dt$

2) Delta function (Dirac)

$$\boxed{\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \quad \text{such that} \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1.}$$

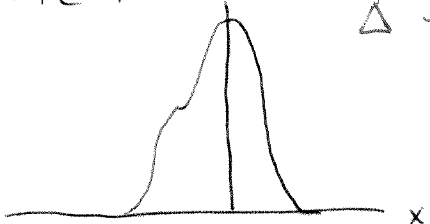


we can build $\delta(x)$ from a function $g(x)$ (say, a Gaussian or a rectangle function) of unit area:



$$\int_{-\infty}^{\infty} g(x)\,dx = 1.$$

Note that $\frac{1}{\Delta} g\left(\frac{x}{\Delta}\right)$, for $0 < \Delta < 1$, also has unit area:



$$\int_{-\infty}^{\infty} g\left(\frac{x}{\Delta}\right)\frac{dx}{\Delta} = 1$$

this is thinner and taller, but with the same area. Then, we can build $\delta(x)$ as

$$\delta(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} g\left(\frac{x}{\Delta}\right)$$

Properties:

- Units. since $\int \delta(x)\,dx$ has no units, δ has units of $\frac{1}{x}$.

• Note that, since $\delta(x-x_0)$ is zero except at $x=x_0$, then $f(x)\,\delta(x-x_0) = f(x_0)\,\delta(x-x_0)$ for any (well-behaved) $f(x)$. Therefore

$$\boxed{\int f(x)\,\delta(x-x_0)\,dx = f(x_0)\int \delta(x-x_0)\,dx = f(x_0)}$$

This is the so-called "sifting property" of the delta function.

Note then that

$$\boxed{f*\delta = \int f(x')\,\delta(x-x')\,dx' = f(x)}$$

so δ is the "unity" element for convolutions.

Finally let us show that we can write

$$\boxed{\delta(x) = \int_{-\infty}^{\infty} e^{i2\pi\nu x}\,d\nu}$$

To show this, we insert 1 in the integrand in the form

$$1 = \lim_{a\to 0} e^{-\pi a\nu^2},$$

so

$$\int_{-\infty}^{\infty} e^{i2\pi\nu x}\,d\nu = \lim_{a\to 0}\int_{-\infty}^{\infty} \overbrace{e^{-\pi a\nu^2}\,e^{i2\pi x\nu}}^{e^{-\pi a\left(\nu^2 - 2i\frac{x\nu}{a}\right)}}\,d\nu$$

but

$$\nu^2 - 2i\frac{x}{a}\nu = \left(\nu - i\frac{x}{a}\right)^2 + \frac{x^2}{a^2}, \text{ so}$$

$$\int_{-\infty}^{\infty} e^{i2\pi\nu x}\, d\nu = \lim_{a\to 0} \int e^{-\pi a \underbrace{(\nu - i\frac{x}{a})}_{\nu'}{}^2} e^{-\frac{\pi x^2}{a}}\, d\nu \qquad , \quad d\nu' = d\nu$$

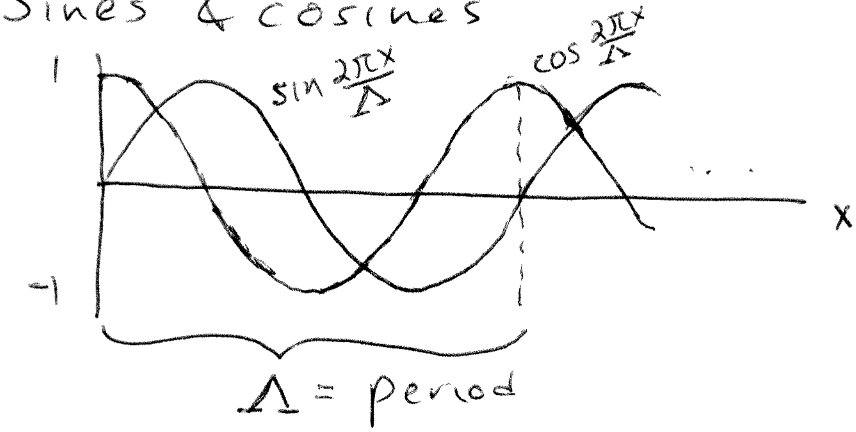
$$= \lim_{a\to 0} e^{-\frac{\pi x^2}{a}} \underbrace{\int e^{-\pi a \nu'^2}\, d\nu'}_{\frac{1}{\sqrt{a}}} = \lim_{a\to 0} \frac{e^{-\frac{\pi x^2}{a}}}{\sqrt{a}}.$$

Let $a = \Delta^2$, so

$$\int_{-\infty}^{\infty} e^{i2\pi\nu x}\, d\nu = \lim_{\Delta\to 0} \frac{e^{-\pi\left(\frac{x}{\Delta}\right)^2}}{\Delta} = \delta(x)$$

Fourier Theory

Sines & cosines



Any (well-behaved) function can be expressed as a continuous superposition of sines and cosines with different amplitudes and periods (Λ).

It is more convenient, though, to use imaginary exponentials. Recall

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

so, instead of $\cos\frac{2\pi x}{\Lambda}$ and $\sin\frac{2\pi x}{\Lambda}$, we use:

$$e^{i2\pi\nu x}, \text{ with } \nu = \pm\frac{1}{\Lambda}$$

The Fourier theorem then states that $f(x)$

can be written as

$$\boxed{f(x) = \int_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i2\pi\nu x}\, d\nu}$$

where $\tilde{f}(\nu)$, known as the <u>Fourier transform</u> of $f(x)$, is the amplitude of the corresponding oscillation.

How do we find $\tilde{f}(\nu)$? Consider the integral

$$\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x} dx = \iint_{-\infty}^{\infty} \tilde{f}(\nu')\, e^{i2\pi\nu' x} d\nu' e^{-i2\pi\nu x} dx$$

Substitute as $\int_{-\infty}^{\infty} \tilde{f}(\nu')\, e^{i2\pi\nu' x} d\nu'$

$$= \int_{-\infty}^{\infty} \hat{f}(\nu') \underbrace{\int_{-\infty}^{\infty} e^{i2\pi(\nu'-\nu)x} dx}_{\delta(\nu'-\nu)}\, d\nu' = \int_{-\infty}^{\infty} \hat{f}(\nu')\, \delta(\nu'-\nu)\, d\nu'$$

$$= \hat{f}(\nu),$$

so

$$\boxed{\tilde{f}(\nu) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x} dx}$$

So in summary

Fourier Transformation $\quad \tilde{f}(\nu) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\nu x} dx$

Inverse Fourier Transformation $\quad f(x) = \int_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i2\pi\nu x} d\nu$

In what follows we use the notation:

$$\tilde{f}(\nu) = \hat{\mathcal{F}}_{x\to\nu}\, f(x)$$

$$f(x) = \hat{\mathcal{F}}_{\nu\to x}\, \tilde{f}(\nu)$$

Properties

- **Parseval-Plancherel theorem**

In many physical applications, $|f(x)|^2$ is a physically significant (and observable) quantity, and the integral of this quantity corresponds to, for example, the total power or energy. Note that

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f^*(x) \underbrace{f(x)}_{=\int_{-\infty}^{\infty} \tilde{f}(\nu) e^{i2\pi\nu x} d\nu} dx$$

$$= \int_{-\infty}^{\infty} f^*(x) \int_{-\infty}^{\infty} \tilde{f}(\nu) e^{i2\pi\nu x} d\nu \, dx = \int_{-\infty}^{\infty} \tilde{f}(\nu) \int_{-\infty}^{\infty} f^*(x) e^{i2\pi\nu x} dx \, d\nu$$

$$= \int_{-\infty}^{\infty} \hat{f}(\nu) \underbrace{\left[\int_{-\infty}^{\infty} f(x) e^{-i2\pi\nu x} dx\right]}_{\hat{f}(\nu)}^{*} d\nu = \int_{-\infty}^{\infty} \tilde{f}(\nu) \tilde{f}^*(\nu) d\nu = \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 d\nu$$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 d\nu}$$

- **Shift-phase**

Consider the FT of a shifted function

$$\hat{\mathcal{F}}_{x\to\nu}\, f(x-x_0) = \int_{-\infty}^{\infty} f(\underbrace{x-x_0}) e^{-i2\pi x\nu} dx$$

$x' = x - x_0 \to x = x' + x_0, \quad dx = dx'$

$$= \int_{-\infty}^{\infty} f(x') e^{-i2\pi(x'+x_0)\nu} dx' = \int_{-\infty}^{\infty} f(x') e^{-i2\pi x'\nu} dx' \, e^{-i2\pi x_0\nu}$$

$$= \tilde{f}(\nu) e^{-i2\pi x_0 \nu}$$

therefore

$$\hat{\mathfrak{F}}_{x\to\nu}\, f(x-x_0) = \tilde{f}(\nu)\, e^{-i2\pi x_0 \nu} = \left[\hat{\mathfrak{F}}_{x\to\nu}\, f(x)\right] e^{-i2\pi x_0 \nu}$$

which implies

$$\hat{\mathfrak{F}}^{-1}_{\nu\to x}\left[\tilde{f}(\nu)\, e^{-i2\pi x_0 \nu}\right] = f(x-x_0)$$

Analogously, multiplying $f(x)$ by a linear phase function leads to the shift of the Fourier transform

$$\hat{\mathfrak{F}}_{x\to\nu}\left[f(x)\, e^{i2\pi x \nu_0}\right] = \int_{-\infty}^{\infty} f(x) e^{i2\pi x\nu_0}\, e^{-i2\pi x\nu}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi(\nu-\nu_0)x}\, dx = \tilde{f}(\nu-\nu_0)$$

and therefore

$$\hat{\mathfrak{F}}^{-1}_{\nu\to x}\, \tilde{f}(\nu-\nu_0) = f(x)\, e^{i2\pi\nu_0 x}$$

- Scaling

Consider the FT of $f\left(\frac{x}{a}\right)$

$$\hat{\mathfrak{F}}_{x\to\nu}\, f\left(\frac{x}{a}\right) = \int_{-\infty}^{\infty} f\left(\frac{x}{a}\right) e^{-i2\pi x\nu}\, dx$$

$x' = \frac{x}{a}$, $x = ax'$, $dx = a\,dx'$

$$= \begin{cases} a\int_{-\infty}^{\infty} f(x')\, e^{-i2\pi a x'\nu}\, dx', & a>0 \\ a\int_{\infty}^{-\infty} f(x')\, e^{-i2\pi a x'\nu}\, dx', & a<0 \end{cases}$$

$$= \underbrace{\mathrm{sgn}(a)\, a}_{|a|} \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi x'(a\nu)}\, dx' = |a|\, \tilde{f}(a\nu)$$

Derivative

$$\hat{\mathcal{F}}_{x\to\nu}\, f'(x) = \int_{-\infty}^{\infty} f'(x)\, e^{-i2\pi x\nu}\, dx = \int_{-\infty}^{\infty} u\, dv$$

Integrate by parts $dv = f'dx$ $\quad u = e^{-i2\pi x\nu}$

$$= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\, du = \underbrace{f(x)\, e^{-i2\pi x\nu}\Big|_{-\infty}^{\infty}}_{\text{assume } f(\pm\infty)=0} + i2\pi\nu \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x\nu} dx$$

$v = f \quad du = -i2\pi\nu\, e^{-i2\pi x\nu}$

$$= i2\pi\nu\, \tilde{f}(\nu)$$

More generally: $\underline{\hat{\mathcal{F}}_{x\to\nu}\, f^{(n)}(x) = (i2\pi\nu)^n\, \tilde{f}(\nu)}$

Similarly

$$\hat{\mathcal{F}}_{x\to\nu}\left[x^n f(x)\right] = \int_{-\infty}^{\infty} f(x)\, \underbrace{x^n e^{-i2\pi x\nu}}_{\left(\frac{1}{-i2\pi}\right)^n \frac{d^n}{d\nu^n} e^{-i2\pi x\nu}} dx$$

$$= \left(\frac{1}{-i2\pi}\right)^n \frac{d^n}{d\nu^n} \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x\nu} dx = \underline{\frac{\tilde{f}^{(n)}(\nu)}{(-i2\pi)^n}}$$

Convolution/product

$$\hat{\mathcal{F}}_{x\to\nu}\left[f * g\right] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'\right] e^{-i2\pi x\nu} dx$$

$$= \int_{-\infty}^{\infty} f(x') \underbrace{\int_{-\infty}^{\infty} g(x-x')\, e^{-i2\pi x\nu} dx}_{\text{From shift/phase: } \tilde{g}(\nu)\, e^{-i2\pi x'\nu}}\, dx' = \tilde{g}(\nu) \int_{-\infty}^{\infty} f(x')\, e^{-i2\pi x'\nu} dx'$$

$$= \tilde{g}(\nu)\, \tilde{f}(\nu) = \underline{\tilde{f}(\nu)\, \tilde{g}(\nu)}$$

Similarly

$$\hat{\mathcal{F}}_{x\to\nu}\left[f(x)g(x)\right] = \int_{-\infty}^{\infty} f(x)\underbrace{g(x)}_{\text{insert: } \int_{-\infty}^{\infty}\tilde{g}(\nu')e^{i2\pi x\nu'}d\nu'} e^{-i2\pi x\nu}dx$$

$$= \int_{-\infty}^{\infty}\tilde{g}(\nu')\int_{-\infty}^{\infty} f(x)e^{-i2\pi x(\nu-\nu')}dx = \int_{-\infty}^{\infty}\tilde{g}(\nu')\tilde{f}(\nu-\nu')d\nu'$$

$$= \underline{\tilde{f}*\tilde{g}}$$

- Space-bandwidth product / uncertainty relation.

The average or "centroid" of $|f(x)|^2$ is defined as

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x|f(x)|^2 dx}{\int_{-\infty}^{\infty}|f(x)|^2 dx}$$

and the rms spread is

$$\Delta x = \left[\frac{\int_{-\infty}^{\infty}(x-\bar{x})^2|f(x)|^2 dx}{\int_{-\infty}^{\infty}|f(x)|^2 dx}\right]^{1/2}.$$

Similarly

$$\bar{\nu} = \frac{\int_{-\infty}^{\infty}\nu|\tilde{f}(\nu)|^2 d\nu}{\int_{-\infty}^{\infty}|\tilde{f}(\nu)|^2 d\nu}, \quad \Delta\nu = \left[\frac{\int_{-\infty}^{\infty}(\nu-\bar{\nu})^2|\tilde{f}(\nu)|^2 d\nu}{\int_{-\infty}^{\infty}|\tilde{f}(\nu)|^2 d\nu}\right]^{1/2}.$$

It is now shown that

$$\boxed{\Delta x\,\Delta\nu \geq \frac{1}{4\pi}}$$

Proof.

Part a) Cauchy-Schwarz-Bunyakowski inequality

consider two functions g, h. then

$$\iint \underbrace{|g(x)h(y)-g(y)h(x)|^2}_{\text{this is always} \geq 0} dx\,dy \geq 0.$$

But we can write this as

$$\iint \left[g^*(x)h^*(y)-g^*(y)h^*(x)\right]\left[g(x)h(y)-g(y)h(x)\right] dx\,dy$$

$$= \iint \Big[|g(x)|^2|h(y)|^2 - g^*(x)h(x)\,h^*(y)g(y)$$
$$- g^*(y)h(y)\,h^*(x)g(x) + |g(y)|^2|h(x)|^2\Big] dx\,dy$$

$$= \int|g(x)|^2dx\int|h(y)|^2dy + \int|g(y)|^2dy\int|h(x)|^2dx$$
$$- \left[\int g^*(x)h(x)\,dx\int h^*(y)g(y)\,dy + \int g^*(y)h(y)dy\int h^*(x)g(x)dx\right]$$

but x & y are now dummy variables, so we can write

$$= 2\left[\int|g(x)|^2dx\right]\left[\int|h(x)|^2dx\right] - 2\left|\int g^*(x)h(x)dx\right|^2.$$

and recall that all this ≥ 0. Therefore

$$\underline{\int|g(x)|^2dx\int|h(x)|^2dx \geq \left|\int g^*(x)h(x)dx\right|^2}$$

Part b)

Let $g(x) = \dfrac{(x-\bar{x})f(x)}{\Phi^{1/2}}$, where $\Phi = \int_{-\infty}^{\infty}|f(x)|^2dx$

then $\displaystyle\int_{-\infty}^{\infty}|g(x)|^2dx = \frac{\int(x-\bar{x})^2|f(x)|^2dx}{\int_{-\infty}^{\infty}|f(x)|^2dx} = \Delta x^2$

Now, $\int_{-\infty}^{\infty} |h(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{h}(\nu)|^2 d\nu$ (Parseval-Plancherel)

Let
$$\tilde{h}(\nu) = \frac{(\nu - \bar{\nu})\tilde{f}(\nu)}{\Phi^{1/2}}, \text{ so } \int_{-\infty}^{\infty} |h(x)|^2 dx = \Delta\nu^2$$

Notice
$$\tilde{h}(\nu) = \frac{1}{\Phi^{1/2}}\left[\nu \tilde{f}(\nu) - \underbrace{\bar{\nu}}_{\text{constant.}} \tilde{f}(\nu)\right]$$

therefore
$$h(x) = \hat{\mathcal{F}}^{-1}_{\nu \to x}\, \tilde{h}(\nu) = \frac{1}{\Phi^{1/2}}\left[\frac{1}{i2\pi} f'(x) - \bar{\nu} f(x)\right].$$

Therefore
$$\int_{-\infty}^{\infty} g^*(x)\, h(x)\, dx = \frac{1}{\Phi}\int_{-\infty}^{\infty} (x - \bar{x}) f^*(x)\left[\frac{1}{i2\pi} f'(x) - \bar{\nu} f(x)\right] dx$$

$$= \frac{1}{i2\pi\Phi}\underbrace{\int_{-\infty}^{\infty} (x-\bar{x}) f^*(x) f'(x)\, dx}_{\text{integrate by parts:}} - \frac{\bar{\nu}}{\Phi}\int_{-\infty}^{\infty} (x - \bar{x})\, |f(x)|^2 dx \qquad (i)$$

$u = (x-\bar{x}) f^*$, $dv = f' dx$, $v = f$, $du = [f^* + (x-\bar{x}) f'^*] dx$

$$= \frac{1}{i2\pi\Phi} \cancel{\underbrace{\left[(x-\bar{x})\, f^*(x) f(x)\right]\Big|_{-\infty}^{\infty}}_{\text{assume this vanishes.}}} - \frac{1}{i2\pi\Phi}\int_{-\infty}^{\infty} \left[f(x) + (x-\bar{x}) f'(x)\right]^* f(x)\, dx$$
$$- \frac{\bar{\nu}}{\Phi}\int_{-\infty}^{\infty} (x - \bar{x})\, |f(x)|^2 dx$$

$$= -\frac{\overbrace{\int_{-\infty}^{\infty} |f(x)|^2 dx}^{\Phi}}{i2\pi\cancel{\Phi}} - \frac{1}{i2\pi\Phi}\left[\int_{-\infty}^{\infty} f^*(x)\, (x-\bar{x})\, f'(x)\, dx\right]^*$$
$$- \frac{\bar{\nu}}{\Phi}\int_{-\infty}^{\infty} (x - \bar{x})\, |f(x)|^2 dx$$

$$\int_{-\infty}^{\infty} g^*(x)h(x)dx = +\frac{i}{2\pi} + \left[\frac{1}{i2\pi\Phi}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)f'(x)dx - \frac{\bar{\nu}}{\Phi}\int_{-\infty}^{\infty}(x-\bar{x})|f(x)|^2dx\right]^* \quad (ii)$$

Note that $\int_{-\infty}^{\infty} g^*(x)h(x)dx$ is given by either the expression in (i) or the one in (ii), therefore also by their average:

$$\int_{-\infty}^{\infty} g^*(x)h(x)dx = \underbrace{\frac{1}{2}\left[\frac{1}{\Phi}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)\left(\frac{f'(x)}{i2\pi}-\bar{\nu}f(x)\right)dx\right]}_{(i)}$$

$$+\frac{1}{2}\left[\underbrace{\frac{i}{2\pi}+\frac{1}{\Phi}\left(\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)\left(\frac{f'(x)}{i2\pi}-\bar{\nu}f(x)\right)dx\right)^*}_{(ii)}\right]$$

$$= \underbrace{\mathrm{Re}\left\{\frac{1}{\Phi}\int_{-\infty}^{\infty}(x-\bar{x})f^*(x)\left(\frac{f'(x)}{i2\pi}-\bar{\nu}f(x)\right)dx\right\}}_{\text{call this } \Delta_{x\nu}} + \frac{i}{4\pi}$$

$$= \Delta_{x\nu} + \frac{i}{4\pi}$$

Real

Therefore:

$$\left|\int_{-\infty}^{\infty} g^*(x)h(x)dx\right|^2 = \left(\Delta_{x\nu}-\frac{i}{4\pi}\right)\left(\Delta_{x\nu}+\frac{i}{4\pi}\right) = \Delta_{x\nu}^2 + \frac{1}{(4\pi)^2}$$

so $\int_{-\infty}^{\infty}|g(x)|^2dx \int_{-\infty}^{\infty}|h(x)|^2dx \geq \left|\int_{-\infty}^{\infty} g^*(x)h(x)dx\right|^2$ gives

$$\Delta x^2 \Delta\nu^2 \geq \Delta_{x\nu}^2 + \frac{1}{(4\pi)^2} \geq \frac{1}{(4\pi)^2} \quad \text{so} \quad \boxed{\Delta x \Delta\nu \geq \frac{1}{4\pi}}$$

- Complex conjugate

$$\hat{\mathfrak{F}}_{x\to\nu}\left[f^*(x)\right] = \int_{-\infty}^{\infty} f^*(x)\, e^{-i2\pi x\nu}\, dx$$

$$= \left[\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi(-\nu)x}\, dx\right]^* = \tilde{f}^*(-\nu)$$

Note then that, if f is real

$$f(x) = f^*(x) \Rightarrow \tilde{f}(\nu) = \tilde{f}^*(-\nu)$$

$$\underbrace{\mathrm{Re}\,\tilde{f}(\nu) = \mathrm{Re}\,\tilde{f}(-\nu)}_{\text{The real part of } \tilde{f} \text{ is even}} \qquad \underbrace{\mathrm{Im}\,\tilde{f}(\nu) = -\mathrm{Im}\,\tilde{f}(-\nu)}_{\text{The imaginary part of } \tilde{f} \text{ is odd.}}$$

Exercise:

$$\hat{\mathfrak{F}}_{x\to\nu}\left[|f(x)|^2\right] =$$

Summary

1D Fourier transform

$$\boxed{\begin{aligned}\tilde{f}(\nu) &= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x\nu}\, dx \\ f(x) &= \int_{-\infty}^{\infty} \tilde{f}(\nu)\, e^{i2\pi x\nu}\, d\nu\end{aligned}}$$

Properties

- Parseval-Plancherel
$$\int_{-\infty}^{\infty} f^*(x)\, g(x)\, dx = \int_{-\infty}^{\infty} \tilde{f}^*(\nu)\, \tilde{g}(\nu)\, d\nu$$
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(\nu)|^2 d\nu$$

- Shift-Phase
$$\hat{\mathfrak{F}}_{x\to\nu}\, f(x-x_0) = \tilde{f}(\nu)\, e^{-i2\pi x_0 \nu}$$
$$\hat{\mathfrak{F}}_{x\to\nu}\left[f(x)\, e^{i2\pi \nu_0 x}\right] = \tilde{f}(\nu-\nu_0)$$

- Scaling
$$\hat{\mathfrak{F}}_{x\to\nu}\, f\left(\tfrac{x}{a}\right) = |a|\, \tilde{f}(a\nu) \qquad (a \text{ real}, \neq 0)$$

- Derivative
$$\hat{\mathfrak{F}}_{x\to\nu}\, f^{(n)}(x) = (i2\pi\nu)^n\, \tilde{f}(\nu)$$
$$\hat{\mathfrak{F}}_{x\to\nu}\left[x^n f(x)\right] = \frac{\tilde{f}^{(n)}(\nu)}{(-i2\pi)^n}$$

- Convolution/product
$$\hat{\mathfrak{F}}_{x\to\nu}\left[f * g\right] = \tilde{f}(\nu)\, \tilde{g}(\nu)$$
$$\hat{\mathfrak{F}}_{x\to\nu}\left[f(x)\, g(x)\right] = \tilde{f} * \tilde{g}$$

- Space-bandwidth product / uncertainty $\qquad \Delta x\, \Delta\nu \geq \frac{1}{4\pi}$

- Complex conjugate $\qquad \hat{\mathfrak{F}}_{x\to\nu}\left[f^*(x)\right] = \tilde{f}^*(-\nu).$

Exercises. Calculate the FT of:

1) $\delta(x)$
2) $\delta(x-x_0)$
3) $\text{rect}(x)$
4) $\text{rect}(x) * \text{rect}(x)$
5) $c\,\text{rect}\left(\frac{x-a}{b}\right)$
6) $e^{-\pi x^2}$
7) $x\, e^{-\pi x^2}$

2 Dimensions

$\underline{x} = (x, y), \quad \underline{\nu} = (\nu_x, \nu_y)$

Convolution

$$f * g = \iint_{-\infty}^{\infty} f(\underline{x}') \, g(\underline{x} - \underline{x}') \, dx' dy'$$

Delta function $\delta(\underline{x})$

$$\iint_{-\infty}^{\infty} \delta(\underline{x}) \underbrace{dx\,dy}_{\text{units of } x^2} = 1, \text{ so } \delta \text{ has units of } \frac{1}{x^2}$$

sifting: $\displaystyle\iint_{-\infty}^{\infty} f(\underline{x}) \, \delta(\underline{x} - \underline{x}_0) \, dx\,dy = f(\underline{x}_0)$

Fourier transform

$$\tilde{f}(\underline{\nu}) = \iint_{-\infty}^{\infty} f(\underline{x}) \, e^{-i2\pi \underline{x} \cdot \underline{\nu}} \, dx\,dy$$

$$f(\underline{x}) = \iint_{-\infty}^{\infty} \tilde{f}(\underline{\nu}) \, e^{i2\pi \underline{x} \cdot \underline{\nu}} \, d\nu_x \, d\nu_y$$

Properties

- Parseval-Plancherel $\displaystyle\iint_{-\infty}^{\infty} f^*(\underline{x}) \, g(\underline{x}) \, dx\,dy = \iint_{-\infty}^{\infty} \tilde{f}^*(\underline{\nu}) \, \tilde{g}(\underline{\nu}) \, d\nu_x \, d\nu_y$
- Shift-Phase $\hat{\mathcal{F}}_{\underline{x} \to \underline{\nu}} \, f(\underline{x} - \underline{x}_0) = \tilde{f}(\underline{\nu}) \, e^{-i2\pi \underline{x}_0 \cdot \underline{\nu}}$

  $\hat{\mathcal{F}}_{\underline{x} \to \underline{\nu}} \left[f(\underline{x}) \, e^{i2\pi \underline{\nu}_0 \cdot \underline{x}} \right] = \tilde{f}(\underline{\nu} - \underline{\nu}_0)$
- Scaling $\hat{\mathcal{F}}_{\underline{x} \to \underline{\nu}} \, f(\underline{x}/a) = a^2 \, \tilde{f}(a\underline{\nu})$
- Derivative $\hat{\mathcal{F}}_{\underline{x} \to \underline{\nu}} \left[\nabla_{\underline{x}} f(\underline{x}) \right] = i2\pi \underline{\nu} \, \tilde{f}(\underline{\nu})$

  $\hat{\mathcal{F}}_{\underline{x} \to \underline{\nu}} \left[\underline{x} \, f(\underline{x}) \right] = \frac{1}{-i2\pi} \nabla_{\underline{\nu}} \tilde{f}(\underline{\nu})$
- Convolution $\hat{\mathcal{F}}_{\underline{x} \to \underline{\nu}} [f * g] = \tilde{f}(\underline{\nu}) \tilde{g}(\underline{\nu})$, $\hat{\mathcal{F}}_{\underline{x} \to \underline{\nu}} [f(\underline{x}) g(\underline{x})] = \tilde{f} * \tilde{g}$
- Uncertainty $\Delta_\rho \Delta_\nu \geq \frac{1}{2\pi}$

2D Fourier transform in polar coordinates:

$$\underline{x} = (\rho\cos\theta, \rho\sin\theta) \quad , \quad \underline{\nu} = (\nu\cos\phi, \nu\sin\phi)$$

$$\tilde{f}(\underline{\nu}) = \int_0^\infty \int_0^{2\pi} f(\underline{x})\, e^{-i2\pi\rho\nu\cos(\theta-\phi)}\, \rho\, d\theta\, d\rho$$

If $f(\underline{x})$ depends only on ρ, i.e. has rotational symmetry: $f(\underline{x}) = f_\rho(\rho)$

$$\tilde{f}(\underline{\nu}) = \int_0^\infty f_\rho(\rho)\,\rho \underbrace{\int_0^{2\pi} e^{-i2\pi\rho\nu\cos(\theta-\phi)}\, d\theta}_{2\pi J_0(2\pi\rho\nu),\ \text{independent of } \phi}\, d\rho$$

so $\tilde{f}(\underline{\nu}) = \tilde{f}_\nu(\nu)$ also has rotational symmetry.

$$\boxed{\begin{array}{ll} \text{Hankel Transf.} & \tilde{f}_\rho(\nu) = 2\pi \int_0^\infty f_\rho(\rho)\, J_0(2\pi\rho\nu)\,\rho\, d\rho \\ \text{Inverse HT} & f_\rho(\rho) = 2\pi \int_0^\infty \tilde{f}_\nu(\nu)\, J_0(2\pi\rho\nu)\,\nu\, d\nu \end{array}}$$

In this case

$$\Delta_\rho = \left[\frac{\int_0^\infty |f_\rho(\rho)|^2 \rho^2\, \rho\, d\rho}{\int_0^\infty |f_\rho(\rho)|^2 \rho\, d\rho} \right]^{1/2}$$

$$\Delta_\nu = \left[\frac{\int_0^\infty |\tilde{f}_\nu(\nu)|^2 \nu^2\, \nu\, d\nu}{\int_0^\infty |\tilde{f}_\nu(\nu)|^2 \nu\, d\nu} \right]^{1/2}$$

$$\Delta_\rho \Delta_\nu \geq \frac{1}{2\pi}.$$

Exercises:

- Calculate the Hankel transforms of

1) $f_\rho(\rho) = \delta(\rho - a)$
2) $f_\rho(\rho) = \begin{cases} 1, & \rho \le a \\ 0, & \rho > a \end{cases}$
3) $f_\rho(\rho) = \begin{cases} 1 - \frac{\rho^2}{a^2}, & \rho \le a \\ 0, & \rho > a \end{cases}$

Formulas you might need

$$\int_0^u u' J_0(u')\,du' = u J_1(u)$$

$$\int_0^u u'^3 J_0(u')\,du' = 2u^2 J_2(u) - u^3 J_3(u)$$

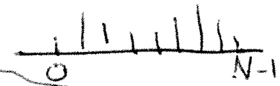
$$J_{n+1} + J_{n-1} = \frac{2n J_n}{u}$$

- Calculate the convolution of 2) with itself.

What is its Fourier transform?

Discrete Fourier transform (DFT)

Instead of $f(x)$ we have f_n, $n=0,\dots,N-1$



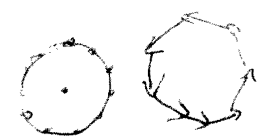
$$\boxed{F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n \, e^{-i 2\pi \frac{mn}{N}}}$$

Discrete Fourier transform

Inverse: try:

$$f_{n'} = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m e^{i 2\pi \frac{mn'}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f_n \underbrace{\sum_{m=0}^{N-1} e^{i 2\pi \frac{(n'-n)m}{N}}}_{N \delta_{n'-n}}$$



so:

$$\boxed{f_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} F_m \, e^{i 2\pi \frac{mn}{N}}}$$

Inverse Discrete Fourier transform

Approximating FT with DFT

(Notice that the sums can be shifted,) if we define

$$f_{n-N} = f_n$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N}{2} \rfloor} f_n \, e^{-i 2\pi \frac{mn}{N}}$$

Let f_n be a sampling of $f(x)$:

$$f_n = f(n\Delta x)$$

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=-\lfloor \frac{N-1}{2} \rfloor}^{\lfloor \frac{N}{2} \rfloor} f(n\Delta x) \, e^{-i 2\pi \frac{mn}{N}}$$

For very large N, and small Δx, can approximate the sum as an integral

$$F_m \approx \frac{1}{\sqrt{N}} \int_{-X_1}^{X_2} f(x)\, e^{\frac{-i2\pi m x}{N\Delta x}} \frac{dx}{\Delta x}$$

where $n\Delta x \to x$

$$X_1 = \left\lfloor \frac{N-1}{2} \right\rfloor \Delta x, \quad X_2 = \left\lfloor \frac{N}{2} \right\rfloor \Delta x$$

Assume $\underset{\text{big}}{N}\ \underset{\text{small}}{\Delta x} = \text{big} \gg$ width of $f(x)$.

note $X_1 \approx X_2 \approx N\frac{\Delta x}{2} = \text{big}$.

Then

$$F_m \approx \frac{1}{\sqrt{N}\,\Delta x} \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi x\left(\frac{m}{N\Delta x}\right)} dx$$

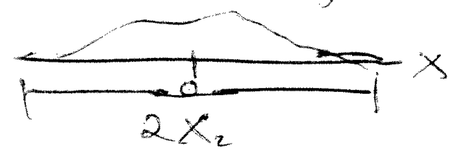
$$= \frac{\tilde{f}\left(\frac{m}{N\Delta x}\right)}{\sqrt{N}\,\Delta X}$$

So the sampling distance in ν is $\frac{1}{N\Delta X} \approx \frac{1}{2X_2}$

where $2X_2$ is the width over which we're sampling $f(x)$.

Therefore:

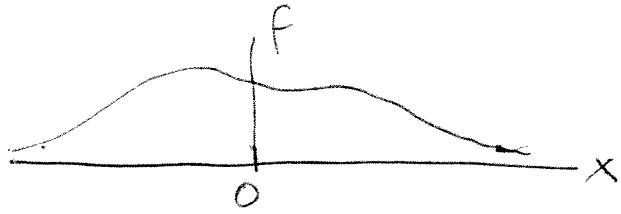
- To increase resolution in $\tilde{f}(\nu)$ $\longrightarrow$ must increase range in $f(x)$



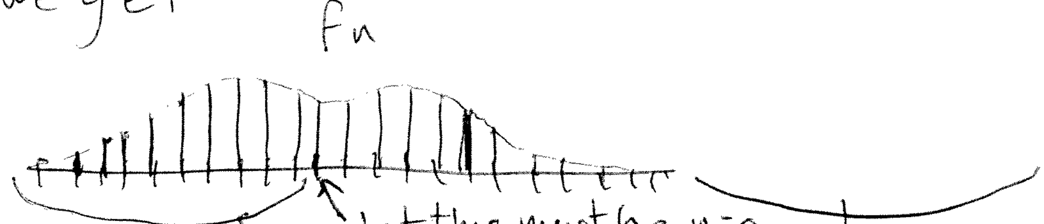
- To increase range in $\tilde{f}(\nu)$ and avoid aliasing $\longrightarrow$ must decrease sampling spacing in $f(x)$

Shifting the functions

Notice that, if we sample:

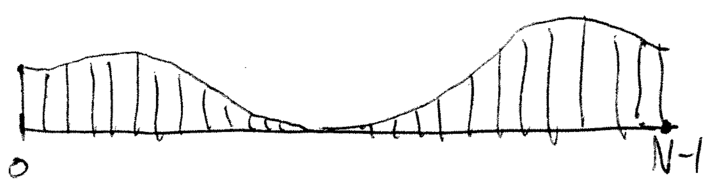


we get



but this must be $n=0$
so we must shift
this to here

so we get



→ this is f_n

Similarly, once we get F_m, it will look like

To reconstruct $\tilde{f}(\nu)$ we must cut the second half and place it before the first. We also need to multiply by $\sqrt{N}\Delta x$.

Fast Fourier transform (FFT)

Notice that the, for each m, the DFT involves the sum of N terms. Since m runs from 0 to N-1, then N^2 must be performed. The time of computation can therefore be expected to be proportional to N^2.

The FFT is an algorithm for performing the DFT whose time of computation is proportional to $N \log N$. While it can work for any N, its simplest form can be understood if $N = 2^M$ (so that $M = \log_2 N$):

$$F_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{\frac{-i2\pi nm}{N}} = \frac{1}{\underset{\text{write as } 2\frac{N}{2}}{\sqrt{N}}} \left[\underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{\frac{-i2\pi(2n')m}{N}}}_{\text{terms with even } n} + \underbrace{\sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{\frac{-i2\pi(2n'+1)m}{N}}}_{\text{terms with odd } n} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{2n'} e^{\frac{-i2\pi n'm}{(N/2)}}} + e^{-i\frac{2\pi m}{N}} \underbrace{\frac{1}{\sqrt{\frac{N}{2}}} \sum_{n'=0}^{\frac{N}{2}-1} f_{(2n'+1)} e^{\frac{-i2\pi n'm}{(N/2)}}} \right]$$

Each of these two sums is itself a DFT of size $\frac{N}{2}$.

They can be joined.

$$F_m = \frac{1}{\sqrt{N}} \sum_{n'=0}^{\frac{N}{2}-1} \left(f_{2n'} + e^{\frac{-i2\pi m}{N}} f_{(2n'+1)} \right) e^{\frac{-i2\pi n'm}{(N/2)}}$$

The same separation can be done M times.

2D DF

$$\boxed{F_{m_1,m_2} = \frac{1}{N} \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} f_{n_1,n_2}\, e^{-i 2\pi \frac{(m_1 n_1 + m_2 n_2)}{N}}}$$

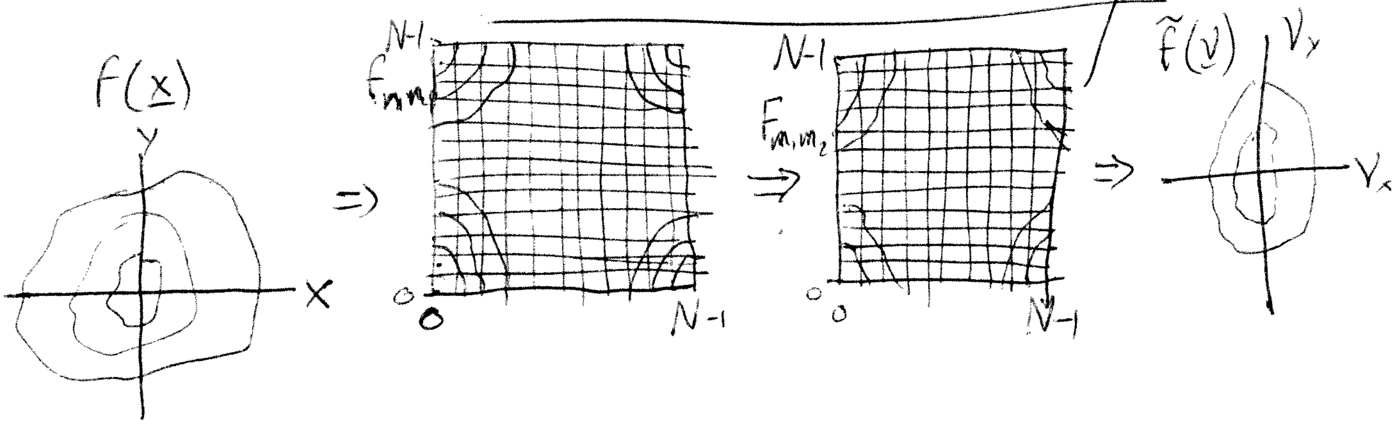
$$\boxed{f_{n_1,n_2} = \frac{1}{N} \sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} F_{m_1,m_2}\, e^{i 2\pi \frac{(m_1 n_1 + m_2 n_2)}{N}}}$$

Using 2D DFT to approximate 2D FT:

if $f_{n_1,n_2} = f(n_1 \Delta x, n_2 \Delta x)$,

and $N\Delta x$ is bigger than width of f, then:

$$F_{m_1,m_2} \approx \frac{1}{N\Delta x^2}\, \tilde{f}\left(\frac{m_1}{N\Delta x}, \frac{m_2}{N\Delta x}\right)$$



Fast Fourier transform: time $\propto N^2 \log N$