

# From Maxwell's equations to the Helmholtz Equation

Free space:

$$\left. \begin{aligned} \nabla \cdot \vec{E} &= 0 & (i) \\ \nabla \cdot \vec{B} &= 0 & (ii) \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} & (iii) \\ \nabla \times \vec{B} &= \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} & (iv) \end{aligned} \right\} \text{Maxwell's equations}$$

Take the curl of (iii):

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \left( \frac{\partial \vec{B}}{\partial t} \right)$$

$$\underbrace{\nabla(\nabla \cdot \vec{E})}_{\text{use (i)}} - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \underbrace{(\nabla \times \vec{B})}_{\text{use (iv)}}$$

$$-\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \implies \boxed{\nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}} \text{ wave equation}$$

(same for  $\vec{B}$ )

$\mu_0 \epsilon_0 = \frac{1}{c^2}$ ,  $c = \text{speed of light in vacuum.}$

Let  $\vec{E}(\vec{r}, t)$  be expressed as an inverse FT in time:

$$\underbrace{\vec{E}(\vec{r}, t)}_{\text{real}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\vec{E}(\vec{r}, \omega)}_{\text{complex}} e^{-i\omega t} d\omega$$

notice that a different sign convention is used for time FT.

Because  $\vec{E}$  is real:  $\vec{E} = \vec{E}^*$ , that is,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{r}, \omega) e^{-i\omega t} d\omega = \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{r}, \omega) e^{-i\omega t} d\omega \right]^* = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}^*(\vec{r}, \omega) e^{i\omega t} d\omega$$

change variables  $\omega' = -\omega, d\omega' = -d\omega$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \vec{E}(\vec{r}, -\omega') e^{-i\omega' t} d\omega' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}^*(\vec{r}, \omega) e^{-i\omega t} d\omega$$

so

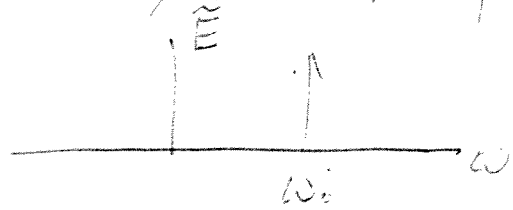
$$\boxed{\vec{E}^*(\vec{r}, -\omega) = \vec{E}(\vec{r}, \omega)}$$

This allows us to simplify:

$$\begin{aligned}
 \vec{E}(\vec{r}, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{\tilde{E}}(\vec{r}, \omega) e^{-i\omega t} d\omega \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^0 \vec{\tilde{E}}(\vec{r}, \omega) e^{-i\omega t} d\omega + \int_0^{\infty} \vec{\tilde{E}}(\vec{r}, \omega) e^{-i\omega t} d\omega \right] \\
 &= \frac{1}{\sqrt{2\pi}} \left[ \int_0^{\infty} \underbrace{\vec{\tilde{E}}(\vec{r}, -\omega)}_{\vec{\tilde{E}}^*(\vec{r}, \omega)} e^{i\omega t} d\omega + \int_0^{\infty} \vec{\tilde{E}}(\vec{r}, \omega) e^{-i\omega t} d\omega \right] \\
 &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[ \vec{\tilde{E}}^*(\vec{r}, \omega) e^{i\omega t} + \vec{\tilde{E}}(\vec{r}, \omega) e^{-i\omega t} \right] d\omega \\
 &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \text{Re} \left\{ \vec{\tilde{E}}(\vec{r}, \omega) e^{-i\omega t} \right\} d\omega
 \end{aligned}$$

Monochromatic field

there is only one frequency:

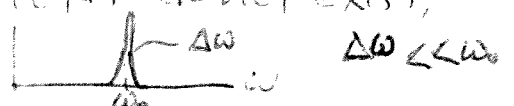


so  $\vec{E}(\vec{r}, t) = \frac{2}{\sqrt{2\pi}} \text{Re} \left\{ \vec{\tilde{E}}(\vec{r}, \omega_0) e^{-i\omega_0 t} \right\}$

Let  $\vec{U}(\vec{r}) = \frac{\vec{\tilde{E}}(\vec{r}, \omega_0)}{\sqrt{2\pi}}$

then  $\vec{E}(\vec{r}, t) = 2 \text{Re} \left\{ \vec{U}(\vec{r}) e^{-i\omega_0 t} \right\}$

In practice, true monochromatic fields do not exist, but lasers can come close.



True monochromatic fields would in principle exist for all time!

Time-dependent intensity:

$$\begin{aligned} \mathcal{I}(\vec{r}, t) &\propto \vec{E} \cdot \vec{E} = 2 \operatorname{Re} \{ \vec{U} e^{-i\omega_0 t} \} \cdot \operatorname{Re} \{ \vec{U} e^{-i\omega_0 t} \} \\ &\quad \xrightarrow{\text{for convenience}} \\ &= 2 \left[ \left( \operatorname{Re} \{ U_x e^{-i\omega_0 t} \} \right)^2 + \left( \operatorname{Re} \{ U_y e^{-i\omega_0 t} \} \right)^2 + \left( \operatorname{Re} \{ U_z e^{-i\omega_0 t} \} \right)^2 \right] \end{aligned}$$

Let  $U_m = |U_m| e^{i\phi_m}$ ,  $m=x, y, z$ . Then

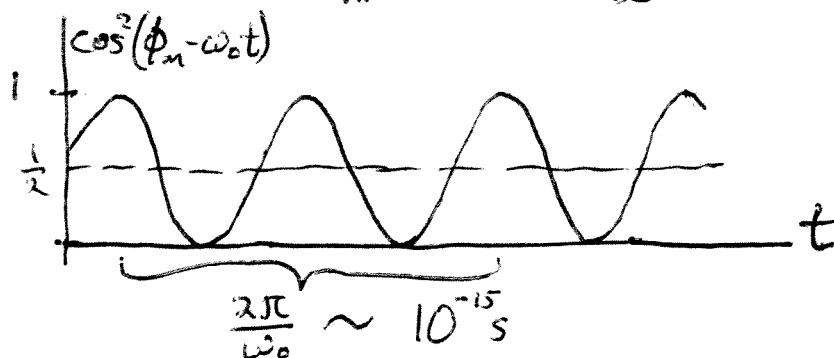
$$\mathcal{I}(\vec{r}, t) \propto 2 \left[ |U_x|^2 \cos^2(\phi_x - \omega_0 t) + |U_y|^2 \cos^2(\phi_y - \omega_0 t) + |U_z|^2 \cos^2(\phi_z - \omega_0 t) \right]$$

In practice, the oscillations are so fast that the eye or a detector only sees an average:

$$\mathcal{I}(\vec{r}) = \langle \mathcal{I}(\vec{r}, t) \rangle_t$$

Notice:

$$\langle \cos^2(\phi_m - \omega_0 t) \rangle_t = \frac{1}{2}$$



$$\text{Then, } \mathcal{I}(\vec{r}) \propto 2 \left[ |U_x|^2 \frac{1}{2} + |U_y|^2 \frac{1}{2} + |U_z|^2 \frac{1}{2} \right] = \underline{\underline{\vec{U}^*(\vec{r}) \cdot \vec{U}(\vec{r}) = |\vec{U}(\vec{r})|^2}}$$

Substitute now  $\vec{E} = 2 \operatorname{Re} \{ \vec{U}(\vec{r}) e^{i\omega_0 t} \}$   
 into wave eq.

$$\nabla^2 \vec{E} = 2 \operatorname{Re} \{ \nabla^2 \vec{U}(\vec{r}) e^{-i\omega_0 t} \}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = 2 \operatorname{Re} \{ -\omega_0^2 \vec{U}(\vec{r}) e^{-i\omega_0 t} \}$$

so

$$\operatorname{Re} \left\{ \left[ \nabla^2 \vec{U} + \underbrace{\frac{\omega_0^2}{c^2}}_{k_0^2} \vec{U} \right] e^{-i\omega_0 t} \right\} = 0 \quad \text{for all } t$$

therefore

$$\boxed{(\nabla^2 + k_0^2) \vec{U}(\vec{r}) = 0} \quad \text{Free-space vector Helmholtz Eq.}$$

For a monochromatic field in a linear, isotropic dielectric:  $\vec{D} = \epsilon(\vec{r}, \omega) \vec{E}$ ,

so

$$(\nabla^2 + \underbrace{\omega_0^2 \mu_0 \epsilon}_{\omega_0^2 \mu_0 \epsilon_0 \frac{\epsilon}{\epsilon_0}}) \vec{U}(\vec{r}) = 0$$

$$\underbrace{\omega_0^2 \mu_0 \epsilon_0}_{k_0^2} \underbrace{\frac{\epsilon}{\epsilon_0}}_{n^2(\vec{r}, \omega_0)}$$

$$\boxed{[\nabla^2 + k_0^2 n^2(\vec{r}, \omega_0)] \vec{U}(\vec{r}) = 0} \quad \text{Vector Helmholtz Eq. for inhomogeneous media.}$$

This is the basis of most of what follows

## Plane waves

One solution of the equation  $\nabla^2 U(\vec{r}) + k^2 U(\vec{r}) = 0$  is

$$\vec{U}(\vec{r}) = \vec{A} e^{i\vec{K} \cdot \vec{r}}$$

check:  $\nabla^2 \vec{U} = (i\vec{K}) \cdot (i\vec{K}) \vec{A} e^{i\vec{K} \cdot \vec{r}} = -\vec{K} \cdot \vec{K} \vec{U}$

so  $(k^2 - \vec{K} \cdot \vec{K}) \vec{U}(\vec{r}) = 0$

$$\vec{K} \cdot \vec{K} = k^2 \text{ or, in other words,}$$

$$K_x^2 + K_y^2 + K_z^2 = k^2, \text{ so only two components}$$

are independent.

Why do we call it plane wave?

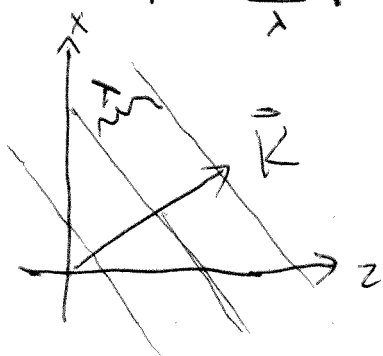
because  $\vec{K} \cdot \vec{r}$  is constant over all points  $\vec{r}$  within any plane perpendicular to  $\vec{K}$ .

Let us write  $\vec{K} = k \hat{u}$ , where  $\hat{u}$  is a unit vector.

Then  $e^{ik(\hat{u} \cdot \vec{r})} = \cos[k(\hat{u} \cdot \vec{r})] + i \sin[k(\hat{u} \cdot \vec{r})]$

oscillates in the direction of  $\hat{u}$ , with period  $\lambda$ , where

$$k = \frac{2\pi}{\lambda}. \quad \lambda \text{ is the wavelength.}$$



Additionally, we have

$$\nabla \cdot \vec{E} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{U} = 0$$

$$\text{but } \nabla \cdot [\vec{A} e^{i\vec{k} \cdot \vec{r}}] = i\vec{A} \cdot \vec{k} e^{i\vec{k} \cdot \vec{r}} = 0$$

so  $\vec{A}$  must be perpendicular to  $\vec{k}$

Therefore, there are two independent polarizations.

Exercise. Find the intensity of a superposition of two plane waves:

$$\vec{U}(\vec{r}) = \vec{A}_1 e^{i\vec{k}_1 \cdot \vec{r}} + \vec{A}_2 e^{i\vec{k}_2 \cdot \vec{r}}$$

## Paraxial approximation

Let us parametrize  $k_z$  in terms of  $k_x$  &  $k_y$ :

$$k_x^2 + k_y^2 + k_z^2 = k^2 \Rightarrow k_z = \pm \sqrt{k^2 - (k_x^2 + k_y^2)}$$

choose + sign for forward propagating waves.

Define  $\underline{k} = (k_x, k_y)$ . Then

$$k_z(\underline{k}) = \sqrt{k^2 - |\underline{k}|^2}$$

Paraxial approximation:  $\vec{k}$  is at small angles from the  $z$  axis, i.e.

$$k_z \approx k \gg |\underline{k}|$$

$$\begin{aligned} \text{Then } k_z(\underline{k}) &= k \sqrt{1 - \frac{|\underline{k}|^2}{k^2}} \approx k \left(1 - \frac{|\underline{k}|^2}{2k^2}\right) \\ &= k - \frac{|\underline{k}|^2}{2k} \end{aligned}$$

since  $\vec{A} \cdot \vec{k} = 0$ ,  $A_z \approx 0$ ,  $\vec{A} \approx A_x \hat{x} + A_y \hat{y}$ .

$$\begin{aligned} \text{so } \vec{U}(\vec{r}) &= (A_x \hat{x} + A_y \hat{y}) e^{i\vec{k} \cdot \vec{r}} \\ &= (A_x \hat{x} + A_y \hat{y}) e^{i\underline{k} \cdot \underline{x}} e^{ikz} e^{-\frac{i|\underline{k}|^2}{2k}z} \end{aligned}$$

where  $\underline{x} = (x, y)$

Consider now a continuous superposition of plane waves in different directions  $\underline{k}$ . Each might have a different complex amplitude  $\bar{A}$ :

$$\vec{U}(\vec{r}) = e^{ikz} \iint \bar{A}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} e^{-i|\underline{k}|z} dK_x dK_y$$

Change variables to  $\underline{k} = 2\pi\underline{v}$ ,  $k = 2\pi\lambda$

$$\begin{aligned} \vec{U}(\vec{r}) &= e^{ikz} \iint \bar{A}(2\pi\underline{v}) e^{i2\pi\underline{v} \cdot \underline{x}} e^{-i\frac{2\pi\lambda|\underline{v}|^2 z}{\lambda}} dV_x dV_y (2\pi)^2 \\ &= e^{ikz} \iint [4\pi^2 \bar{A}(2\pi\underline{v})] e^{i2\pi\underline{v} \cdot \underline{x}} e^{-i\pi\lambda z |\underline{v}|^2} dV_x dV_y \end{aligned}$$

Note that for  $z=0$

$$\vec{U}(\underline{x}, 0) = \vec{U}_0(\underline{x}) = \iint [4\pi^2 \bar{A}(2\pi\underline{v})] e^{i2\pi\underline{v} \cdot \underline{x}} dV_x dV_y$$

$$\text{so } \tilde{U}_0(\underline{v}) = \hat{F}_{\underline{x} \rightarrow \underline{v}} \vec{U}_0(\underline{x}) = 4\pi^2 \bar{A}(2\pi\underline{v})$$

so

$$\begin{aligned} \vec{U}(\underline{x}, z) &= e^{ikz} \iint \tilde{U}_0(\underline{v}) e^{-i\pi\lambda z |\underline{v}|^2} e^{i2\pi\underline{v} \cdot \underline{x}} dV_x dV_y \\ &= e^{ikz} \hat{F}_{\underline{v} \rightarrow \underline{x}}^{-1} \left[ e^{-i\pi\lambda z |\underline{v}|^2} \hat{F}_{\underline{x} \rightarrow \underline{v}} [\vec{U}_0(\underline{x})] \right] \end{aligned}$$

$$\text{Let } \tilde{G}(\underline{v}) = e^{-i\pi\lambda z |\underline{v}|^2}$$

$$\begin{aligned} \text{Then } \vec{U}(\underline{x}, z) &= e^{ikz} \hat{F}_{\underline{v} \rightarrow \underline{x}}^{-1} [\tilde{U}_0(\underline{v}) \tilde{G}(\underline{v})] \\ &= e^{ikz} \vec{U}_0(\underline{x}) * \tilde{G}(\underline{x}) \end{aligned}$$



where

$$\begin{aligned} G(\underline{x}) &= \hat{F}_{\underline{v} \rightarrow \underline{x}}^{-1} \tilde{G}(\underline{v}) = \iint_{-\infty}^{\infty} e^{-i\pi\lambda z |\underline{v}|^2} e^{i2\pi\underline{x} \cdot \underline{v}} d\underline{v}_x d\underline{v}_y \\ &= \iint_{-\infty}^{\infty} e^{-i\pi\lambda z \underbrace{|\underline{v} - \underline{x}/\lambda z|^2}_{\underline{v}'}} d\underline{v}_x d\underline{v}_y e^{i\pi |\underline{x}|^2 / \lambda z} \\ &\quad \underbrace{\qquad\qquad\qquad}_{\underline{v}', d\underline{v}_x d\underline{v}_y = d\underline{v}'_x d\underline{v}'_y} \\ &= \frac{e^{i\pi \frac{|\underline{x}|^2}{\lambda z}}}{i\lambda z} \end{aligned}$$

so

$$\vec{U}(\underline{x}, z) = e^{ikz} \vec{U}_0(\underline{x}) * G(\underline{x})$$

$$= \iint_{-\infty}^{\infty} \vec{U}_0(\underline{x}') \underbrace{\frac{e^{ik\left[z + \frac{|\underline{x}' - \underline{x}|^2}{2z}\right]}}{i\lambda z}}_{\text{Paraxial approximation to a spherical wave}} dx' dy'$$

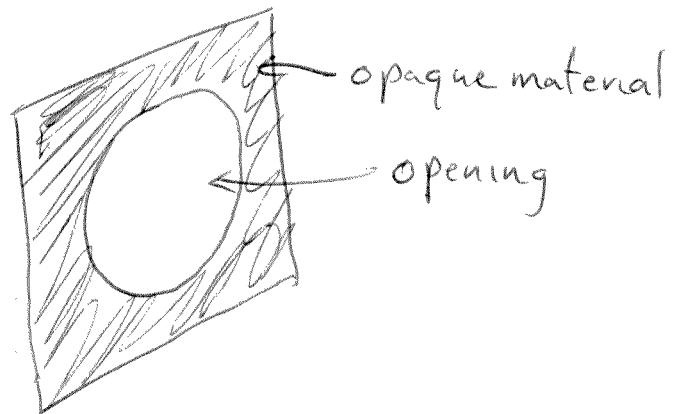
Paraxial approximation  
to a spherical wave.

Fresnel diffraction formula

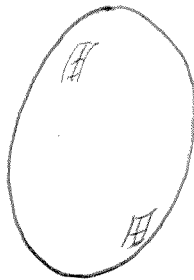
# "Planar" obstacles

These are optical elements at planes normal to the optical axis, e.g.

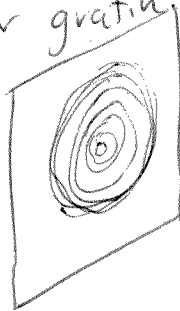
- planar apertures



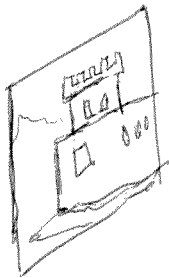
- thin lenses



- diffractive lenses or gratings



- transparencies



## Kirchhoff approximation:

The field right after the planar object equals the field right before the planar object times a transmission function  $t(x, y)$ .

Consider a "scalar field" (e.g. only one polarization)

$$U(\vec{r})$$

and place an element described by  $t(x, y)$  at  $z$ .

then:

$$U(x, z_+) = t(x) U(x, z_-)$$

↑ just after                      ↓ just before.

The intensity right after is

$$I(x, z_+) = |U(x, z_+)|^2 = |t(x)|^2 |U(x, z_-)|^2 = |t(x)|^2 I(x, z_-)$$

$$\text{so } |t(x)| \leq 1$$

If the element is transparent,  $|t(x)| = 1$

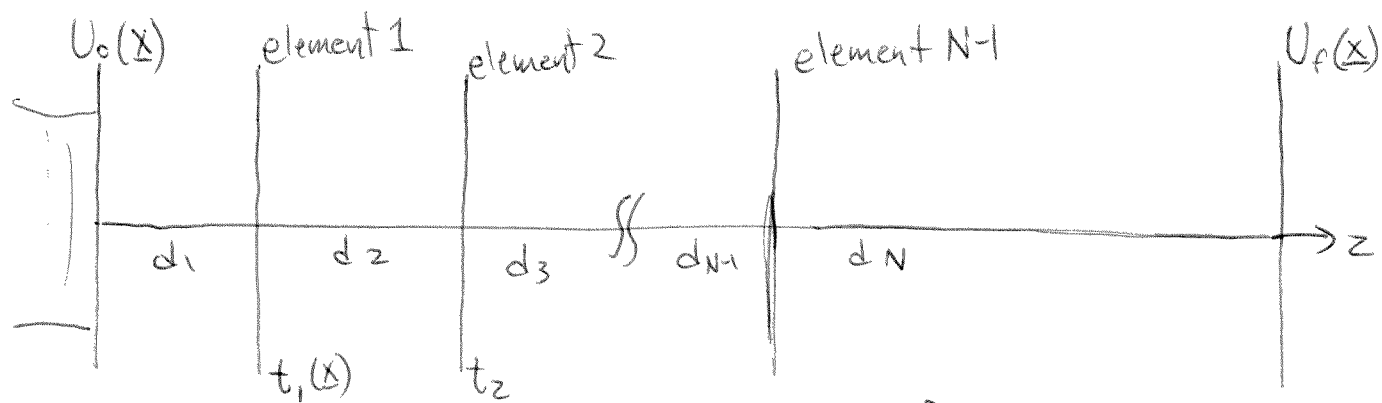
$$t(x) = e^{i\phi(x)},$$

for a lens,  $\phi(x) = k \left[ n \Delta_0 - \frac{|x|^2}{2f} \right]$ , where  $f$  = focal length.

For an aperture:  $t(x) = \begin{cases} 1 & \text{inside aperture} \\ 0 & \text{outside aperture} \end{cases}$

$$\text{In general } t(x) = |t(x)| e^{i\phi(x)}$$

# Propagation across a system



Let  $\tilde{h}(k, d) = e^{ikd} e^{-i\pi \lambda d |k|^2}$

Then

$$U_f(x) = \hat{\mathcal{F}}^{-1} \tilde{h}(k, d_N) \hat{\mathcal{F}} t_{N-1}(x) \hat{\mathcal{F}}^{-1} \tilde{h}(k, d_{N-1}) \hat{\mathcal{F}} t_{N-2}(x) \hat{\mathcal{F}}^{-1} \tilde{h}(k, d_1) \hat{\mathcal{F}} U_0(x) /$$

# Resolution

The image of a point object is not a point but a "blur" called the Airy pattern:

$$U_o(\underline{x}) = \delta(\underline{x} - \underline{x}_o) \Rightarrow U_i(\underline{x}) \propto \frac{J_1(kNA_i |\underline{x} - M\underline{x}_o|)}{kNA_i |\underline{x} - M\underline{x}_o|} e^{i\phi}$$

where it was assumed that the system is paraxial and there are no aberrations, and where

$NA_i$  = numerical aperture in image space  
 $\phi$  = some phase  
 $M$  = transverse magnification.

Consider two object points  $\underline{x}_1$  &  $\underline{x}_2$  with ideal image locations  $\bar{\underline{x}}_1 = M\underline{x}_1$ ,  $\bar{\underline{x}}_2 = M\underline{x}_2$ .

- If the illumination is coherent (a collimated laser), then

$$U_i(\underline{x}) \propto \frac{J_1(kNA_i |\underline{x} - \bar{\underline{x}}_1|)}{kNA_i |\underline{x} - \bar{\underline{x}}_1|} e^{i\phi_1} + \frac{J_2(kNA_i |\underline{x} - \bar{\underline{x}}_2|)}{kNA_i |\underline{x} - \bar{\underline{x}}_2|} e^{i\phi_2}$$
$$I_i(\underline{x}) = \left| \frac{J_1(kNA_i |\underline{x} - \bar{\underline{x}}_1|)}{kNA_i |\underline{x} - \bar{\underline{x}}_1|} + e^{i(\phi_2 - \phi_1)} \frac{J_2(kNA_i |\underline{x} - \bar{\underline{x}}_2|)}{kNA_i |\underline{x} - \bar{\underline{x}}_2|} \right|^2$$

this depends strongly on  $\phi_2 - \phi_1$

If the illumination is spatially incoherent, then the interference terms average to zero, and

$$I_i \propto \left| \frac{J_1(kNA_i |\underline{x} - \bar{x}_1|)}{kNA_i |\underline{x} - \bar{x}_1|} \right|^2 + \left| \frac{J_1(kNA_i |\underline{x} - \bar{x}_2|)}{kNA_i |\underline{x} - \bar{x}_2|} \right|^2$$

### Rayleigh Resolution Criterion

We can distinguish the images of two points if their separation is more than the distance to the first zero:

$$|\bar{x}_2 - \bar{x}_1| \geq 0.609 \lambda / NA_i$$

