QUANTUM UNCERTAINTY RELATIONS: ENGAGING THE MEDIAN WHEN MEAN IS NOT LICIT

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TO BE DISCUSSED

- Motivation
- Quantum uncertainty relation
- Problems with mean-based QUR
- What next? Median!
- New, median-based, QUR
- Several distributions
- Title of most classical state → CHANGED!!
- Conclusion

MOTIVATION

- QUR: A cornerstone in differentiating quantum from classical theories.
- Mean as measure of central tendency and standard deviation as measure of dispersion not always relevant. Also they may not exist.
- Will be good if such a fundamental relation is true and relevant for all quantum states. Not true currently.

Quantum Uncertainty relation

- Usually expressed using standard deviations of two observables, say the position (x) and momentum (p) of quantum system.

- $$\Delta x \, \Delta p \geq \hbar / r$$

  where $\hbar = h/2\pi$ and $r=2$

- Refer to such QUR as mean-based QUR. Note that SD is defined via mean.

Problems with the mean-based QUR

- Mean is pretty standard measure of central tendency. But not useful in certain cases.

  Ex. (a) when there are large outliers in a distribution.

  (b) when the distribution is skewed.

  (c) (and obviously) when the mean of a distribution is non-existent.

- Similarly troubles appear for SD as a measure of dispersion/spread.

What next? Median!

The median of a probability distribution: A measure of central tendency.

- Unaffected by large outlier values of r.v.
- Good estimation of central tendency for skewed distributions.
- Importantly, always exists.

What next? Median!

For a probability distribution $P(A=a)$, due median, M, is given by

$$\int_{-\infty}^{M} P(A=a)\,da = 1/2$$

Median-based dispersion

- A measure of dispersion:
  The median of the moduli of the deviations around the median—
  Semi interquartile range

- It exists for all probability distributions. In particular, for position and momentum distributions of all quantum states.

The Quartiles

- The first and third quartiles, Q_1^A and Q_3^A, of the probability distribution $P(A=a)$:

$$\int_{-\infty}^{Q_{2m+1}^A} P(A=a)\,da = \frac{2m+1}{4}$$

for $m=0,1$.

Semi-interquartile range

- The "semi-interquartile range" is defined as

$$\Delta A = 1/2\ (Q_{3}^{A} - Q_{1}^{A})$$

Semi-interquartile range

- Median signals the half-way point of a probability distribution.
- "Quartiles" signal when the distribution is quarter-way and three-quarter way.
- Semi-interquartile range in the median-based world is what SD is in the mean-based one. It quantifies dispersion or spread of the probability distribution.

The question addressed

Does quantum mechanics places a "median-based" uncertainty relation of position and momentum for arbitrary states, which is independent of the mean-based one?

Setting the stage

Position probability distribution:

For the pure quantum state, $\psi(x)$, in coordinate representation, of a system moving on a straight line, the quartiles, Q_1^x and Q_3^x can be obtained from

$$\int_{-\infty}^{Q_{2m+1}^x} |\psi(x)|^2 \, dx = 2m+1/4$$

for $m=0, 1.$

Setting the stage

<u>Momentum probability distribution:</u>

For the same system, the momentum representation of $\psi(x)$, is given by the Fourier transform

$$\phi(p)=1/2\pi\hbar \int_{-\infty}^{\infty} e^{-ixp/\hbar}\,\psi(x)dx$$

Setting the stage

The quartiles, Q_1^p and Q_3^p, are obtained from the following:

$$\int_{-\infty}^{Q_{2m+1}^p} |\phi(p)|^2 \, dp = \frac{2m+1}{4}$$

for $m = 0, 1$.

Median-based QUR

- We want to analyze the following quantity:

$\Delta x \, \Delta p$

where $\Delta x = 1/2 \; (Q_3^x - Q_1^x)$, and

$\Delta p = 1/2 \; (Q_3^p - Q_1^p)$.

Median-based QUR

Aim:

To find a lower bound of $\Delta x \Delta p$ for ***arbitrary*** quantum states.

Distributions considered

- Cauchy distribution
- Gaussian distribution
- Student's t-distribution

[Have also considered F distribution and numerical simulation on space of all functions. Will not discuss these here.]

Cauchy distribution

- Cauchy probability distribution:

$$f_C(a; x_0, \gamma) = \frac{\gamma}{\pi} \frac{1}{(a - x_0)^2 + \gamma^2}$$

where $a \in \mathbb{R}$ and $\gamma > 0$ and x_0 are the distribution parameters.

Cauchy distribution, cont.

- A quantum state in 1D, whose wave function,

$\psi_C(x) = (f_C(x; x_0, \gamma))^{1/2}$, $x \in \mathbb{R}$

- The function is square integrable and continuous. Hence valid quantum mechanical wave function.

- Neither mean nor SD exist.

Cauchy distribution, cont.

- For the position probability distribution,

$$Q_{1,C}^{x} = x_0 - \gamma,$$ and

$$Q_{3,C}^{x} = x_0 + \gamma,$$

So, semi-interquartile range $= \gamma$

Cauchy distribution, cont.

- For the momentum probability distribution, we have used

  a) Gaussian Quadrature and

  b) Van Wijngaarden-Dekker-Brent methods

to find $Q_{1,C}^{p}$ and $Q_{1,C}^{p}$.

Cauchy distribution, cont.

- The semi interquartile ranges for momentum probability distributions:

$$0.094\hbar,\ 0.047\hbar,\ 0.032\hbar,\ 0.024\hbar$$

for $\gamma = 1, 2, 3, 4.$

Therefore, $\Delta x\, \Delta p \approx \hbar / 10.6$

Gaussian distribution

- The probability distribution:

$$f_G(a;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(a-\mu)^2/2\sigma^2}$$

Here $a\in\mathbb{R}$ and μ and σ are the distribution parameters.

- Both mean and SD exist.

Gaussian distribution, cont.

- Consider a quantum system, for which the wave function in coordinate representation is given by

$$\psi_G(x) = \left(f_G(x{:}\mu,\sigma^2)\right)^{1/2}, \quad x \in \mathbb{R}$$

Gaussian distribution, cont.

- The first and third quartiles are

  $Q_{1,G}^{x} = \mu - 2/3\ \sigma$, and

  $Q_{3,G}^{x} = \mu + 2/3\ \sigma$

So, the semi-interquartile range $= 2/3\ \sigma$.

Gaussian distribution, cont.

- For the momentum probability distribution, the semi-interquartile range $= \hbar/3\sigma$.

- So, $\Delta x \, \Delta p = \hbar/4.5$

Gaussian distribution, cont.

- Cauchy distribution is more classical than Gaussian, for median-based QUR.

- Gaussian distribution provides most classical state in quantum mechanics, i.e., provides minimum uncertainty state, for mean-based QUR.

Student's t-distribution

- The probability distribution:

$$f_s(a;n)=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{a^2}{n}\right)^{-\frac{n+1}{2}}$$

where $a\in\mathbb{R}$, and n is the number of "degrees of freedom".

Student's t-distribution n=2

- Mean exists and is vanishing.
- Standard deviation diverges to infinity.

Student's t-distribution n=2, cont.

- Consider a quantum system in 1D whose wave function in coordinate representation is

$$\psi_S(x)=(f_S(x:2))^{1/2}\quad , \quad x\in\mathbb{R}$$

Student's t-distribution n=2, cont.

For position probability distribution

$Q{\downarrow}1, S{\downarrow}{\uparrow}x = -\sqrt{2}/3$ and

$Q{\downarrow}1, S{\downarrow}{\uparrow}x = -\sqrt{2}/3$.

So, semi-interquartile range $= \sqrt{2}/3$

Student's t-distribution n=2, cont.

For momentum probability distribution

$$Q_{1,S}^{p} = -0.161\hbar \quad \text{and}$$

$$Q_{3,S}^{p} = 0.161\hbar.$$

So, semi-interquartile range $= 0.161\hbar$.

Student's t-distribution n=2, cont.

- So, the median based quantum uncertainty product is

$$\Delta x \, \Delta p \approx 0.131\hbar = \hbar/7.63$$

Student's t-distribution n=2, cont.

- Student's t-distribution is somewhat midway between Cauchy and Gaussian distributions w.r.t. the bound on the median-based uncertainty product.

- Cauchy and t-distribution (n=2) not considered in race for minimum uncertainty state, for mean-based QUR, as none have finite variance.

Student's t-distribution n=2, cont.

- Both win over Gaussian distribution, for median-based QUR.
- Is there a quantum state that has finite variance, and yet provide minimum value over the Gaussian , for median-based QUR.
- This is what we consider next.

Student’s t-distribution n=3

- The wave function in coordinate representation is given by

$$\psi_{S'}(x)=(f_{S}(x:3))^{1/2} \quad , \quad x\in\mathbb{R}$$

- Both mean and SD exist for the position and momentum distribution.

Student's t-distribution n=3, cont.

- For position probability distribution,

$$Q_{1}, S'_{}^{x} = -0.765 \quad \text{and}$$

$$Q_{3}\ S'_{}^{x} = 0.765.$$

So, semi-interquartile range $= 0.765$

Student's t-distribution n=3, cont.

- For momentum probability distribution,

$$Q_{1,S'^{p}} = -0.200\hbar$$ and

$$Q_{3,S'^{p}} = 0.200\hbar.$$

So, semi-interquartile range $= 0.200\hbar$.

Student's t-distribution n=3, cont.

- Therefore, in this case,

$$\Delta x\, \Delta p \approx 0.153\hbar = \hbar/6.54$$

- Clearly, it is more classical than the Gaussian distribution.

The race for the most classical state

Distribution	
Gaussian	$\hbar/4.5$
Student's t (n=3)	$\hbar/6.54$
Student's t (n=2)	$\hbar/7.63$
Cauchy	$\hbar/10.6$

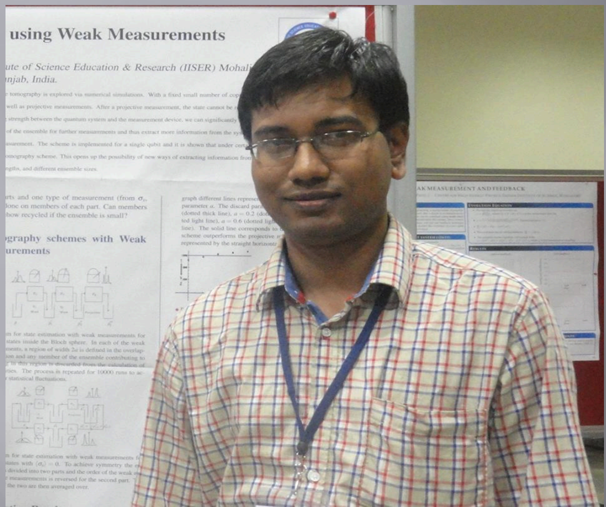
Take home message

- Mean is not always relevant measure of central tendency. Also can be non-existent.
- Standard deviation as a measure of dispersion may be inefficient or no-existent, even if mean exists.
- Median always exists.

Take home message

- We propose a median-based quantum uncertainty relation with semi-interquartile range as a measure of dispersion.
- Valid for all quantum states.
- Among the ***distribution*** studied, Cauchy is the most classical quantum state rather than the Gaussian one.

Collaborators



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Thank You!