Simultaneous estimation of parameters encoded in coherent states

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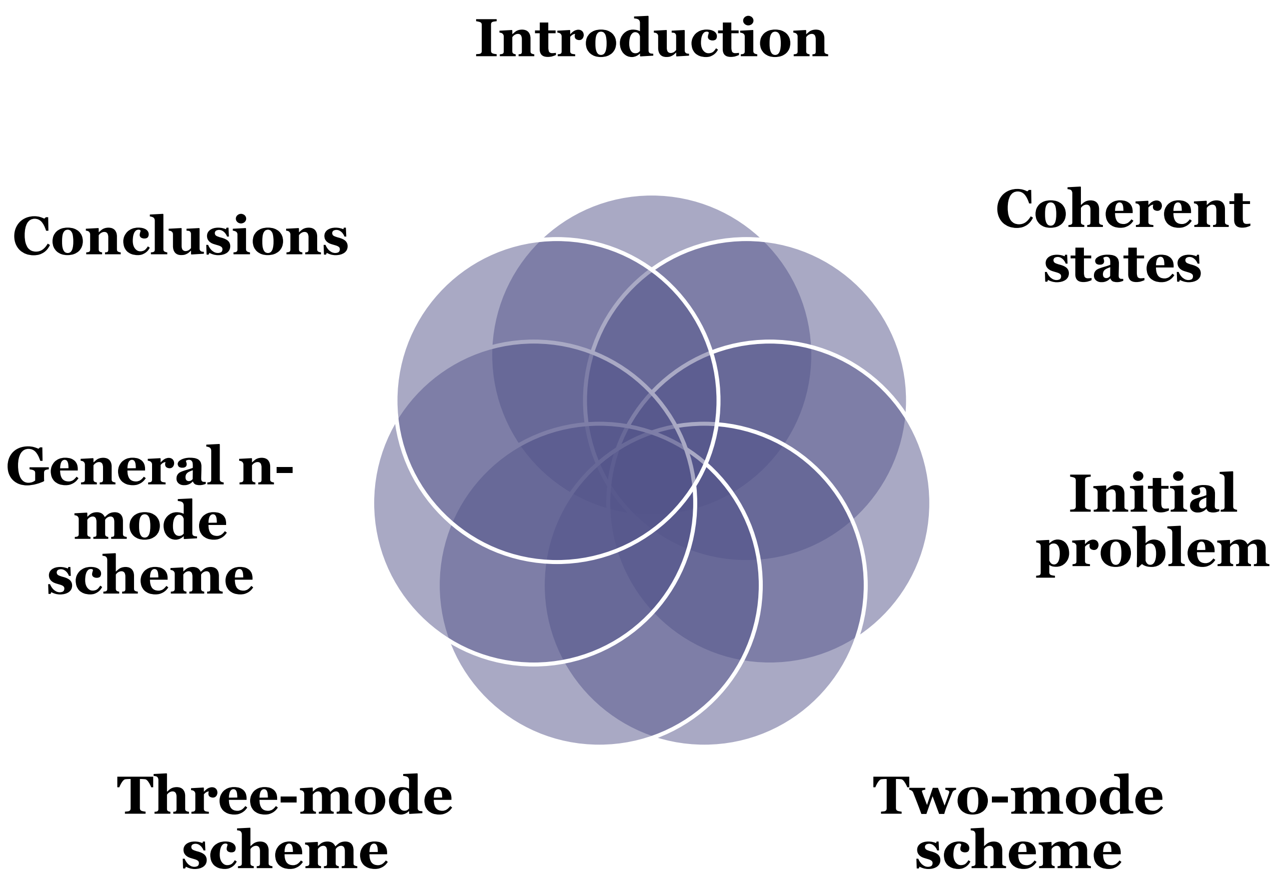
Trieste, september 2017

Outline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Introduction

$$\left[\hat{x}, \hat{p}\right] = i\hbar \quad \Rightarrow \quad \Delta x \Delta p \geq \frac{\hbar}{2}$$ **Heisenberg's inequality**

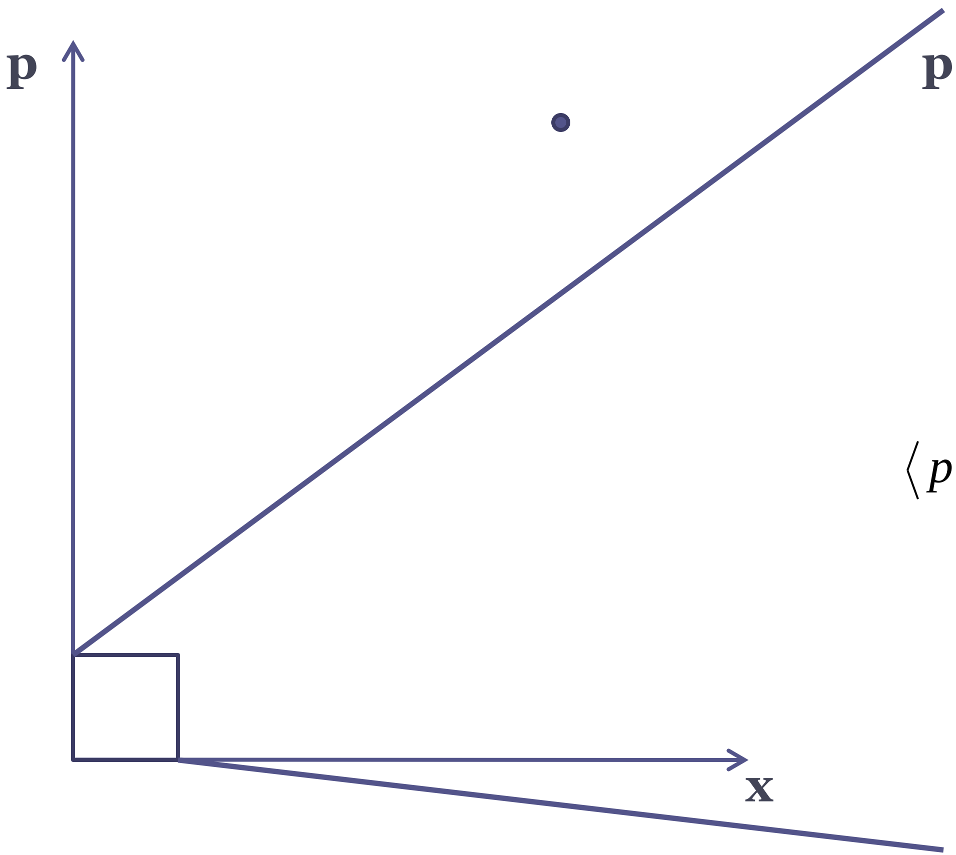
Question : How can we access classical information encoded in conjugated variables ?

Good states :	**Encoding procedure :**	**Measurement scheme :**
Coherent states of light $\Delta x \Delta p = \frac{1}{2}, \quad \hbar = 1$	Classical information is encoded in the quadratures of the coherent states	Homodyne detection Heterodyne detection

Coherent states

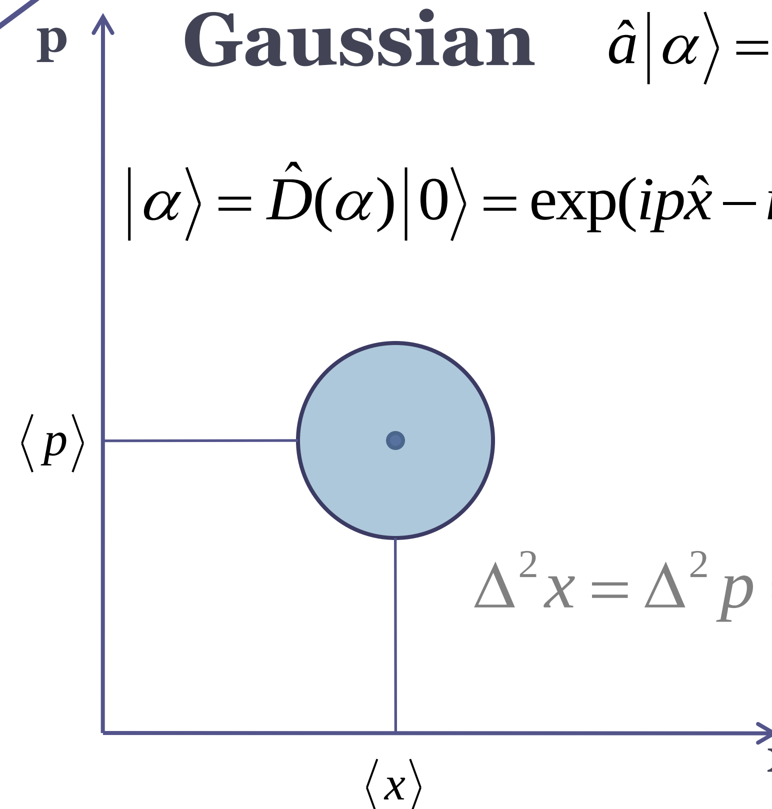
Quantum state of light produced by a coherent laser beam.

Classical laser beam	Coherent state of light



Gaussian $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

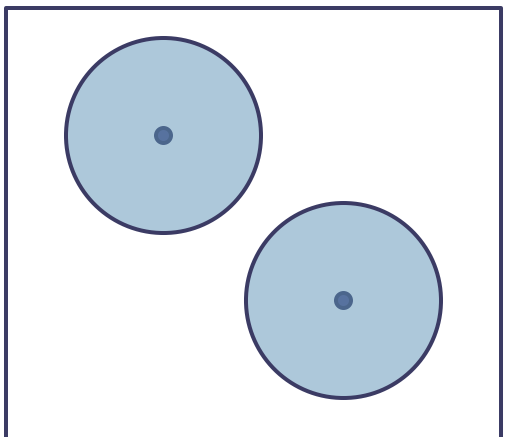
$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = \exp(ip\hat{x} - ix\hat{p})|0\rangle$$



$$\Delta^2 x = \Delta^2 p = 1/2$$

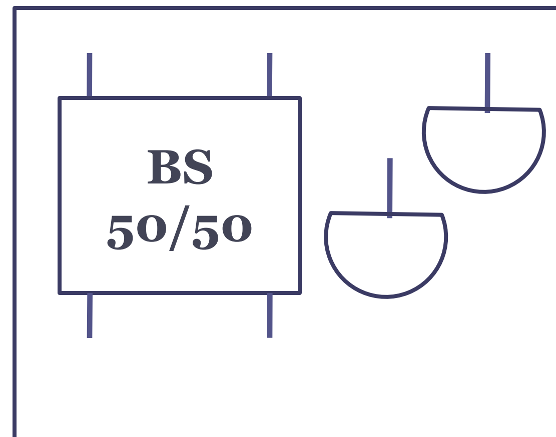
Initial Problem

Alice $z = \frac{a + ib}{\sqrt{2}}$ → **Bob**



Constraint :

$$x_1^2 + p_1^2 = x_2^2 + p_2^2$$



$$a, b$$

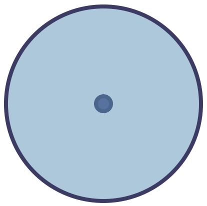
$$x_1, p_1, x_2, p_2$$

$$\tilde{z} = \frac{\tilde{a} + i\tilde{b}}{\sqrt{2}}$$

Individual measurement

One coherent state

$$z = \frac{a+ib}{\sqrt{2}}$$



Measurement

Encoding

$$x = a,$$
$$p = b.$$

Homodyne

$$\Delta^2 x = \frac{1}{2} \quad OR \quad \Delta^2 p = \frac{1}{2}$$

Heterodyne

$$\Delta^2 x \geq \frac{1}{2} \quad AND \quad \Delta^2 p \geq \frac{1}{2}$$

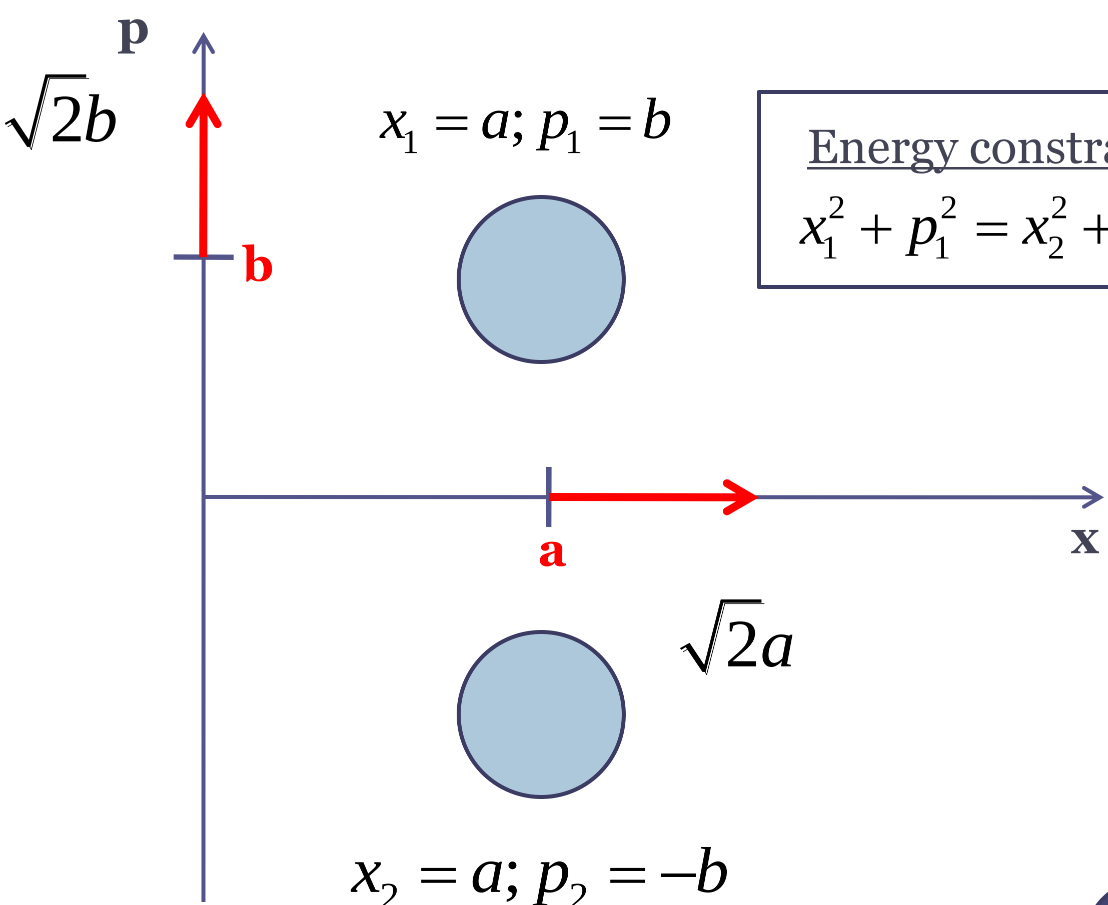
Two identical coherent state

$$\Delta^2 x = \frac{1}{2} \quad AND \quad \Delta^2 p = \frac{1}{2}$$

$$\Delta^2 x \geq \frac{1}{4} \quad AND \quad \Delta^2 p \geq \frac{1}{4}$$

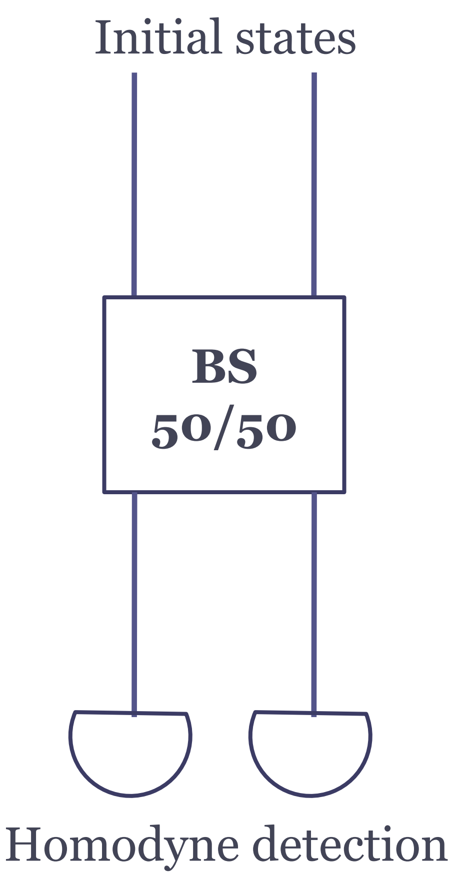
Cerf-Iblisdir scheme [1]

- Phase conjugate coherent states



Energy constraint

$$x_1^2 + p_1^2 = x_2^2 + p_2^2$$



$$\Delta^2 a = \Delta^2 b = 1/4$$

Experimentally confirmed ! [2]

[1] N. J. Cerf & S. Iblisdir, Phys. Rev. A **64**, 032307 (2001)
[2] J. Niset *et al.*, Phys. Rev. Lett. **98**, 260404 (2007)

Proof of optimality

Cerf – Iblisdir Scheme :

- **Set of parameters :** $\Theta = (a, b)$
- **Initial states :** $|\psi\rangle = |\alpha\rangle \otimes |\alpha^*\rangle$

Quantum Fisher Information Matrix : $F^{(\alpha\alpha^*)} = F^{(\alpha)} + F^{(\alpha^*)}$

$$F^{(\alpha)} = \begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix}, \quad F^{(\alpha^*)} = \begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix}.$$

$$F^{(\alpha\alpha^*)} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

Quantum Cramér-Rao bound

$$\Delta^2 a \geq \frac{1}{4}, \quad \Delta^2 b \geq \frac{1}{4}.$$

General two-mode scheme

Question : Are the phase conjugated states the only optimal encoding setup of initial coherent states ?

Answer : No, if we relax the equally distributed energy constraint !

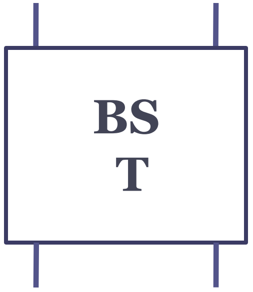
En… …raint (crossed out)

$x_1^2 + \ldots + p_2^2$

Total energy constraint

$$x_1^2 + p_1^2 + x_2^2 + p_2^2 = 2(a^2 + b^2)$$

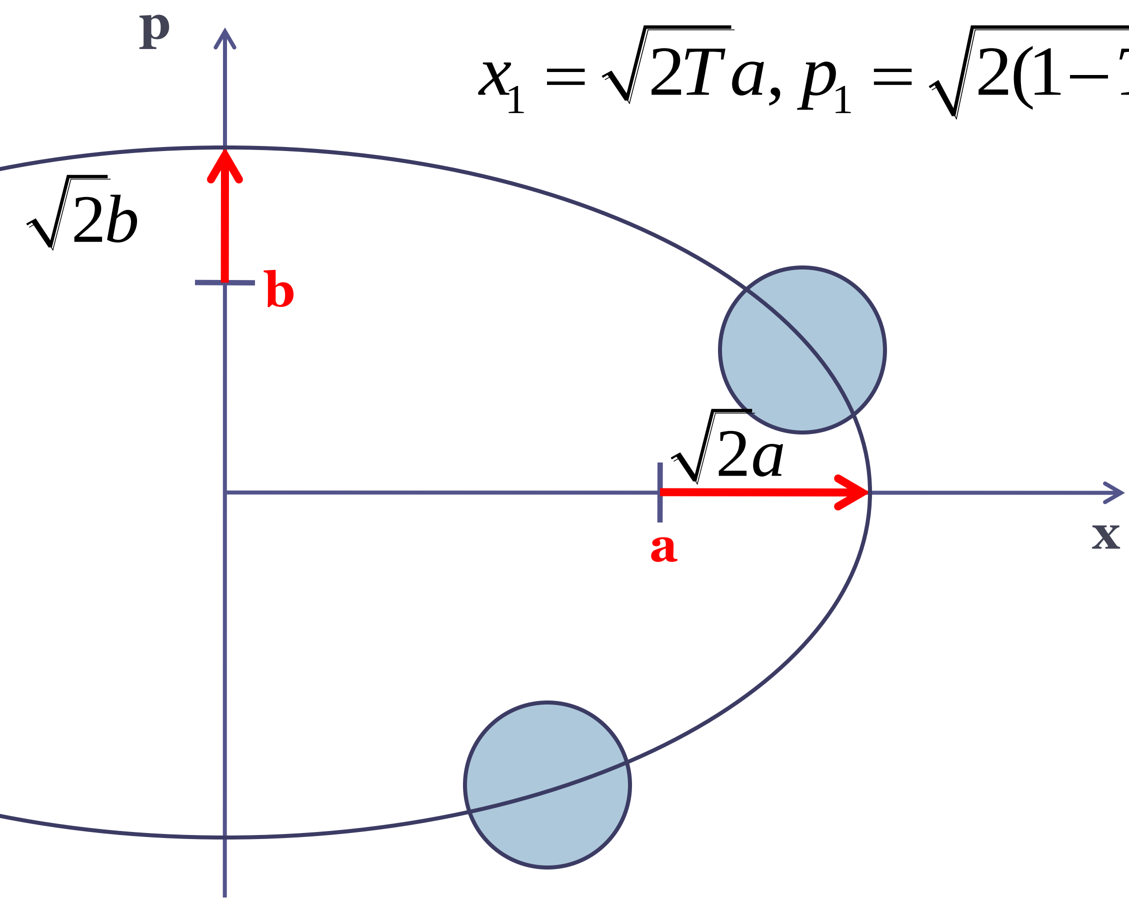
One extra free parameter : Transmittance T



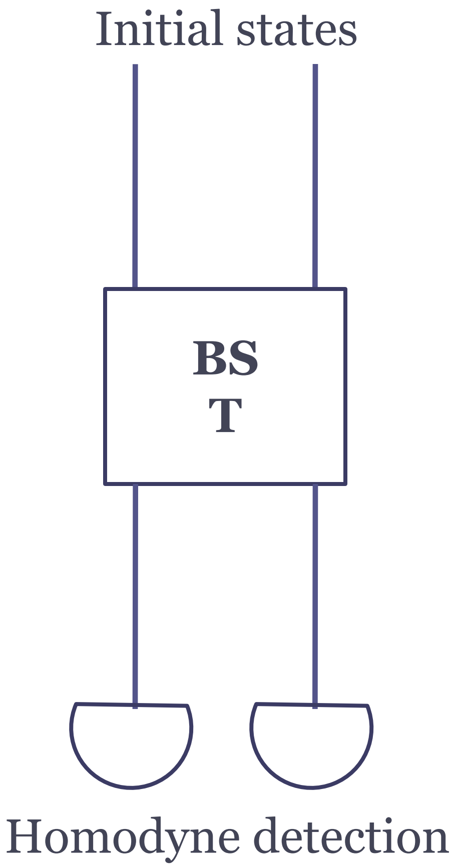
Familly of optimal input states parametrized by T

General two-mode scheme

$$x_1 = \sqrt{2T}a,\ p_1 = \sqrt{2(1-T)}b$$

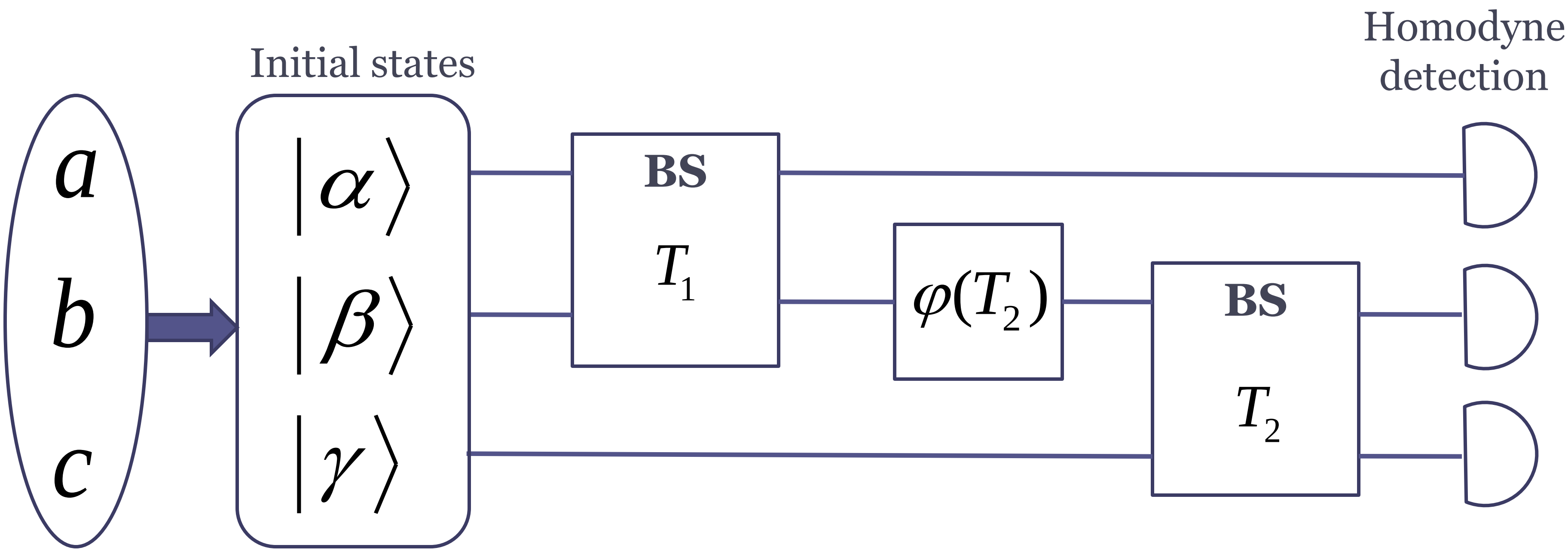


$$x_2 = \sqrt{2(1-T)}a,\ p_2 = -\sqrt{2T}b$$



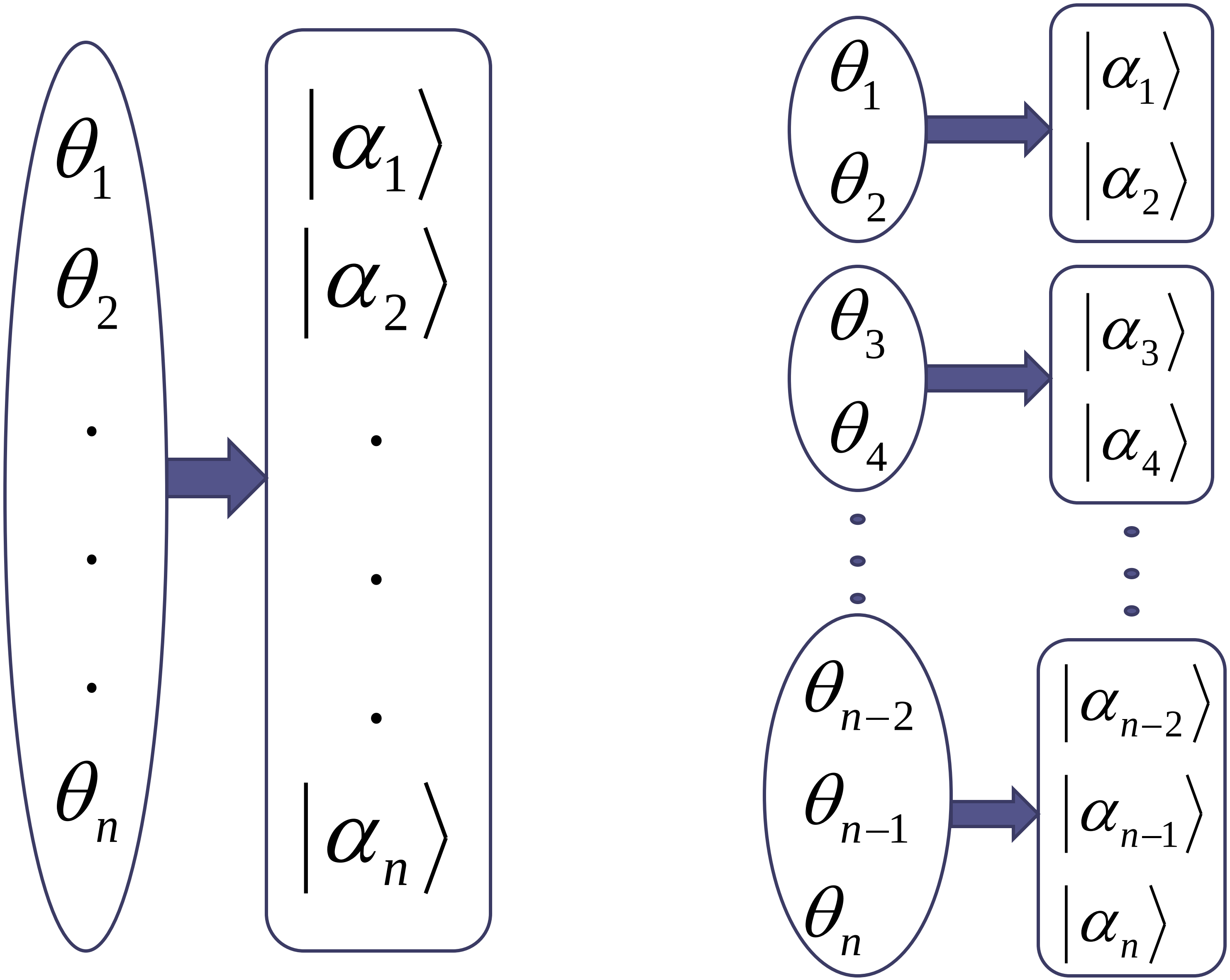
$$\Delta^2 a = \Delta^2 b = \frac{1}{4}$$

General Three-mode Scheme



Optimal measurement $$\Delta^2 a = \Delta^2 b = \Delta^2 c = 1/4$$

General n-mode Scheme



Conclusions & Perspectives

- Optimality proof of the Cerf-Iblisdir scheme
- Generalization for two- and three modes
- General scheme for encoding n real parameters into the quadratures of n coherent states which allows optimal decoding with homodyne detection
- Extension to other states (squeezed states…) ?
- Potential usage in cryptographic applications ?

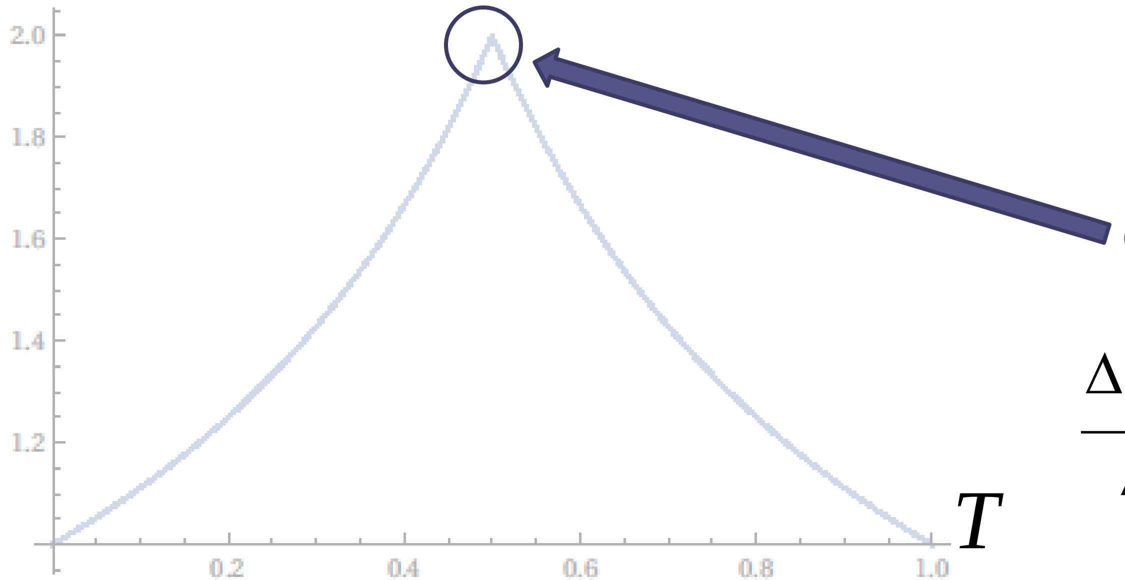
Thank you for your attention

Performance

We compare the performance of our joint measurement and of individual measurement techniques

$$\frac{\Delta^2 a_{ind}(T)}{\Delta^2 a_{QCRB}} = \frac{\Delta^2 b_{ind}(T)}{\Delta^2 b_{QCRB}} = \frac{2}{1+|1-2T|}$$

$$\frac{\Delta^2 a_{ind}(T)}{\Delta^2 a_{QCRB}}$$



$$\frac{\Delta^2 a_{ind}(\tfrac{1}{2})}{\Delta^2 a_{QCRB}} = \frac{\Delta^2 b_{ind}(\tfrac{1}{2})}{\Delta^2 b_{QCRB}} = 2$$

Optimal ?

Quantum Cramér-Rao bound : (QCRB)

$$\Delta^2(\theta) \geq \left[N F(\theta) \right]^{-1}$$

Single parameter

Quantum Fisher Information (QFI) :

$$F(\theta) = 4\left[\langle \partial_\theta \psi_\theta | \partial_\theta \psi_\theta \rangle - \langle \partial_\theta \psi_\theta | \psi_\theta \rangle \langle \psi_\theta | \partial_\theta \psi_\theta \rangle \right]$$

Pure state

For multiple parameters : $\Theta = \left(\theta_1, \theta_2, \ldots, \theta_n \right)$

QFI Matrix : $F_{ij}(\Theta) = 4\left[\langle \partial_{\theta_i} \psi | \partial_{\theta_j} \psi \rangle - \langle \partial_{\theta_i} \psi | \psi \rangle \langle \psi | \partial_{\theta_j} \psi \rangle \right]$

QCRB :

$$\Delta^2(\theta_i) \geq \frac{(F^{-1}(\Theta))_{ii}}{N}$$