The Abdus Salam International Centre for Theoretical Physics

Entangled Hypergraphs vs. Hypergraph States and Their Role in Classification of Multipartite Entanglement

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Outline

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Motivation

Motivation i

Entangled State: a pure state is called entangled if it is not separable.

$$|\Psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$

Equivalent relation:

- LOCC: equivalency based on LUT (Local Unitary Transforamations):

$$|\Psi\rangle \sim |\Phi\rangle \qquad (P = 1) \qquad \text{iff} \qquad |\Psi\rangle = U_1 \otimes U_2 \otimes \cdots \otimes U_n |\Phi\rangle$$

LOCC $\longrightarrow$ infinite orbits even in the simplest bipartite systems!

- SLOCC: equivalency based on LIT(Local Invertible Transforamations):

$$|\Psi\rangle \sim |\Phi\rangle \qquad (0 < P < 1) \qquad \text{iff} \qquad |\Psi\rangle = GL_1 \otimes GL_2 \otimes \cdots \otimes GL_n |\Phi\rangle$$

C.H. Bennett et al., PRA 63, 012307 (2000)

W. Dür, G. Vidal, J.I. Cirac, PRA 62, 062314 (2000)

Motivation ii

SLOCC classification:

3 qubits: 6 classes (A-B-C, A-BC, B-AC, C-AB, W, & GHZ)

$n \geqslant 4$ qubits: infinite classes!

SLOCC classification into families criteria:

- Every SLOCC class must belong to only one family
- Separable states must be in one family
- SLOCC classes belonging to the same family must show common physical (mathematical) properties
- The classification into families must be efficient in the sense that
    1. The number of families must grow slowly with the number of qubits
    2. Classifying N qubits should be useful for classifying $N+1$ qubits

M. Sanz et al., Sci. Rep. 6, 30188 (2016)

Introduction

Entanglement Measures

- Concurrence: for a general 2-qubit state, Wootters defines the concurrence as below

$$\mathcal{C} = |\langle\Psi|\sigma_y \otimes \sigma_y|\Psi^*\rangle|$$

- Tangle: for a 3-qubit state, CKW introduce a measure as below

$$\tau = \mathcal{C}^2_{A(BC)} - \mathcal{C}^2_{AB} - \mathcal{C}^2_{AC}$$

- Global entanglement: consider an N-qubit pure state partitioned into two blocks S and $\overline{S}$ comprising m and $N-m$ qubits respectively.

entanglement of block S to the rest: $\eta_{S\overline{S}} = \frac{2^m}{2^m - 1}\left(1 - Tr(\varrho_S^2)\right)$

geomtric mean: $\mathcal{C}_g = \left(\prod \eta_{S\overline{S}}\right)^{\frac{1}{2^{N-1}-1}}$

W.K. Wootters, PRL 80, 2245 (1998)

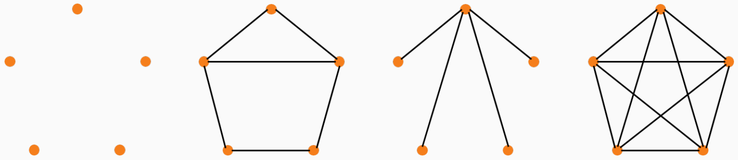
V. Coffman, J. Kundu, W.K. Wootters, PRA 61, 052306 (2000)

P.J. Love et al., QIP 6, 187 (2007) - M. G G & S.J. Akhtarshenas, EPJD 70, 54 (2016)

Map: States $\longleftrightarrow$ Graphs

Graph: a simple & undirected graph G is an ordered pair $G = (V, E)$ where:

- V is a set of elements called vertices
- E is a set of edges, which are 2-element subsets of V



Cardinality of a graph:

- $|V|$ = number of vertices, is called the order of graph
- $|E|$ = number of edges, is called the size of graph

Connected graph: existence of a path between every pair of vertices

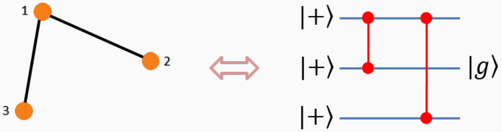
Tree: connected graph by exactly one path between every pair of vertices

Graph States

Goal: to create REW (Real Equally Weighted) states

- Vertex $\longleftrightarrow$ Qubit
- Edge $\longleftrightarrow$ Two-body interaction

$$|g\rangle = \prod_{\{i_1,i_2\}\in E} C^2 Z_{i_1 i_2} |+\rangle^{\otimes n}$$



$$|g\rangle = \frac{1}{\sqrt{8}}\big(+ |000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle \big)$$

$$\begin{cases} \mathcal{C}_{12} = \mathcal{C}_{13} = \mathcal{C}_{23} = 0 \\ \tau = 1 \\ \mathcal{C}_g = 1 \end{cases}$$

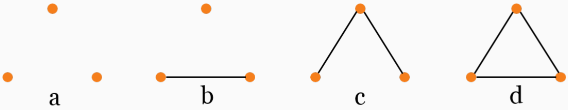
1. No one-to-one correspondence between the graph and entanglement!
2. No W state exist!

M. Hein, J. Eisert, H.J. Briegel, PRA 69, 062311 (2004)

Entangled Graphs

Goal: to write a pure state for every possible graph where:

- Vertex $\longleftrightarrow$ Qubit
- Edge $\longleftrightarrow$ Bipartite entanglement



$$\begin{cases} a) \left.\begin{cases} |\text{Sep}\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle \\ |\text{GHZ}\rangle = \alpha|000\rangle + \beta|111\rangle \end{cases}\right\} \text{ambiguity!} \\ b) \quad |\text{BS}\rangle = |\text{Bell State}\rangle \otimes |\varphi\rangle \\ c) \quad |\text{Star}\rangle = \alpha|000\rangle + \beta|100\rangle + \gamma|110\rangle + \delta|111\rangle \\ d) \quad |\text{W}\rangle = \alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle \end{cases}$$

Weighted entangled graphs: Edges are weighted by concurrence

M. Plesch & V. Bužek, PRA 67, 012322 (2003)

Case Study

Classification of 3-qubit entanglement

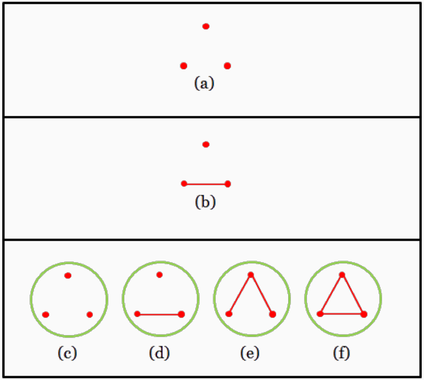
The generalized Schmidt decomposition for 3-qubit pure state is as follow

$$|\Psi\rangle_3 = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$$

$$\lambda_i \geq 0, \quad 0 \leq \phi \leq \pi, \quad \sum \lambda_i^2 = 1$$

$\mathcal{C}_{12}$	$\mathcal{C}_{13}$	$\mathcal{C}_{23}$
$(\lambda_0\ ,\ \lambda_3)$	$(\lambda_0\ ,\ \lambda_2)$	$(\lambda_1\ ,\ \lambda_4)$
		$(\lambda_2\ ,\ \lambda_3)$

$$\begin{cases} |A\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle \\ |B\rangle = (\lambda_0|00\rangle + \lambda_3|11\rangle) \otimes |0\rangle \\ |C\rangle = \lambda_0|000\rangle + \lambda_4|111\rangle \\ |D\rangle = \lambda_0|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \\ |E\rangle = \lambda_0|000\rangle + \lambda_1|100\rangle + \lambda_3|110\rangle \\ \qquad +\lambda_4|111\rangle \\ |F\rangle = \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle \end{cases}$$



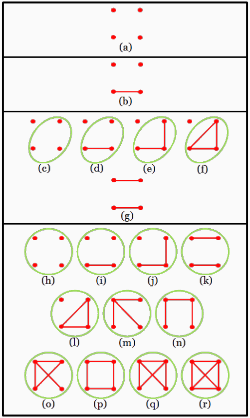
M. G G & S.J. Akhtarshenas, EPJD 70, 54 (2016)

Classification of 4-qubit entanglement

We have a classification of 4-qubit entanglement as follow:

$$|\Psi\rangle_4 = \alpha|0000\rangle + \beta|0100\rangle + \gamma|0101\rangle + \delta|0110\rangle + \epsilon|1000\rangle + \zeta|1001\rangle$$
$$+\eta|1010\rangle + \kappa|1011\rangle + \lambda|1100\rangle + \mu|1101\rangle + \nu|1110\rangle + \omega|1111\rangle$$

$\mathcal{C}_{12}$	$\mathcal{C}_{13}$	$\mathcal{C}_{14}$	$\mathcal{C}_{23}$	$\mathcal{C}_{24}$	$\mathcal{C}_{34}$
(α, λ)	(α, η)	(α, ζ)	(α, δ)	(α, γ)	(ϵ, κ)
(β, ϵ)	(β, ν)	(β, μ)	(ϵ, ν)	(ϵ, μ)	(λ, ω)
(γ, ζ)	(γ, ω)	(δ, ω)	(ζ, ω)	(η, ω)	(γ, δ)
(δ, η)	(δ, λ)	(γ, λ)	(η, λ)	(ζ, λ)	(ζ, η)
			(κ, μ)	(κ, ν)	(μ, ν)



How we can relate this classification to SLOCC?

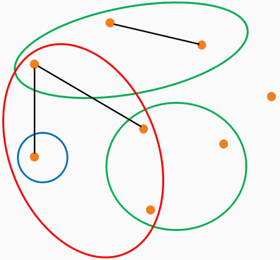
M. G G & S.J. Akhtarshenas, EPJD 70, 54 (2016)

Generalization & Future Works

Map: States $\longleftrightarrow$ Hypergraphs

Hypergraph: a hypergraph is a generalization of a graph in which an hyperedge can join any number of vertices. Mathematically $H = (V, E)$ where:

- V is a set of elements called vertices
- E is a subset of $\mathcal{P}(V)$ called hyperedges ($\mathcal{P}$ is the power set of V)

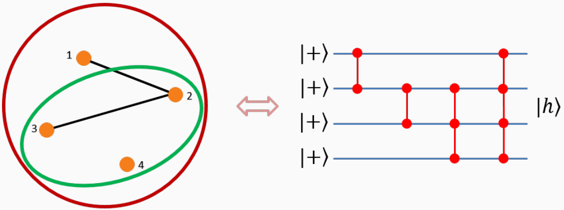


Connected hypergraph: existence of a path between every pair of vertices

Hypergraph States i

- Vertex $\longleftrightarrow$ Qubit
- Hyperedge $\longleftrightarrow$ Many-body interaction

$$|h\rangle = \prod_{k=1}^{n} \prod_{\{i_1, i_2, \cdots, i_k\} \in E} C^k Z_{i_1 i_2 \cdots i_k} |+\rangle^{\otimes n}$$



$$|h\rangle = C^4 Z_{1234}\ C^3 Z_{234}\ C^2 Z_{13}\ C^2 Z_{12}\ |+\rangle \otimes |+\rangle \otimes |+\rangle \otimes |+\rangle$$

$$|h\rangle = \frac{1}{4}\big(+ |0000\rangle + |0001\rangle + |0010\rangle + |0011\rangle + |0100\rangle + |0101\rangle + |0110\rangle - |0111\rangle$$
$$+ |1000\rangle + |1001\rangle - |1010\rangle - |1011\rangle - |1100\rangle - |1101\rangle + |1110\rangle + |1111\rangle\big)$$

M. Rossi et al., NJP 15, 113022 (2013)

Hypergraph States ii

Consider all 3-vertex hypergraphs splited into six LU-equivalent classes:

$H_0 = \{(V,E) | E \in \mathcal{P}(\{\{\Phi\},\{A\},\{B\},\{C\}\})\}$ $\quad H_1 = \{h + \{\{A,B\}\} | h \in H_0\}$

$H_2 = \{h + \{\{A,C\}\} | h \in H_0\}$ $\quad H_3 = \{h + \{\{B,C\}\} | h \in H_0\}$

$H_4 = \{h + E | E \subseteq \{\{A,B\},\{A,C\},\{B,C\}\} \wedge |E| \geqslant 2 \wedge h \in H_0\}$

$H_5 = \{(V,E) | V \in E\}$

	$\mathcal{C}_{AB}$	$\mathcal{C}_{AC}$	$\mathcal{C}_{BC}$	τ
H_0	0	0	0	0
H_1	1	0	0	0
H_2	0	1	0	0
H_3	0	0	1	0
H_4	0	0	0	1
H_5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$

$H_4 \cup H_5 \longrightarrow$ connected hypergraphs

$$\begin{cases} H_0 \longrightarrow A - B - C \\ H_1 \longrightarrow AB - C \\ H_2 \longrightarrow AC - B \\ H_3 \longrightarrow BC - A \\ H_4 \cup H_5 \longrightarrow \text{GHZ-type} \end{cases}$$

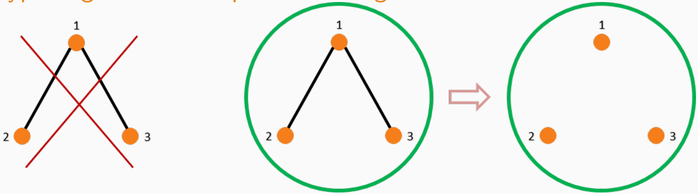
W-type states are missing!

R. Qu et al., PRA 87, 032329 (2013)

Entangled Hypergraphs

Entangled hypergraph is a generalization of entangled graph where:

- Vertex $\longleftrightarrow$ Qubit
- Hyperedge $\longleftrightarrow$ Multipartite entanglement



$$|\psi\rangle = \text{Cos}^2(\alpha)|000\rangle + \text{i Sin}(\alpha)\text{Cos}(\alpha)(|011\rangle + |101\rangle) - \text{Sin}^2(\alpha)|110\rangle$$

$$\begin{cases} \mathcal{C}_{12} = 2\,|\text{Sin}^3(\alpha)\text{Cos}(\alpha) - \text{Sin}(\alpha)\text{Cos}^3(\alpha)| \\ \mathcal{C}_{13} = 2\,|\text{Sin}^3(\alpha)\text{Cos}(\alpha) - \text{Sin}(\alpha)\text{Cos}^3(\alpha)| \\ \mathcal{C}_{12} = 0 \\ \tau = 16\ \text{Sin}^4(\alpha)\text{Cos}^4(\alpha) \end{cases} \quad \text{if } \alpha = \frac{\pi}{4} \Rightarrow \quad \begin{cases} \mathcal{C}_{12} = 0 \\ \mathcal{C}_{13} = 0 \\ \mathcal{C}_{12} = 0 \\ \tau = 1 \end{cases}$$

Soon in arXiv

Conclusion

Summary

- W state is missed in both graph & hypergraph states.
- Both entangled graphs & hypergraphs comprising W state.
- The corresponding pure states to connected entangled hypergraphs are completely entangled. (It is like hypergraph states)
- Entangled hypergraphs seem to be fruitful for classification of multipartite entanglement.

Thanks.