Notes on Large N Limits

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Abstract

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1 Introduction

In order to focus on the combinatorial properties of large N theories, instead of d-dimensional QFT we will consider some $d = 0$ examples, which are simply integrals. These examples also provide good practice for deriving the symmetry factors of various Feynman diagrams.

2 ϕ^4 model in $d = 0$

As a warm-up let us consider an $N = 1$ example, which is the $d = 0$ ϕ^4 theory. Here the partition function is simply an integral over one real variable:

$$Z(g) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{\phi^2}{2} - g\frac{\phi^4}{24}} \,. \tag{2.1}$$

This integral may be expanded in powers of g using the integral

$$I_n = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} (\phi^2/2)^n e^{-\alpha\phi^2/2} = (-1)^n \partial_\alpha^n \alpha^{-1/2} \,, \tag{2.2}$$

giving

$$Z(g) = 1 - \frac{g}{8} + \frac{35g^2}{384} - \frac{385g^3}{3072} + O(g^4) \,. \tag{2.3}$$

In fact, in this simple example the integral may be evaluated exactly:

$$Z(g) = \sqrt{\frac{3}{2\pi g}} e^{\frac{3}{4g}} K_{1/4}\left(\frac{3}{4g}\right) \,, \tag{2.4}$$

where K is the modified Bessel function.

The Feynman rules corresponding to (2.1) assign the factor 1 to the propagator and $-g$ to the quartic vertex. The "free energy" $\log Z$ should be given by expansion in connected vacuum amplitudes. The only diagram appearing at order g is the "figure eight" graph which has symmetry factor 1/8. This graph may also be thought of as a snail diagram with two legs connected.

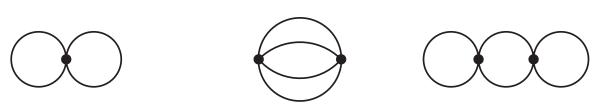

Figure 1: The three vacuum diagrams up to order g^2: "figure eight," "melon," and "triple bubble."

At order g^2 there are two contributing connected graphs: the melon and the triple bubble graph. In the melon the two vertices are connected by 4 lines, and we find the symmetry factor $\frac{1}{2\cdot 4!}$. The triple bubble graph may be thought of as two snail diagrams with their legs connected, and it has symmetry factor $\frac{1}{2^4}$. Adding up this connected vacuum graphs we therefore find

$$\log Z(g) = -\frac{g}{8} + \frac{g^2}{48} + \frac{g^2}{16} + O(g^3) = -\frac{g}{8} + \frac{g^2}{12} + O(g^3) \ . \tag{2.5}$$

This agrees with the expansion of the logarithm of the exact result (2.4:

$$\log Z(g) = -\frac{g}{8} + \frac{g^2}{12} - \frac{11g^3}{96} + \frac{17g^4}{72} - \frac{619g^5}{960} + \frac{709g^6}{324} - \frac{858437g^7}{96768} + O(g^8) \ . \tag{2.6}$$

We note that the coefficients alternate in sign but their magnitude grows rapidly. From the properties of Bessel functions we know that at high orders it grows as $n!$. Therefore, the small g expansion is only asymptotic.

3 Multi-Field Examples

Now let us extend this discussion to the case of multiple real variables ϕ^i where $i = 1, \ldots, n$. The partition function may in general be written as

$$Z = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{d\phi_i}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\phi^i\phi^i - \frac{1}{24}C_{ijkl}\phi^i\phi^j\phi^k\phi^l\right) \ , \tag{3.1}$$

where C_{ijkl} is a fully symmetric tensor. Various particular models may be obtained by imposing special symmetries on this tensor.

Using the connected vacuum Feynman graphs with propagator

$$\langle \phi^i \phi^j \rangle = \delta^{ij} \ , \tag{3.2}$$

we obtain the following general expansion:

$$\log Z = -\frac{C_{iijj}}{8} + \frac{C_{ijkl}C_{ijkl}}{48} + \frac{C_{iikl}C_{jjkl}}{16} + O(C^3) \ . \tag{3.3}$$

The first term comes from the figure eight diagram; the second from the melon; and the third from the triple bubble.

3.1 Vector model with $O(N)$ symmetry

Let us impose the $O(N)$ symmetry on the model and require ϕ^i to be in the fundamental representation. We will then take

$$C_{ijkl} = \frac{g}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \ , \tag{3.4}$$

which turns the model into

$$Z^{\text{vector}}(g) = \prod_{i=1}^{N} \int_{-\infty}^{\infty} \frac{d\phi_i}{\sqrt{2\pi}} e^{-\phi^i\phi^i/2 - g(\phi^i\phi^i)^2/24} \tag{3.5}$$

From (3.3) we obtain the expansion

$$\frac{\log Z^{\text{vector}}(g)}{N} = -\frac{N+2}{24}g + g^2\frac{N+2}{144} + g^2\frac{(N+2)^2}{144} + O(g^3) \ , \tag{3.6}$$

where the second term comes from the melon graph, and the third from the bubble graph. In order to keep $\frac{\log Z^{\text{vector}}(g)}{N}$ finite, we keep $\lambda = gN$ fixed in the large N limit. Then the melon graph is suppressed by $1/N$ while the bubble graphs survive. In fact, by drawing the index structure of the graphs it is not hard to see that the only surviving ones involve chains of bubbles.

In the large N limit the free energy has the structure

$$\frac{\log Z^{\text{vector}}(g)}{N} = f_0(\lambda) + N^{-1}f_1(\lambda) + \dots \ , \tag{3.7}$$

where $f_0(\lambda) = -\lambda/24 + \lambda^2/144 + O(\lambda^3)$.

The function $f_0(\lambda)$ may be determined using the standard method of introducing an auxiliary variable σ so that

$$Z^{\text{vector}}(g) = \int_{-\infty}^{\infty} \prod_{j=1}^{N} \frac{d\phi_j}{\sqrt{2\pi}} \int d\sigma \sqrt{\frac{6}{\pi g}} \exp\left(-\frac{6N\sigma^2}{\lambda} - \frac{\phi^k\phi^k(1+2i\sigma)}{2}\right) \ . \tag{3.8}$$

After performing the Gaussian integral over ϕ^j we find

$$Z^{\text{vector}}(g) = \sqrt{\frac{6}{\pi g}} \int d\sigma \exp\left(-\frac{6N\sigma^2}{\lambda} - \frac{N}{2}\log(1+2i\sigma)\right) \ . \tag{3.9}$$

For large N the integral is dominated by the saddle point located at $\sigma = -i\tilde{\sigma}$ where

$$\frac{12\tilde{\sigma}}{\lambda} = \frac{1}{1+2\tilde{\sigma}} \,. \tag{3.10}$$

The solution of this quadratic equation which matches onto the perturbation theory is

$$\tilde{\sigma}(\lambda) = \frac{\sqrt{1+\frac{2\lambda}{3}}-1}{4} \tag{3.11}$$

and we find

$$f_0^{\text{vector}}(\lambda) = \frac{6\tilde{\sigma}^2}{\lambda} - \frac{1}{2}\log(1+2\tilde{\sigma}) = -\frac{\lambda}{24} + \frac{\lambda^2}{144} - \frac{5\lambda^3}{2592} + \frac{7\lambda^4}{10368} - \frac{7\lambda^5}{25920} + O(\lambda^6) \ , \tag{3.12}$$

which agrees with our Feynman graph calculations. To all orders in λ,

$$f_0^{\text{vector}}(\lambda) = \sum_{k=1}^{\infty} (-\lambda)^k \frac{1}{4k(k+1)6^k} \binom{2k}{k} \tag{3.13}$$

In this series the coefficients decrease, so it is convergent for sufficiently small $|\lambda|$. This is one of the advantages of the large N limit – the functions that appear order by order in $1/N$ have perturbation series with a finite radius of convergence.

Let us note a remarkable fact: $f_0^{\text{vector}}(\lambda)$ makes sense even for negative λ, so long as it is greater than $\lambda_c = -3/2$ (for $\lambda < \lambda_c$ it is ambiguous due a branch cut). Thus, large N expansions may be formally defined even for potentials that are not bounded from below. The expansion for $\lambda > \lambda_c$ is

$$f_0^{\text{vector}}(\lambda) = \frac{2\log 2 - 1}{4} - \frac{1}{6}(\lambda-\lambda_c) + \frac{2}{9}\sqrt{\frac{2}{3}}(\lambda-\lambda_c)^{3/2} + O\big((\lambda-\lambda_c)^2\big) \ . \tag{3.14}$$

It is common to parametrize the leading singular term as $(\lambda - \lambda_c)^{2-\gamma}$, and here we find $\gamma = 1/2$. This is a critical exponent characteristic of the branched polymers.

3.2 Real matrix model with $O(N)^2$ symmetry

Now let us consider $n = N^2$ real degrees of freedom ϕ^{ab}, $a, b = 1, \ldots, N$, and impose $O(N) \times O(N)$ symmetry, so that ϕ^{ab} are in the bi-fundamental representation. The 2 indices of the matrix are distinguishable, and each one is acted on by a different $O(N)$ group:

$$\phi^{ab} = M_1^{aa'} M_2^{bb'} \phi^{a'b'}, \tag{3.15}$$

$$M_1 \in O(N)_1, \quad M_2 \in O(N)_2 \,. \tag{3.16}$$

We will study the matrix integral

$$Z^{\text{matrix}}(g) = \prod_{a,b} \int_{-\infty}^{\infty} \frac{d\phi^{ab}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\phi^{ab}\phi^{ab} - \frac{g}{24}\phi^{a_1b_1}\phi^{a_1b_2}\phi^{a_2b_1}\phi^{a_2b_2}\right) . \tag{3.17}$$

Using the matrix notation, we may write

$$\phi^{ab}\phi^{ab} = \text{tr}(\phi\phi^T) \ , \qquad \phi^{a_1b_1}\phi^{a_1b_2}\phi^{a_2b_1}\phi^{a_2b_2} = \text{tr}(\phi\phi^T\phi\phi^T) \ . \tag{3.18}$$

This demonstrates the invariance of the model under $\phi \to M_1 \phi M_2^T$. We also have

$$\text{tr}(\phi\phi^T\phi\phi^T) = \phi^{a_1b_1}\phi^{a_2b_2}\phi^{a_3b_3}\phi^{a_4b_4}\delta^{a_1a_2}\delta^{a_3a_4}\delta^{b_1b_3}\delta^{b_2b_4} \ . \tag{3.19}$$

Appropriately symmetrizing this product of Kronecker symbols we obtain the version of tensor C^{ijkl} appropriate for this model.

Using the connected vacuum Feynman graphs with propagator

$$\langle \phi^{a_1b_1}\phi^{a_2b_2} \rangle = \delta^{a_1a_2}\delta^{b_1b_2} \ , \tag{3.20}$$

we obtain the expansion

$$\frac{\log Z^{\text{matrix}}(g)}{N^2} = -\frac{1}{24}(2N+1)g + \frac{1}{288}\left(N^2+2N+3\right)g^2 + \frac{1}{144}(2N+1)^2 g^2 + O(g^3) \ , \tag{3.21}$$

where the second term comes from the melon graph, and the third from the bubble graph. In order to keep $\frac{\log Z^{\text{matrix}}(g)}{N^2}$ finite, we keep $\lambda = gN$ fixed in the large N limit. Now we see that both the melon and bubble graphs contribute at leading order.

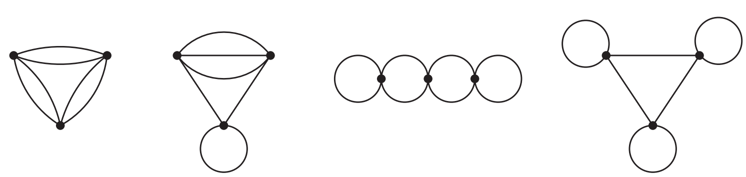

Figure 2: g^3 order vacuum diagrams.

Continuing to the diagrams of order g^3, we find

$$\left.\frac{\log Z^{\text{matrix}}(g)}{N^2}\right|_{g^3\text{order}} = -\frac{1}{48}\cdot\frac{1}{27}\left(N^3+4N^2+13N+9\right)g^3 - \frac{1}{24}\cdot\frac{1}{18}(2N+1)\left(N^2+2N+3\right)g^3 - \frac{1}{32}\cdot\frac{1}{27}(2N+1)^3g^3 - \frac{1}{48}\cdot\frac{1}{27}(2N+1)^3g^3\,, \tag{3.22}$$

where the four terms correspond to the four diagrams shown in Figure 2, respectively. All of them contribute in the large N limit. In fact, 't Hooft proved [1] that all the planar graphs, i.e. the graphs of spherical topology, are dominant. To demonstrate this, it is convenient to rescale the matrix $\phi^{ab} \to \sqrt{N}\phi^{ab}$. Then Then

$$Z^{\text{matrix}} \sim \prod_{a,b}\int_{-\infty}^{\infty}\frac{d\phi^{ab}}{\sqrt{2\pi}}\exp\left(-\frac{N}{2}\phi^{ab}\phi^{ab} - \frac{N\lambda}{24}\phi^{a_1b_1}\phi^{a_1b_2}\phi^{a_2b_1}\phi^{a_2b_2}\right)\,. \tag{3.23}$$

Now each propagator carries a factor $1/N$, and each vertex factor $N\lambda$. Also, each face of the graph contains an index loop and contributes a factor of N. So, the net power of N for a Feynman graph with V vertices, F faces and E edges is $N^{V+F-E} = N^\chi$, where χ is the Euler characteristic. Since $\chi = 2-2g$, where g is the genus of the graph, we see that the graphs contributing at order N^2 are the graphs of genus 0, i.e. of spherical topology.

In the large N limit the free energy has the structure

$$\frac{\log Z^{\text{matrix}}(g)}{N^2} = f_0^{\text{matrix}}(\lambda) + N^{-1}f_{1/2}^{\text{matrix}}(\lambda) + N^{-2}f_1^{\text{matrix}}(\lambda) + \ldots \tag{3.24}$$

where $f_g^{\text{matrix}}(\lambda)$ is the sum over graphs of genus g. We see that the leading correction is due to the non-orientable surface of genus $1/2$, which is RP_2. For the hermitian matrix model such non-orientable surfaces of fractional genus do not appear. Our perturbative calculation gives $f_0^{\text{matrix}}(\lambda) = -\lambda/12 + \lambda^2/32 - \lambda^3/48 + O(\lambda^4)$.

For the one-matrix model it is possible to obtain an exact expression for $f_0(\lambda)$, which is analogous to the one we obtained for the vector model. To do this we represent the real symmetric matrix as $\phi = L\kappa R^T$ where κ is a diagonal matrix of real eigenvalues κ_a, and R and L are two independent $O(N)$ matrices. Integrating them out, we find

$$Z^{\text{matrix}}(g) \sim \prod_a\int_{-\infty}^{\infty}d\kappa_a\Delta^2(\kappa)e^{-\sum_{b=1}^{N}\left(\frac{1}{2}\kappa_b^2+\frac{g}{24}\kappa_b^4\right)}\,, \tag{3.25}$$

where $\Delta(\kappa) = \prod_{a<b}(\kappa_a-\kappa_b)$ is the Vandermonde determinant.

Now, following the classic paper by [2] we introduce the density of eigenvalues $\rho(\kappa)$. In the large N limit it satisfies a singular integral equation which may be solved explicitly:

$$\rho(\kappa) = \frac{1}{\pi}\left(\frac{1}{2} + \frac{\lambda}{6}a^2 + \frac{\lambda}{12}\kappa^2\right)\sqrt{4a^2 - \kappa^2}\ , \tag{3.26}$$

$$a^2(\lambda) = \frac{\sqrt{1+2\lambda} - 1}{\lambda} \tag{3.27}$$

We see that $\rho(\kappa)$ is a symmetric function with support between $-2a(\lambda)$ and $2a(\lambda)$. As $\lambda \to 0$, $a \to 1$, and the eigenvalue density approaches the classic Wigner semicircle law found for the Gaussian matrix models.

The non-Gaussian effects deform the density to the more general function (3.27). It is then found that

$$f_0^{\text{matrix}}(\lambda) = \frac{1}{24}(a^2(\lambda)-1)(9-a^2(\lambda)) - \frac{1}{2}\log a^2(\lambda) = -\frac{\lambda}{12} + \frac{\lambda^2}{32} - \frac{\lambda^3}{48} + \frac{7\lambda^4}{384} + O(\lambda^5)\ , \tag{3.28}$$

Similarly to the free energy in the vector case, $f_0^{\text{matrix}}(\lambda)$ is well defined for $\lambda > \lambda_c$ where $\lambda_c = -1/2$. Expanding $f_0^{\text{matrix}}(\lambda)$ near λ_c we find that the leading singular term is now $\sim (\lambda - \lambda_c)^{5/2}$ corresponding to $\gamma = -1/2$. This is the well-known critical exponent for pure two-dimensional quantum gravity.

3.3 Hermitian matrix model with $SU(N)$ symmetry

Now let us consider a somewhat different matrix integral, which involves a Hermitian matrix Φ^i_j, $i, j = 1, \ldots, N$. Under the $SU(N)$ symmetry it transforms as

$$\Phi = U\Phi' U^\dagger\ , \tag{3.29}$$

where $U \in SU(N)$. An integeresting integral to consider is

$$Z^{\text{Hermitian}}(g) = \prod_{i,j}\int_{-\infty}^{\infty} \frac{d\text{Re}\Phi^i_j}{\sqrt{2\pi}}\frac{d\text{Im}\Phi^i_j}{\sqrt{2\pi}} \exp\text{tr}\left(-\frac{1}{2}\Phi^2 - \frac{g}{6}\Phi^3\right)\ . \tag{3.30}$$

This may be viewed as a toy model for interactions of gluons, and the 't Hooft coupling is $\lambda = g\sqrt{N}$.

The propagator

$$\langle \Phi^{i_1}_{j_1}\Phi^{i_2}_{j_2}\rangle = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\ , \tag{3.31}$$

may be represented using double lines with opposite directions. If we impose the condition that Φ is traceless, so that it is in the adjoint representation of $SU(N)$, then the propagator becomes

$$\langle \Phi^{i_1}_{j_1} \Phi^{i_2}_{j_2} \rangle = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1} - \frac{1}{N}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} \ . \tag{3.32}$$

The graphs dual to the Feynman graphs are now made of triangles, so that this integral may be interpreted in terms of orientable triangulated surfaces. Representing $\Phi = V\kappa V^\dagger$, where κ is a diagonal matrix of real eigenvalues κ_a, which add up to zero if Φ is traceless, and V is an $SU(N)$ matrix. We may integrate V out to obtain

$$Z^{\text{Hermitian}}(g) \sim \prod_a \int_{-\infty}^{\infty} d\kappa_a \Delta^2(\kappa) e^{-\sum_{b=1}^{N}\left(\frac{1}{2}\kappa_b^2 + \frac{g}{6}\kappa_b^3\right)} \ , \tag{3.33}$$

Now, the critical behavior is $f_0(\lambda) \sim (\lambda_c^2 - \lambda^2)^{5/2}$.

3.4 $O(N)^3$ tensor model

Now let us consider N^3 real degrees of freedom ϕ^{abc}, $a, b, c = 1, \ldots, N$, and impose $O(N)^3$ symmetry, so that ϕ^{abc} are in the tri-fundamental representation. The 3 indices of a tensor are distinguishable, and each one is acted on by a different $O(N)$ group:

$$\begin{aligned} &\phi^{abc} = M_1^{aa'} M_2^{bb'} M_3^{cc'} \phi^{a'b'c'}, \\ &M_1 \in O(N)_1, \quad M_2 \in O(N)_2, \quad M_3 \in O(N)_3 \,. \end{aligned} \tag{3.34}$$

We will study the integral

$$Z^{\text{tensor}}(g) = \prod_{a,b,c} \int_{-\infty}^{\infty} \frac{d\phi^{abc}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\phi^{abc}\phi^{abc} - \frac{g}{24}\phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_2b_2c_1} \right) \ . \tag{3.35}$$

For such a theory one can draw the stranded graphs using the triple-line notation (we may draw each propagator as containing strands of three different colors), and it is possible to prove the melon dominance [3, 4].

First, let us study the low orders in perturbation theory, as we did for the vector and matrix models. Using the connected vacuum Feynman graphs with propagator

$$\langle \phi^{a_1b_1c_1}\phi^{a_2b_2c_2} \rangle = \delta^{a_1a_2}\delta^{b_1b_2}\delta^{c_1c_2} \ , \tag{3.36}$$

we obtain the expansion

$$\frac{\log Z^{\text{tensor}}(g)}{N^3} = -\frac{1}{8}Ng + \frac{1}{288}\left(N^3+3N+2\right)g^2 + \frac{1}{16}N^2g^2 + O(g^3)\ , \tag{3.37}$$

where the second term comes from the melon graph, and the third from the bubble graph. In order to keep $\frac{\log Z^{\text{tensor}}(g)}{N^3}$ finite, we keep $\lambda = gN^{3/2}$ fixed in the large N limit. Now we see that the melon contributes while the figure eight and triple bubble graphs are suppressed. So, finally the melons are winning!

In the large N limit the free energy has the structure

$$\frac{\log Z^{\text{tensor}}(g)}{N^3} = f_0^{\text{tensor}}(\lambda) + O(N^{-1/2})\,, \tag{3.38}$$

where $f_0^{\text{tensor}}(\lambda)$ contains the contributions of melonic vacuum diagrams only (see Figure 3).

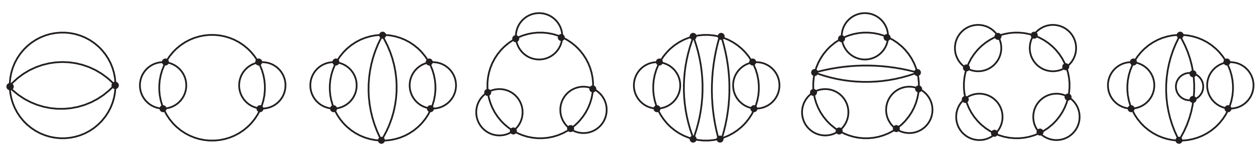

Figure 3: All the melonic vacuum diagrams up to order g^8.

To solve for $f_0^{\text{tensor}}(\lambda)$ one uses the Schwinger-Dyson equation for the full two-point function $G(\lambda)$ implied by the diagram for self-energy [5]

$$G^{-1}(\lambda) = 1+\Sigma\ , \qquad \Sigma = -\frac{\lambda^2}{36}G_{\text{melons}}(\lambda)^3\,. \tag{3.39}$$

This may be written as (see Figure 4)

$$G(\lambda) = 1 + \frac{\lambda^2}{36}G(\lambda)^4\,. \tag{3.40}$$

The solution is

$$G = \frac{\sqrt{3}}{(2\lambda)^{1/2}v^{1/4}}\left((1+4v)^{1/4} - (2-(1+4v)^{1/2})^{1/2}\right)\ ,$$
$$v(\lambda) = \frac{(\lambda/3)^{2/3}}{2}\left[\left(1+\sqrt{1-\frac{2^6\lambda^2}{3^5}}\right)^{1/3} + \left(1-\sqrt{1-\frac{2^6\lambda^2}{3^5}}\right)^{1/3}\right]. \tag{3.41}$$

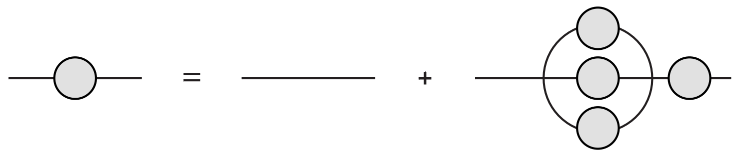

Figure 4: Schwinger-Dyson equation for the two-point function.

Now we use the constraint equation [5]

$$\frac{1}{Z^{\rm tensor}(g)}\prod_{a,b,c}\int_{-\infty}^{\infty}\frac{d\phi^{abc}}{\sqrt{2\pi}}\frac{\partial}{\partial\phi^{a'b'c'}}\left(\phi^{a'b'c'}\exp\left(-\frac{1}{2}\phi^{abc}\phi^{abc}-\frac{g}{24}\phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_2b_2c_1}\right)\right)=0\,. \tag{3.42}$$

Applying the derivative gives

$$N^3 - N^3 G + \frac{4g}{Z^{\rm tensor}(g)}\frac{\partial}{\partial g}Z^{\rm tensor}(g) = 0\ . \tag{3.43}$$

Dividing by N^3 gives the relation

$$G(\lambda) = 1 + 4\lambda\partial_\lambda f_0^{\rm tensor}(\lambda)\ , \tag{3.44}$$

which may be used to obtain $f_0^{\rm tensor}(\lambda)$. The explicit series is [5, 6]

$$f_0^{\rm tensor}(\lambda) = \sum_{n=1}^{\infty} a_{2n}\left(\frac{\lambda}{6}\right)^{2n}\ , \tag{3.45}$$

where

$$a_2 = \frac{1}{8}\ ,\quad a_4 = \frac{1}{4}\ ,\quad a_6 = \frac{11}{12}\ ,\quad a_8 = \frac{35}{8},\quad \ldots,\quad a_{2n} = \frac{1}{8n(4n+1)}\binom{4n+1}{n}\,. \tag{3.46}$$

From the exact large N solution we find that the leading singular behavior of $f_0^{\rm tensor}(\lambda)$ as λ approaches a critical value is $(\lambda_c^2-\lambda^2)^{3/2}$, where $\lambda_c^2 = 3^5/2^6$, and the susceptibility exponent is $\gamma = 1/2$, just as in the vector model.

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