Lectures on Large N Models

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Large N Limits

- An important theoretical tool: some models simplify in the limit of a large number of degrees of freedom.
- One class of such large N limits is for theories where fields transform as vectors under O(N) symmetry with actions like

$$S_{\mathrm{Wilson-Fisher}} = \int d^d x \left(\frac{1}{2} (\partial_\mu \phi^i)^2 + \frac{g}{4} (\phi^i \phi^i)^2 \right)$$

- Describes magnets with O(N) symmetry, which have second-order phase transitions in d<4.

- The O(N) vector model is solvable in the limit where N is sent to infinity while keeping gN fixed.
- Flow from the free d<4 scalar model in the UV to the Wilson-Fisher interacting one in the IR.
- For N=1 it describes the critical Ising model; for N=2 the superfluid transition; for N=3 the critical Heisenberg model.
- The 1/N expansion is generated using the Hubbard-Stratonovich auxiliary field.

$$S = \int d^d x \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{4\lambda} \right)$$

- In d<4 the quadratic term may be ignored in the IR:

$$Z = \int D\phi D\sigma \, e^{-\int d^dx \left(\frac{1}{2}(\partial\phi^i)^2 + \frac{1}{2\sqrt{N}}\sigma\phi^i\phi^i\right)}$$

$$= \int D\sigma \, e^{\frac{1}{8N}\int d^dx d^dy \, \sigma(x)\sigma(y) \, \langle\phi^i\phi^i(x)\phi^j\phi^j(y)\rangle_0 + \mathcal{O}(\sigma^3)}$$

- Induced dynamics for the auxiliary field endows it with the propagator

$$\langle\sigma(p)\sigma(-p)\rangle = 2^{d+1}(4\pi)^{\frac{d-3}{2}}\Gamma\left(\frac{d-1}{2}\right)\sin(\frac{\pi d}{2})\,(p^2)^{2-\frac{d}{2}} \equiv \tilde{C}_\sigma(p^2)^{2-\frac{d}{2}}$$

$$\langle\sigma(x)\sigma(y)\rangle = \frac{2^{d+2}\Gamma\left(\frac{d-1}{2}\right)\sin(\frac{\pi d}{2})}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}-2\right)}\frac{1}{|x-y|^4} \equiv \frac{C_\sigma}{|x-y|^4}$$

- The 1/N corrections to operator dimensions are calculated using this induced propagator. For example,

$$\Delta_\phi = \frac{d}{2} - 1 + \frac{1}{N}\eta_1 + \frac{1}{N^2}\eta_2 + \dots$$

- For the leading correction need

$$\frac{1}{N}\int \frac{d^d q}{(2\pi)^d}\frac{1}{(p-q)^2}\frac{\tilde{C}_\sigma}{(q^2)^{\frac{d}{2}-2+\delta}}$$

- δ is the regulator later sent to 0.

$$\eta_1 = \frac{\tilde{C}_\sigma(d-4)}{(4\pi)^{\frac{d}{2}} d\Gamma(\frac{d}{2})} = \frac{2^{d-3}(d-4)\Gamma\left(\frac{d-1}{2}\right)\sin\left(\frac{\pi d}{2}\right)}{\pi^{\frac{3}{2}}\Gamma\left(\frac{d}{2}+1\right)}$$

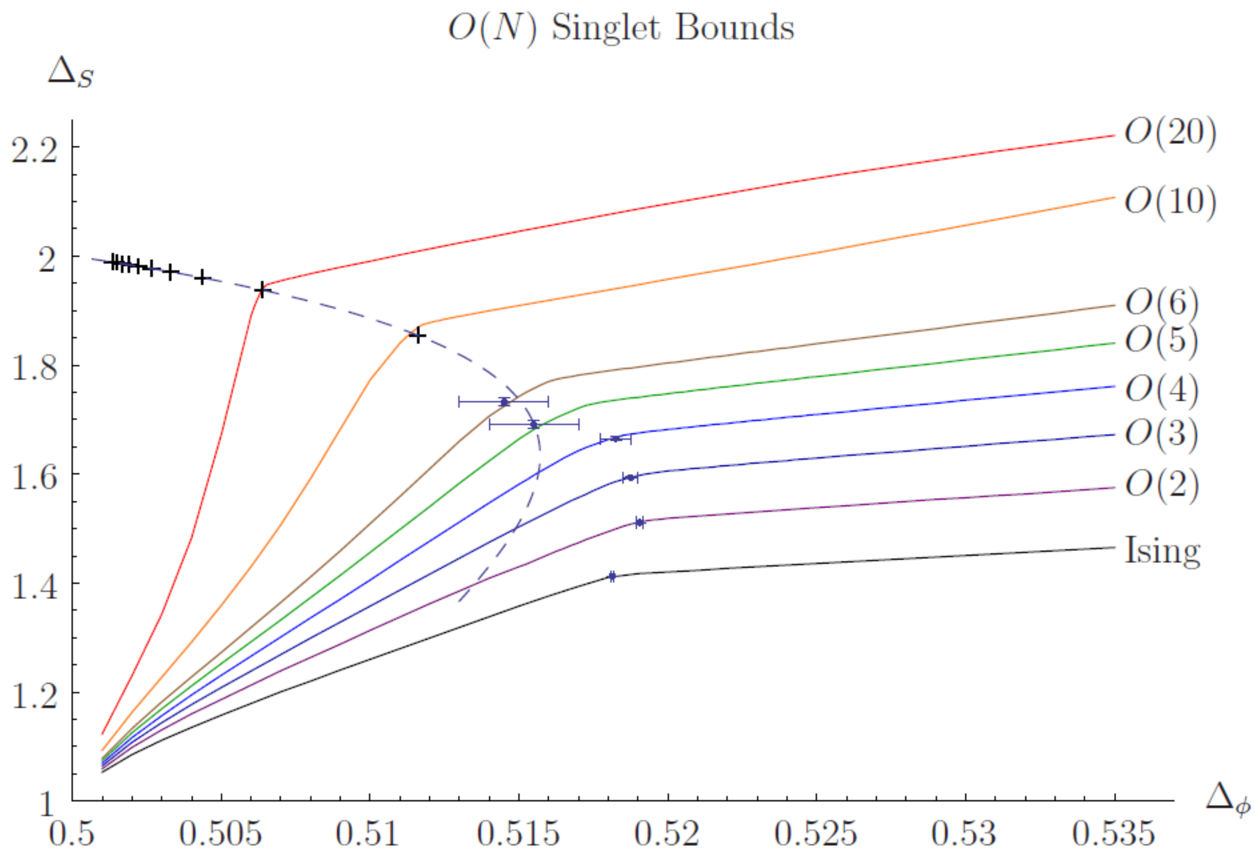
Operator Dimensions in d=3

- S is the O(N) singlet quadratic operator.
- T is the symmetric traceless tensor:

$$
\begin{aligned}
\Delta_\phi &= \frac{1}{2} + \frac{4}{3\pi^2}\frac{1}{N} - \frac{256}{27\pi^4}\frac{1}{N^2} \\
&\quad + \frac{32\left(-3188 + 3\pi^2(-61 + 108\log(2) - 3402\zeta(3))\right)}{243\pi^6}\frac{1}{N^3} + \mathcal{O}\left(\frac{1}{N^4}\right) \\
\Delta_S &= 2 - \frac{32}{3\pi^2}\frac{1}{N} + \frac{32(16 - 27\pi^2)}{27\pi^4}\frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) \\
\Delta_T &= 1 + \frac{32}{3\pi^2}\frac{1}{N} - \frac{512}{27\pi^4}\frac{1}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right).
\end{aligned}
$$

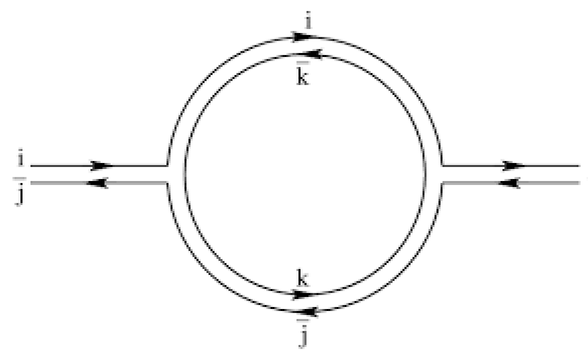
Conformal Bootstrap Results

- From Kos, Poland, Simmons-Duffin, arxiv: 1307.6856

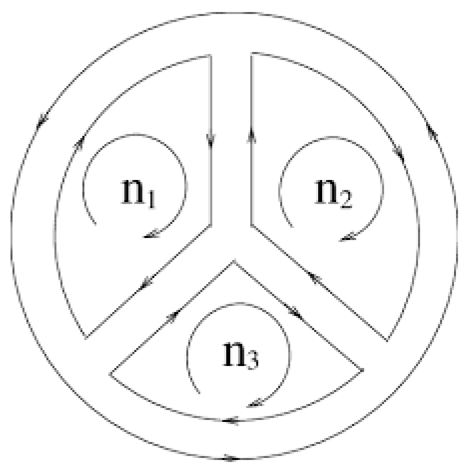


‘t Hooft Limit and Planar Graphs

- Another famous large N limit is for “planar” theories of N x N matrices with single-trace interactions.
- This has been explored widely in the context of large N QCD: SU(N) gauge theory coupled to matter.
- $g_{YM} N^{1/2}$ must be held fixed.
- The ‘t Hooft double line notation is very helpful:

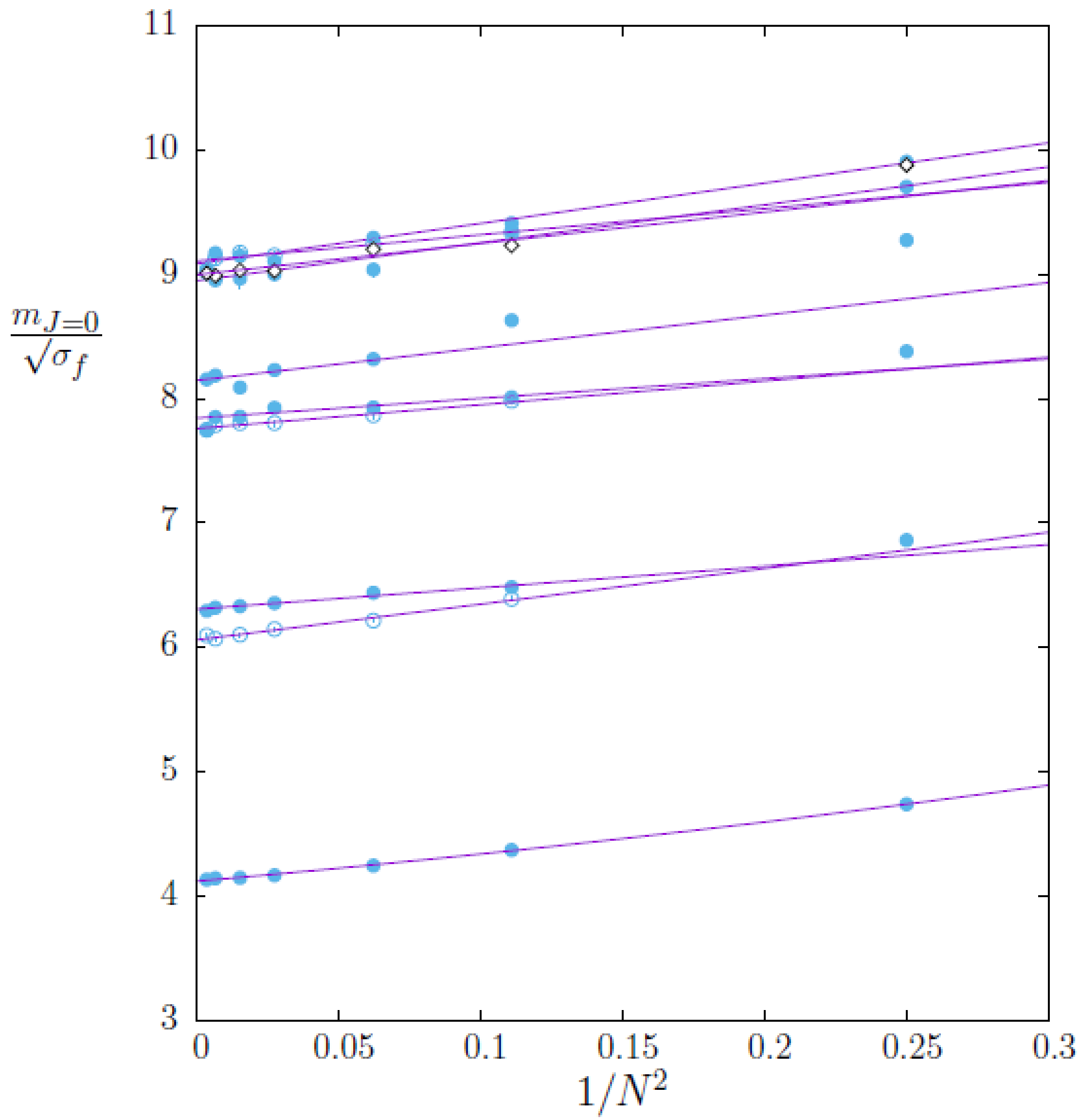


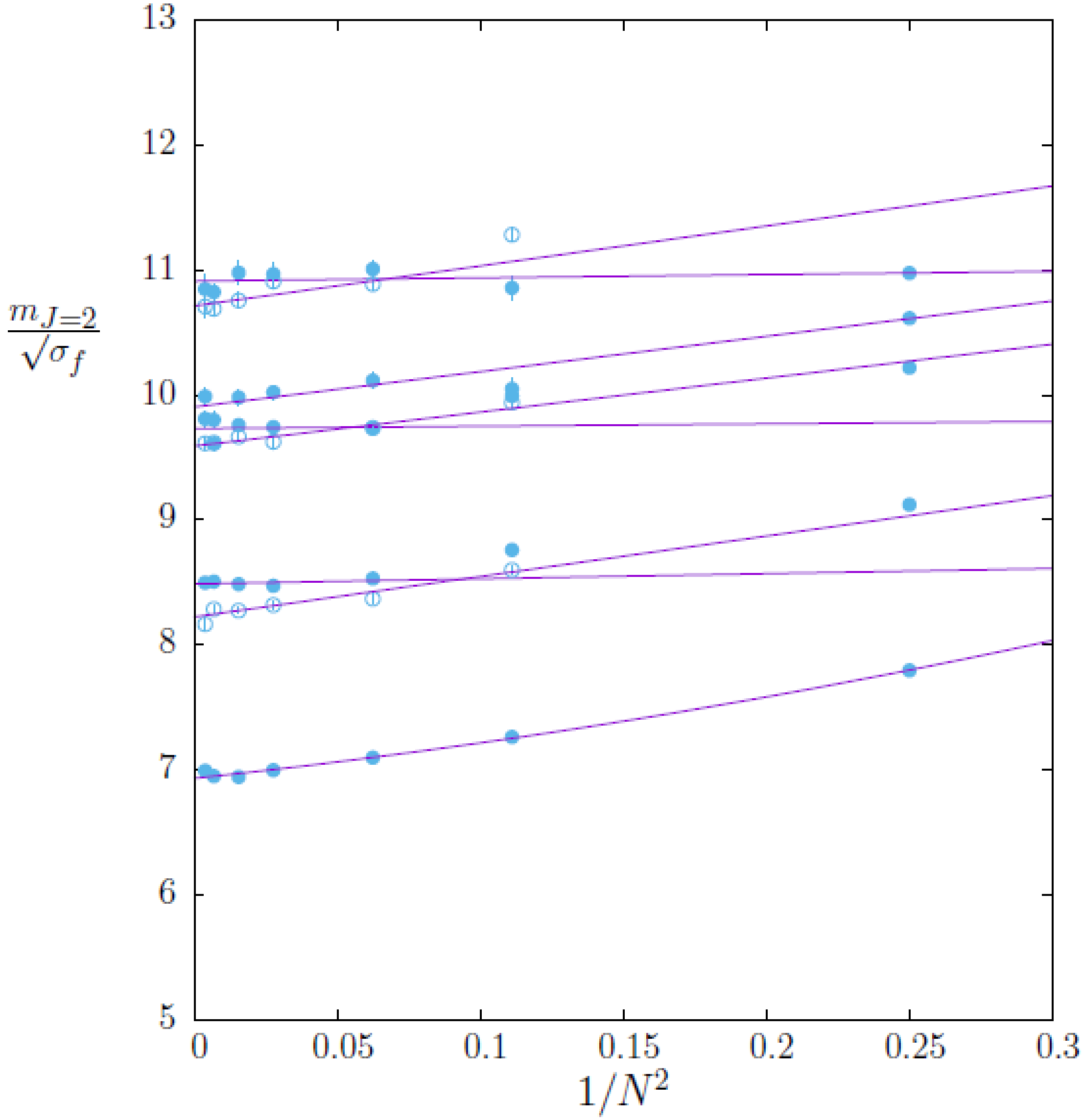
- Each vertex contributes factor ~N, each edge (propagator) ~1/N, each face (index loop)~N.
- The contribution to free energy of the Feynman graphs which can be drawn on a two-dimensional surfaces of genus g scales as $N^{2(1-g)}$



Glueballs in 3d SU(N) Theory

- For SU(N) the corrections are in powers of $1/N^2$
- Direct lattice evidence from Athenodorou,Teper, arXiv: 1609.03873

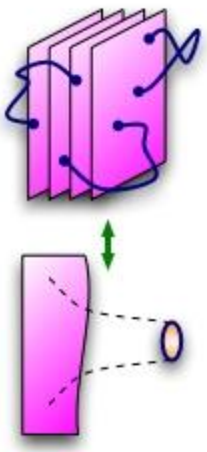




13
12
11
10
9
8
7
6
5
$\frac{m_{J=2}}{\sqrt{\sigma_f}}$
0
0.05
0.1
0.15
0.2
0.25
0.3
$1/N^2$

20 years of AdS/CFT Correspondence

- Starting in 1995 -- D-brane/black hole and D-brane/black brane correspondence. Polchinski; Strominger, Vafa; Callan, Maldacena; …
- A stack of N Dirichlet 3-branes realizes $\mathcal{N}$=4 supersymmetric SU(N) gauge theory in 4 dimensions. It also creates a curved RR charged background of type IIB theory of closed superstrings



$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-(dx^0)^2 + (dx^i)^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right)$$

Large N is Important

- Matching the brane tensions gives $L^4 = g_{\rm YM}^2 N \alpha'^2$

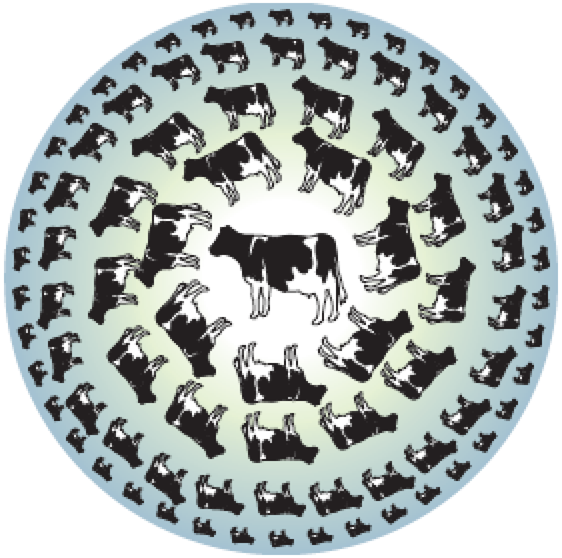
  Gubser, IK, Peet; IK; …
- The ‘t Hooft coupling makes a crucial appearance. In the large N limit, the effects of quantum gravity are suppressed by powers of $1/N^2$
- A serendipitous simplification for $g_{\rm YM}^2 N \gg 1$:

  the background has a small curvature.
- This permitted calculation of two-point functions in strongly coupled gauge theory using classical gravitational absorption. IK
- In the r->0 limit, which corresponds to low energies, approaches $AdS_5 \times S^5$. Maldacena

The AdS/CFT Duality

Maldacena; Gubser, IK, Polyakov; Witten

- The low-energy limit taken directly in the geometry. Maldacena
- Relates conformal gauge theory in 4 dimensions to string theory on 5-d Anti-de Sitter space times a 5-d compact space. For the $\mathcal{N}$=4 SYM theory this compact space is a 5-d sphere.



- The geometrical symmetry of the AdS_5 space realizes the conformal symmetry of the gauge theory.
- Allows us to "solve" strongly coupled gauge theories, e.g. find operator dimensions $\Delta_{\pm} = 2 \pm \sqrt{4 + m^2 L^2}$

Some Tests of AdS/CFT

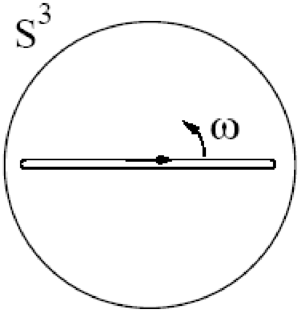
- String theory can make definite, testable predictions!
- The dimensions of unprotected operators, which are dual to massive string states, grow at strong coupling as $2\left(n g_{\mathrm{YM}} \sqrt{N}\right)^{1/2}$
- Verified for the Konishi operator dual to the lightest massive string state (n=1) using the exact integrability of the planar $\mathcal{N}$=4 SYM theory. Gromov, Kazakov, Vieira; …
- Similar successes for the dimensions of high-spin operators, which are dual to spinning strings in AdS space.

Higher-Spin Operators and Spinning Strings

- The dual of a high-spin operator of S>>1

$$\mathrm{Tr}\, F_{+\mu} D_+^{S-2} F_+{}^{\mu}$$

is a folded string spinning around the center of AdS$_5$. Gubser, IK, Polyakov



- The structure of dimensions of high-spin operators is

$$\Delta - S = f(g) \ln S \; + O(S^0), \qquad g = \frac{\sqrt{g_{YM}^2 N}}{4\pi}$$

- Weak coupling expansion of the function f(g)

Kotikov, Lipatov, Onishchenko, Velizhanin; Bern, Dixon, Smirnov; …

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 + O(g^8)$$

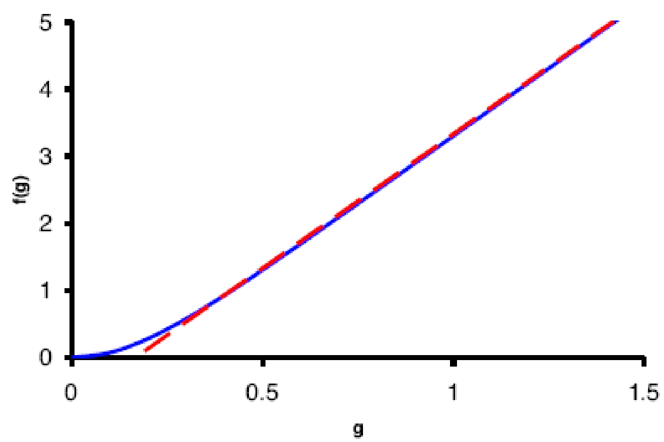
- At strong coupling, the AdS/CFT correspondence predicts via the spinning string energy calculation
- Gubser, IK, Polyakov; Frolov, Tseytlin

$$f(g) = 4g - \frac{3\ln 2}{\pi} + \ldots$$

- Methods of exact integrability allow to match them smoothly.

Beisert, Eden, Staudacher;

Benna, Benvenuti, IK, Scardicchio



Matrix Quantum Mechanics

- A well-known solvable model is the QM of a hermitian NxN matrix with SU(N) symmetry

$$\Phi(t) \rightarrow V^\dagger \Phi(t) V$$

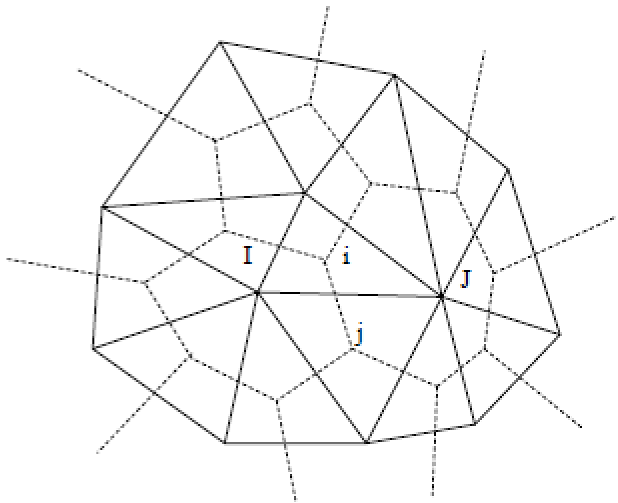
- The partition function is

$$Z \sim \int D^{N^2}\Phi(x) \exp\left[-N \int_{-T/2}^{T/2} dx \ \mathrm{Tr}\left(\frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + \frac{1}{2\alpha'}\Phi^2 - \frac{\kappa}{3!}\Phi^3\right)\right]$$

- Originally solved by Brezin, Itzykson,Parisi, Zuber. Eigenvalues become free fermions!
- Reviewed in my 1991 Trieste Spring School lectures, hep-th/9108019, the 19th paper to appear in hep-th.

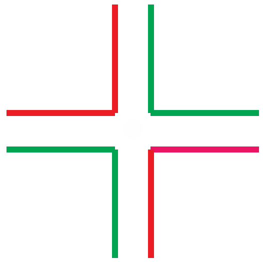
Discretized Random Surfaces

- The dual graphs are made of triangles. The limit where Feynman graphs become large describes two-dimensional quantum gravity coupled to a massless scalar field.
- The conformal factor of 2-d metric, the quantum Liouville field, acts as an extra dimension of non-critical string theory. Polyakov

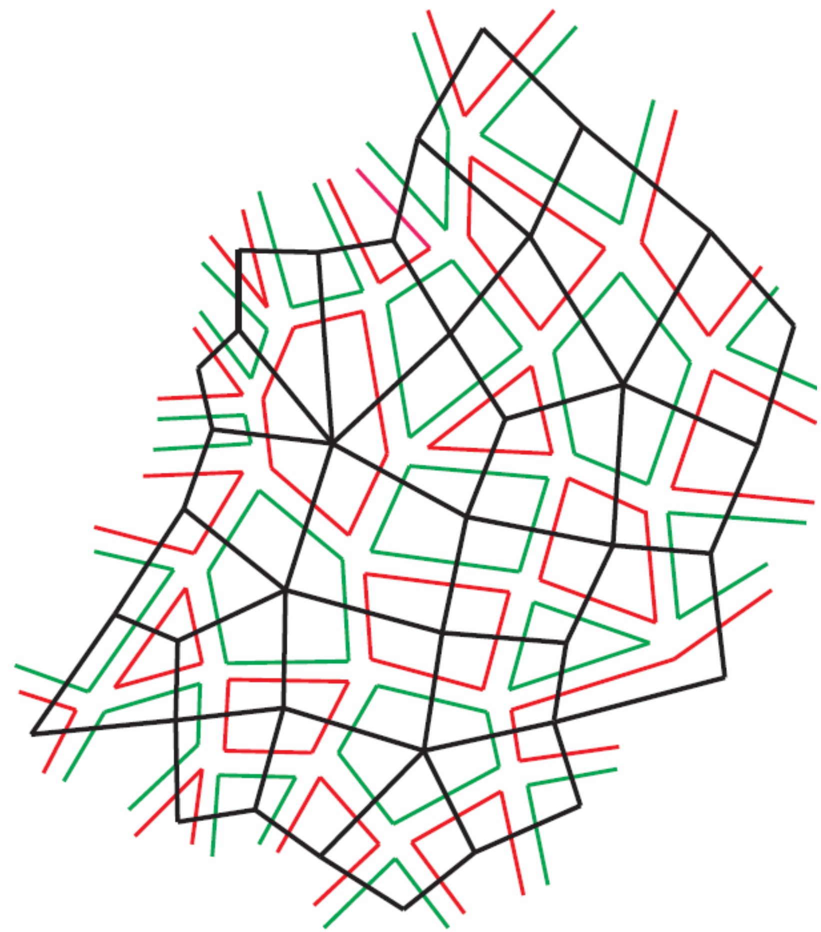


Product Groups

- Another class of matrix models: theories of real matrices ϕ^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a x O(N)_b$ symmetry.
- The interaction is at least quartic: $g \text{ tr } \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is



- In the large N limit where gN is held fixed we again find planar Feynman graphs, but now each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

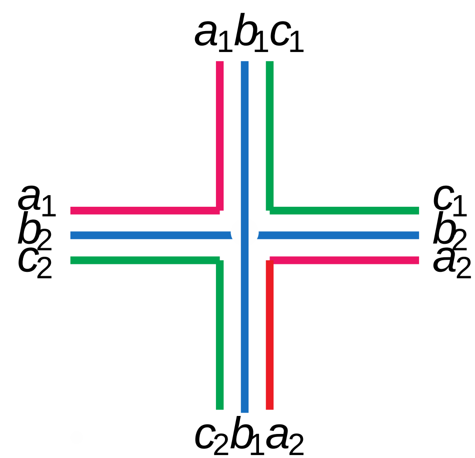
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

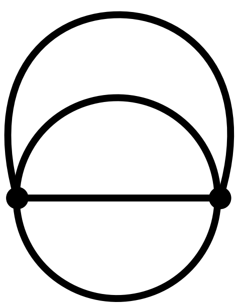
Carrozza, Tanasa; IK, Tarnopolsky

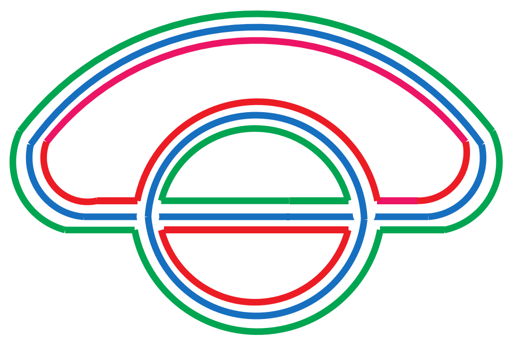
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



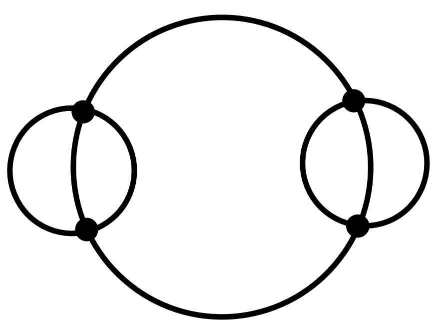
Cables and Wires

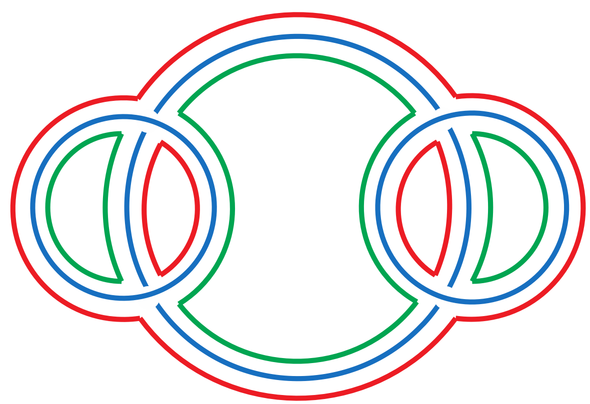
- The Feynman graphs of the quartic field theory may be resolved in terms of the colored wires (triple lines) $\lambda = gN^{3/2}$





$$g^2N^6 \sim N^3\lambda^2$$

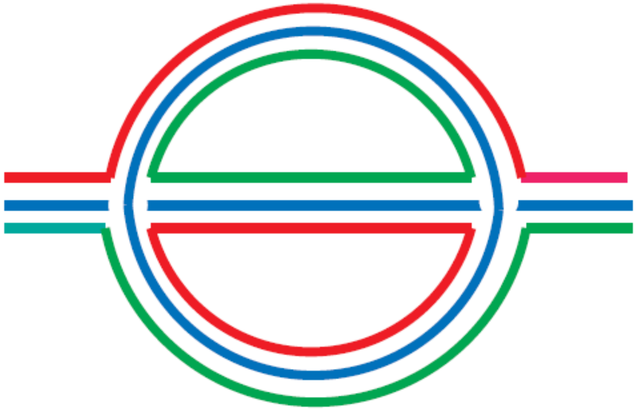




$$g^4N^9 \sim N^3\lambda^4$$

A New Large N Limit

- Leading correction to the propagator has 3 index loops

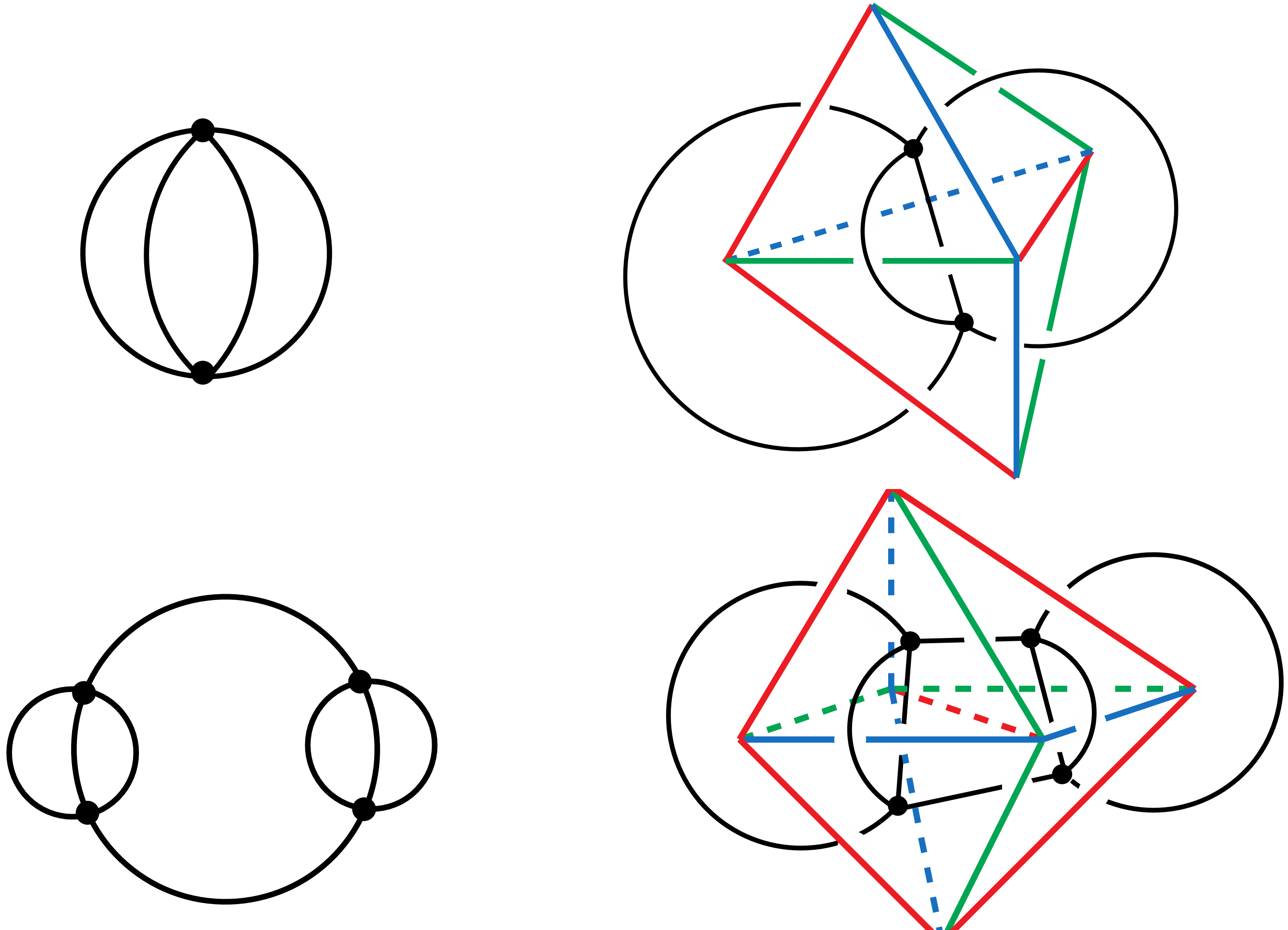


- Requiring that this “melon” insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.

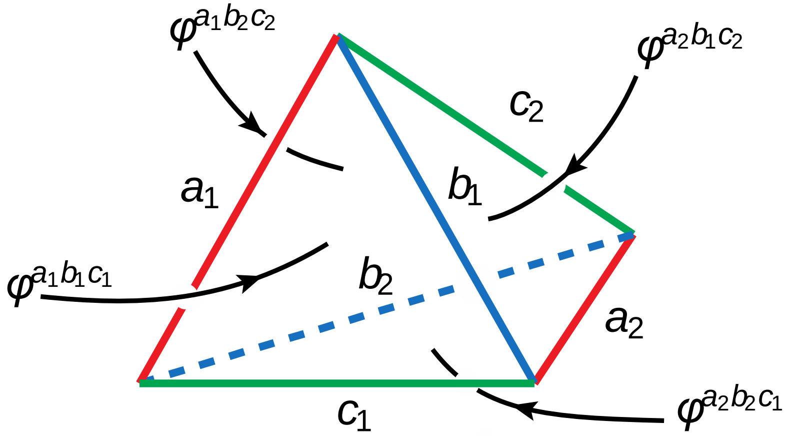
Discretized 3-Geometries

- The study of similar Random Tensor Models was initiated long ago with the goal of generating a class of discretized Euclidean 3-dimensional geometries. Ambjorn, Durhuus, Jonsson; Sasakura; M. Gross
- The original models involved 3-index tensors transforming under a single U(N) or O(N) group. Their large N limit seemed hard to analyze.
- Since 2009 major progress was achieved by Gurau, Rivasseau and others, who found models with multiple O(N) symmetries which possess a new "melonic" large N limit. Gurau, Rivasseau, Bonzom, Ryan, Tanasa, Carrozza, …

- The dual graphs may be represented by tetrahedra glued along the triangular faces. The sides of each triangle have different colors.



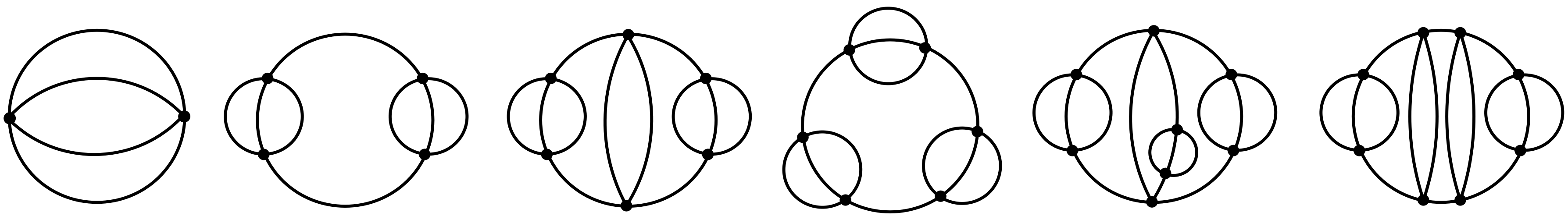
- The 3-geometry interpretation emerges directly is we associate each 3-index tensor with a face of a tetrahedron



- Wick contractions glue a pair of triangles in a special orientation: red to red, blue to blue, green to green.

Melonic Graphs

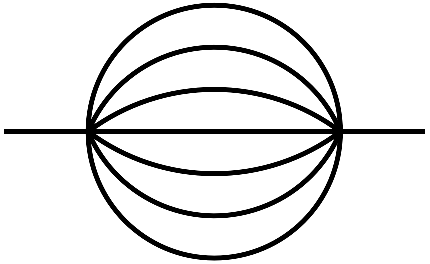
- In some models with multiple O(N) or U(N) symmetries only melon graphs survive in the large N limit where λ is held fixed.



- Remarkably, these graphs may be summed explicitly, so the “melonic” large N limit is exactly solvable!
- The dual structure of glued tetrahedra is dominated by the branched polymers, which is only a tiny subclass of 3-geometries.

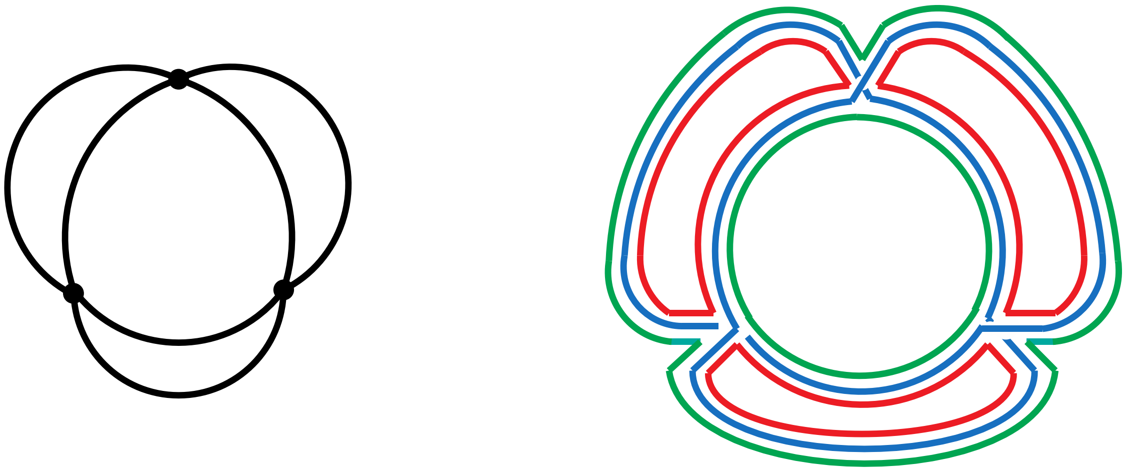
Why Call Them Melons?

- The term seems to have been coined in the 2011 paper by Bonzom, Gurau, Riello and Rivasseau.
- Perhaps because watermelons and some melons have stripes.
- For a tensor with q-1 indices the interaction is ϕ^q so a melon insertion has q-1 lines.
- Much earlier related ideas for ϕ^4 theory by de Calan and Rivasseau in 1981 (they called them "blobs") and by Patashinsky and Pokrovsky in 1964.



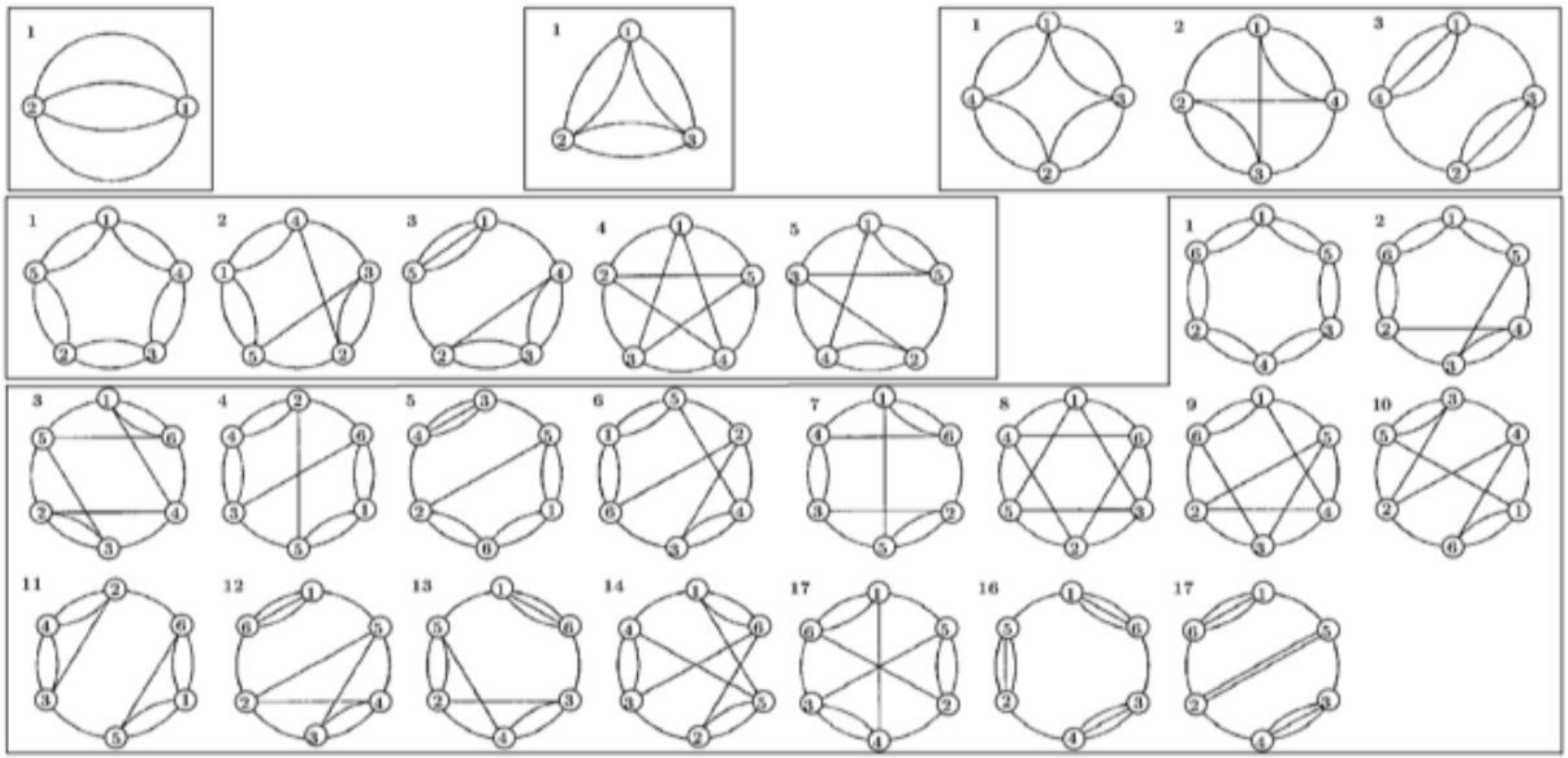
Non-Melonic Graphs

- Most Feynman graphs in the quartic field theory are not melonic are therefore subdominant in the new large N limit, e.g.



- Scales as $g^3 N^6 \sim N^3 \lambda^3 N^{-3/2}$
- None of the graphs with an odd number of vertices are melonic.

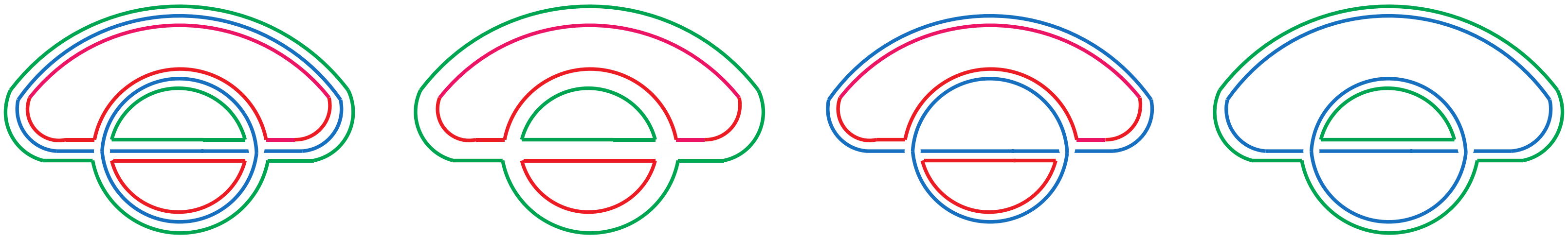
- Here is the list of snail-free vacuum graphs up to 6 vertices Kleinert, Schulte-Frohlinde



- Only 4 out of these 27 graphs are melonic.
- The number of melonic graphs with p vertices grows as C^p Bonzom, Gurau, Riello, Rivasseau

Large N Scaling

- "Forgetting" one color we get a double-line graph.



- The number of loops in a double-line graph is

$f = \chi + e - v$ where χ is the Euler characteristic, e is the number of edges, and v is the number of vertices, $e = 2v$

- If we erase the blue lines we get $f_{rg} = \chi_{rg} + v$

- Adding up such formulas, we find

$$f_{bg} + f_{rg} + f_{br} = 2(f_b + f_g + f_r) = \chi_{bg} + \chi_{br} + \chi_{rg} + 3v$$

- The total number of index loops is

$$f_{\text{total}} = f_b + f_g + f_r = \frac{3v}{2} + 3 - g_{bg} - g_{br} - g_{rg}$$

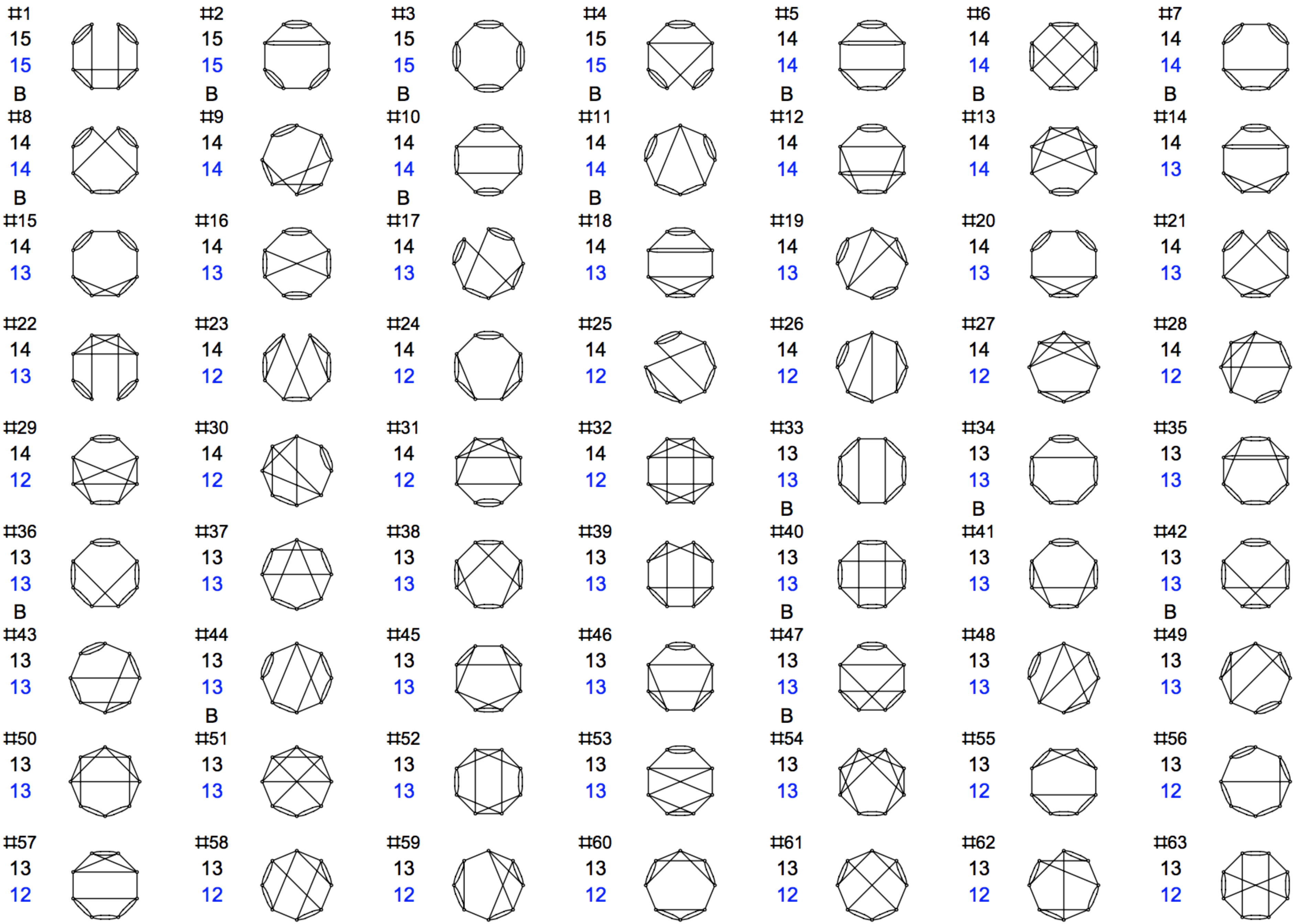
- The genus of a graph is $g = 1 - \chi/2$
- Since $g \geqslant 0$, for a "maximal graph" which dominates at large N all its subgraphs must have genus zero: $f_{\text{total}} = 3 + 3v/2$
- Scales as $N^3(gN^{3/2})^v$
- In the 3-tensor models $\lambda = gN^{3/2}$ must be held fixed in the large N limit.

Bosonic Symmetric Traceless Tensors

- Consider a symmetric traceless bosonic tensor of O(N) with tetrahedron interaction: IK, Tarnopolsky

$$V = \phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c}$$

- Similar to the models considered in the early 90's but the tracelessness condition is crucial. IK, Tarnopolsky; Azeyanagi, Ferrari, Gregori, Leduc, Valette
- Explicit checks of combinatorial factors up to 8^{th} order show that they do dominate. There are 177 diagrams without "snails."



#1 15 15 B
#2 15 15 B
#3 15 15 B
#4 15 15 B
#5 14 14 B
#6 14 14 B
#7 14 14 B
#8 14 14 B
#9 14 14
#10 14 14 B
#11 14 14 B
#12 14 14
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#62 13 12
#63 13 12

- The propagator has the more complicated index structure IK, Tarnopolsky

$$\langle \phi^{abc}\phi^{a'b'c'}\rangle_0 = \frac{1}{6}\Big(\delta^{aa'}\delta^{bb'}\delta^{cc'} + \delta^{ab'}\delta^{bc'}\delta^{ca'} + \delta^{ac'}\delta^{ba'}\delta^{cb'} + \delta^{ab'}\delta^{ba'}\delta^{cc'} + \delta^{ac'}\delta^{bb'}\delta^{ca'} + \delta^{aa'}\delta^{bc'}\delta^{cb'}$$
$$- \frac{2}{N+2}\big(\delta^{ab}\delta^{ca'}\delta^{b'c'} + \delta^{ab}\delta^{cb'}\delta^{a'c'} + \delta^{ab}\delta^{cc'}\delta^{a'b'} + \delta^{ac}\delta^{ba'}\delta^{b'c'} + \delta^{ac}\delta^{bb'}\delta^{a'c'}$$
$$+ \delta^{ac}\delta^{bc'}\delta^{a'b'} + \delta^{bc}\delta^{aa'}\delta^{b'c'} + \delta^{bc}\delta^{ab'}\delta^{a'c'} + \delta^{bc}\delta^{ac'}\delta^{a'b'}\big)\Big)$$

- Similarly, the theory of antisymmetric tensor of O(N) with propagator

$$\langle \phi^{abc}\phi^{a'b'c'}\rangle_0 = \frac{1}{6}\big(\delta^{aa'}\delta^{bb'}\delta^{cc'} + \delta^{ab'}\delta^{bc'}\delta^{ca'} + \delta^{ac'}\delta^{ba'}\delta^{cb'} - \delta^{ab'}\delta^{ba'}\delta^{cc'} - \delta^{ac'}\delta^{bb'}\delta^{ca'} - \delta^{aa'}\delta^{bc'}\delta^{cb'}\big)$$

is also dominated by the melon diagrams.

- Recent combinatorial proof. Benedetti, Carrozza, Tanasa, Kolanowski

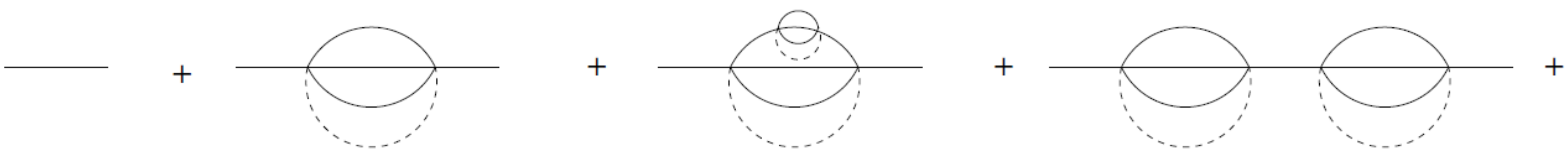
The Sachdev-Ye-Kitaev Model

- Quantum mechanics of a large number N_{SYK} of anti-commuting variables with action

$$I = \int \mathrm{d}t \left(\frac{\mathrm{i}}{2} \sum_i \psi_i \frac{\mathrm{d}}{\mathrm{d}t} \psi_i - \mathrm{i}^{q/2} j_{i_1 i_2 \ldots i_q} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_q} \right)$$

- Random couplings j have a Gaussian distribution with zero mean.
- The model flows to strong coupling and becomes nearly conformal. Georges, Parcollet, Sachdev; Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; …

- The simplest interesting case is q=4.
- Exactly solvable in the large N_{SYK} limit because only the melon Feynman diagrams contribute



- Solid lines are fermion propagators, while dashed lines mean disorder average.
- The exact solution shows resemblance with physics of certain two-dimensional black holes.

SYK-Like Tensor Models

- E. Witten, “An SYK-Like Model Without Disorder,” arXiv: 1610.09758
- Appeared on the evening of Halloween: October 31, 2016.

- It is tempting to change the term “melon diagrams” to “pumpkin diagrams.”

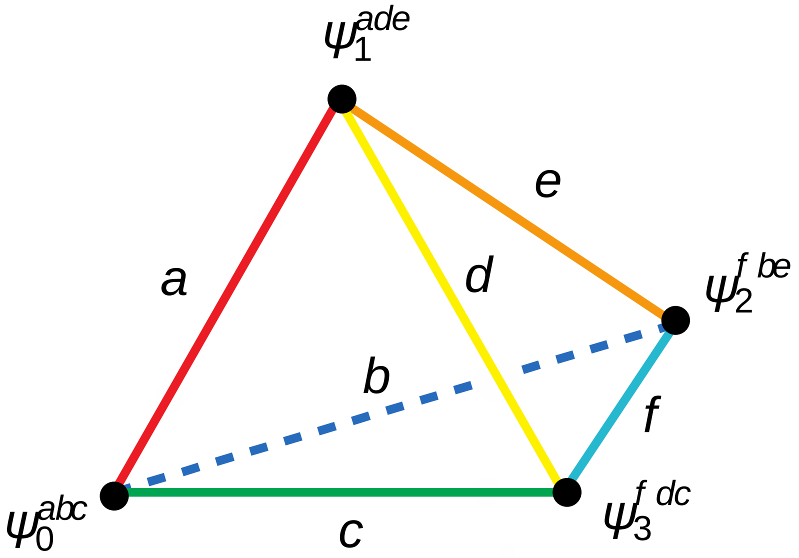
The Gurau-Witten Model

- This model is called "colored" in the random tensor literature because the anti-commuting 3-tensor fields ψ_A^{abc} carry a color label A=0,1,2,3.

$$S_{\mathrm{Gurau-Witten}} = \int dt \left(\frac{i}{2} \psi_A^{abc} \partial_t \psi_A^{abc} + g \psi_0^{abc} \psi_1^{ade} \psi_2^{fbe} \psi_3^{fdc} \right)$$

- The model has $O(N)^6$ symmetry with each tensor in a tri-fundamental under a different subset of the six symmetry groups.

- The 4 different fields may be associated with 4 vertices of a tetrahedron, and the 6 edges correspond to the different symmetry groups:



- As stressed by Witten, gauging the symmetry gets rid of the non-singlet states in the QM.
- This makes it a gauge/gravity correspondence.

This part mostly based on

- IK, G. Tarnopolsky, “Uncolored Random Tensors, Melon Diagrams, and the SYK models,” arXiv:1611.08915

The O(N)^3 Model

- Remove the extra flavor label, so that there are N^3 anticommuting components IK, Tarnopolsky

$$S = \int dt \Big(\frac{i}{2} \psi^{abc} \partial_t \psi^{abc} + \frac{1}{4} g \psi^{a_1 b_1 c_1} \psi^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \psi^{a_2 b_2 c_1} \Big)$$

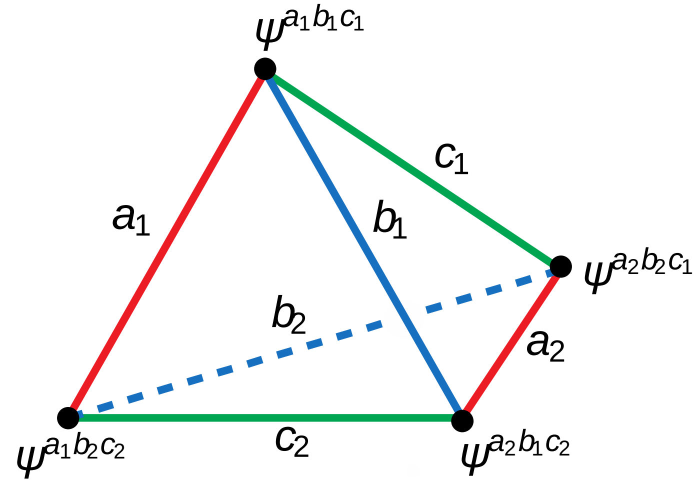
- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \qquad M_1, M_2, M_3 \in O(N)$$

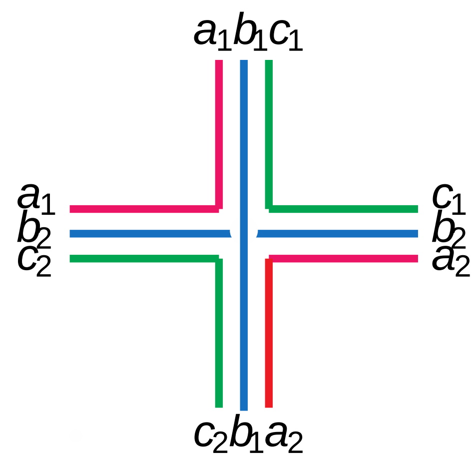
- May be gauged by replacing

$$\partial_t \psi^{abc} \to (D_t \psi)^{abc} = \partial_t \psi^{abc} + A_1^{aa'} \psi^{a'bc} + A_2^{bb'} \psi^{ab'c} + A_3^{cc'} \psi^{abc'}$$

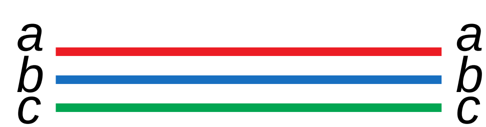
- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.



- This is equivalent to

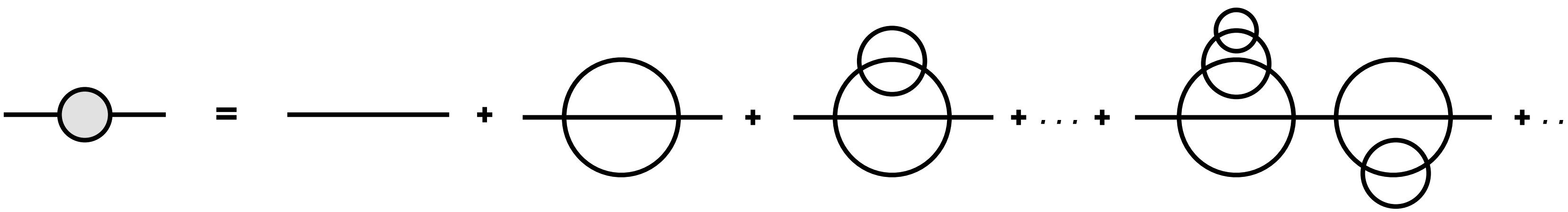


- The 3-line Feynman graphs are produced using the propagator

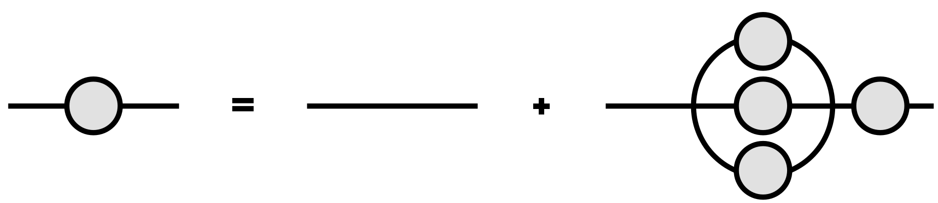


Schwinger-Dyson Equations

- The two-point function obeys the Schwinger-Dyson equation like in SYK model Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon



$$G(t_1 - t_2) = G_0(t_1 - t_2) + g^2 N^3 \int dt dt' G_0(t_1 - t) G(t - t')^3 G(t' - t_2)$$

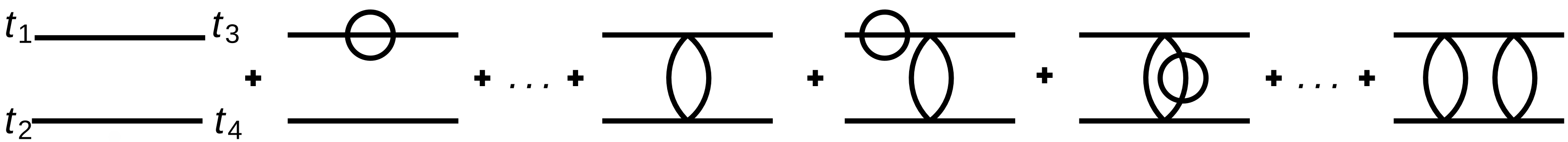


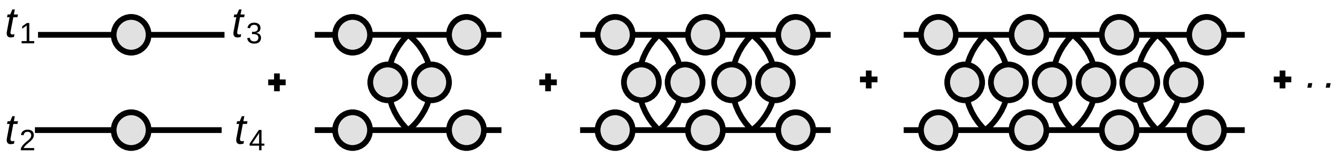
- Neglecting the left-hand side in IR we find

$$G(t_1 - t_2) = -\left(\frac{1}{4\pi g^2 N^3}\right)^{1/4} \frac{\mathrm{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2}}$$

- Four point function

$$\langle \psi^{a_1 b_1 c_1}(t_1)\psi^{a_1 b_1 c_1}(t_2)\psi^{a_2 b_2 c_2}(t_3)\psi^{a_2 b_2 c_2}(t_4)\rangle = N^6 G(t_{12})G(t_{34}) + \Gamma(t_1,\ldots,t_4)$$





- If we denote by Γ_n the ladder with n rungs

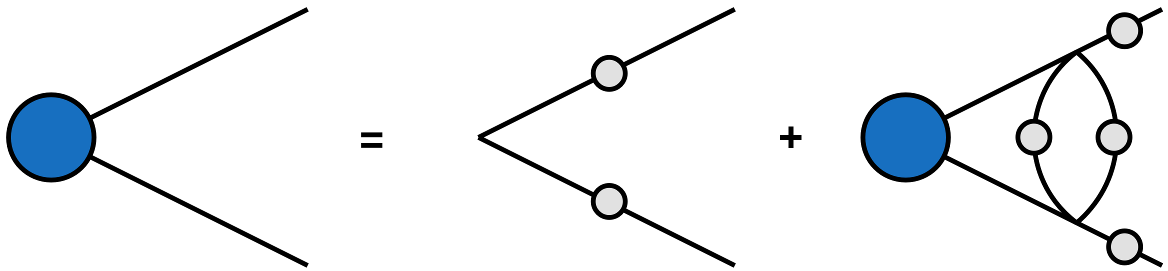
$$\Gamma = \sum_n \Gamma_n$$

$$\Gamma_{n+1}(t_1,\ldots,t_4) = \int dt dt' K(t_1,t_2;t,t')\Gamma_n(t,t',t_3,t_4)$$

$$K(t_1,t_2;t_3,t_4) = -3g^2 N^3 G(t_{13})G(t_{24})G(t_{34})^2$$

Spectrum of two-particle operators

- S-D equation for the three-point function Gross, Rosenhaus



$$v(t_0, t_1, t_2) = g(h) \int dt_3 dt_4 K(t_1, t_2; t_3, t_4) v(t_0, t_3, t_4)$$

$$v(t_0, t_1, t_2) = \langle O_2^n(t_0) \psi^{abc}(t_1) \psi^{abc}(t_2) \rangle = \frac{\mathrm{sgn}(t_1 - t_2)}{|t_0 - t_1|^h |t_0 - t_2|^h |t_1 - t_2|^{1/2-h}}$$

- Scaling dimensions determined by $g(h) = 1$

- Can use SL(2) invariance to take t_0 to infinity and consider eigenfunctions of the form

$$v(t_1, t_2) = \frac{\mathrm{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2-h}}$$

- Two basic integrals

$$\int_{-\infty}^{+\infty} du \frac{\mathrm{sgn}(u - t_1)\,\mathrm{sgn}(u - t_2)}{|u - t_1|^a |u - t_2|^b} = l^+_{a,b} \frac{1}{|t_1 - t_2|^{a+b-1}}\,,$$

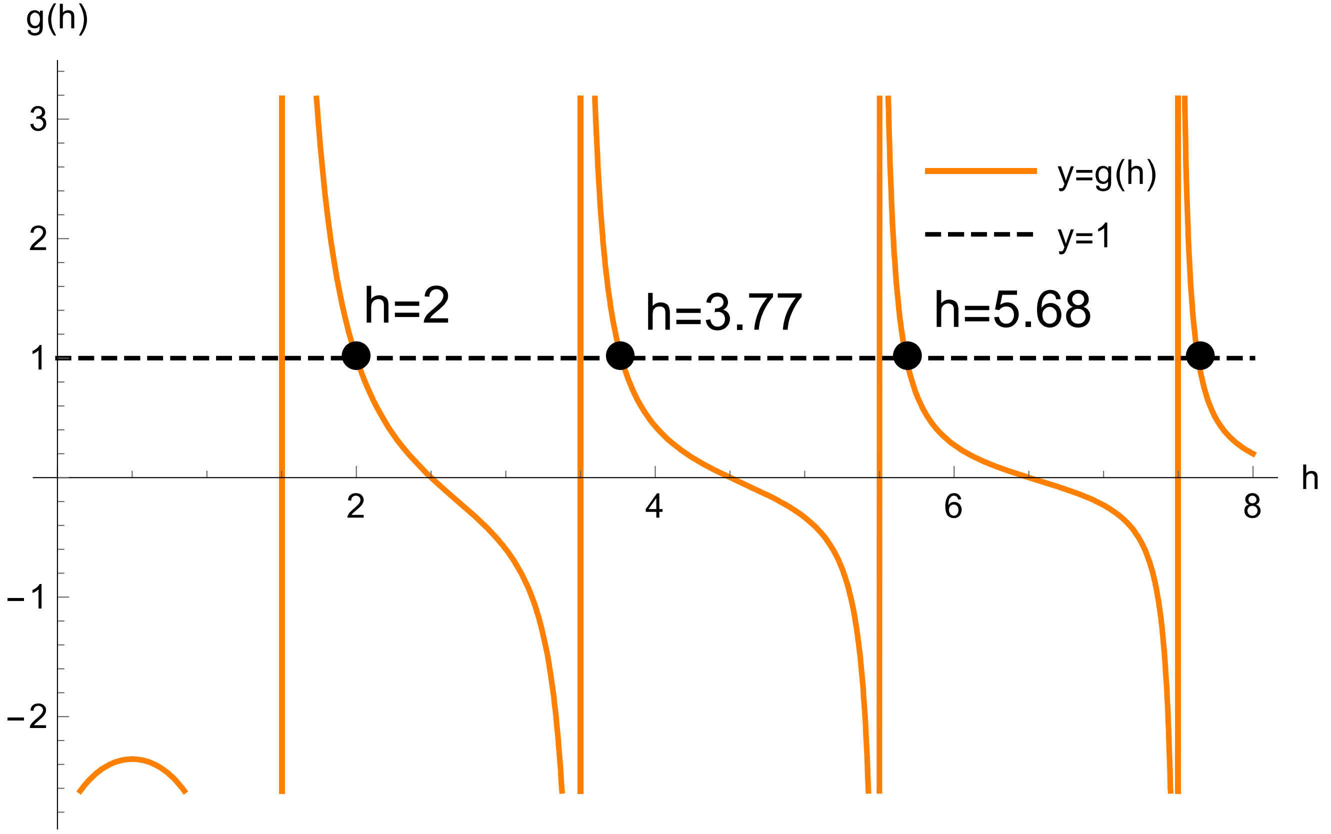
$$\int_{-\infty}^{+\infty} du \frac{\mathrm{sgn}(u - t_2)}{|u - t_1|^a |u - t_2|^b} = l^-_{a,b} \frac{\mathrm{sgn}(t_1 - t_2)}{|t_1 - t_2|^{a+b-1}}\,,$$

$$l^{\pm}_{a,b} = \beta(1-a, a+b-1) \pm (\beta(1-b, a+b-1) - \beta(1-a, 1-b))$$

- Find the result

$$g(h) = -\frac{3}{4\pi} l^+_{\frac{3}{2}-h,\frac{1}{2}} l^-_{1-h,\frac{1}{2}} = -\frac{3}{2} \frac{\tan(\frac{\pi}{2}(h - \frac{1}{2}))}{h - 1/2}$$

- The first solution is h=2; dual to gravity.

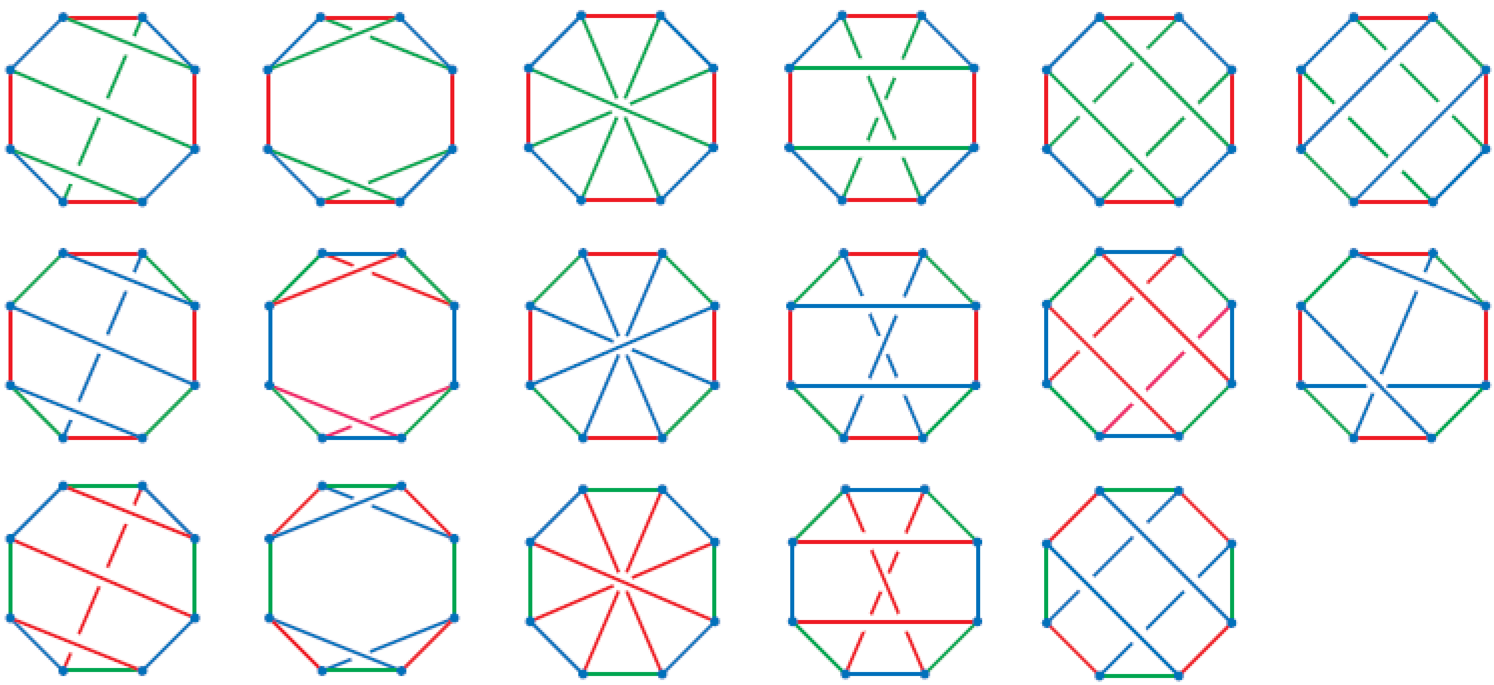


- The higher scaling dimensions are

$h \approx 3.77,\ 5.68,\ 7.63,\ 9.60$ approaching $h_n \to n + \frac{1}{2}$

Gauge Invariant Operators

- Two-particle operators, which are analogous to a "single Regge trajectory" $O_2^n = \psi^{abc}(D_t^n \psi)^{abc}$
- There is a growing number of multi-particle operators. Bulycheva, IK, Milekhin, Tarnopolsky



Model with a Complex Fermion

- The action

$$S = \int dt \Big(i\bar{\psi}^{abc}\partial_t \psi^{abc} + \frac{1}{4} g \psi^{a_1 b_1 c_1} \bar{\psi}^{a_1 b_2 c_2} \psi^{a_2 b_1 c_2} \bar{\psi}^{a_2 b_2 c_1} \Big)$$

has enhanced $U(N) \times O(N) \times U(N)$ symmetry

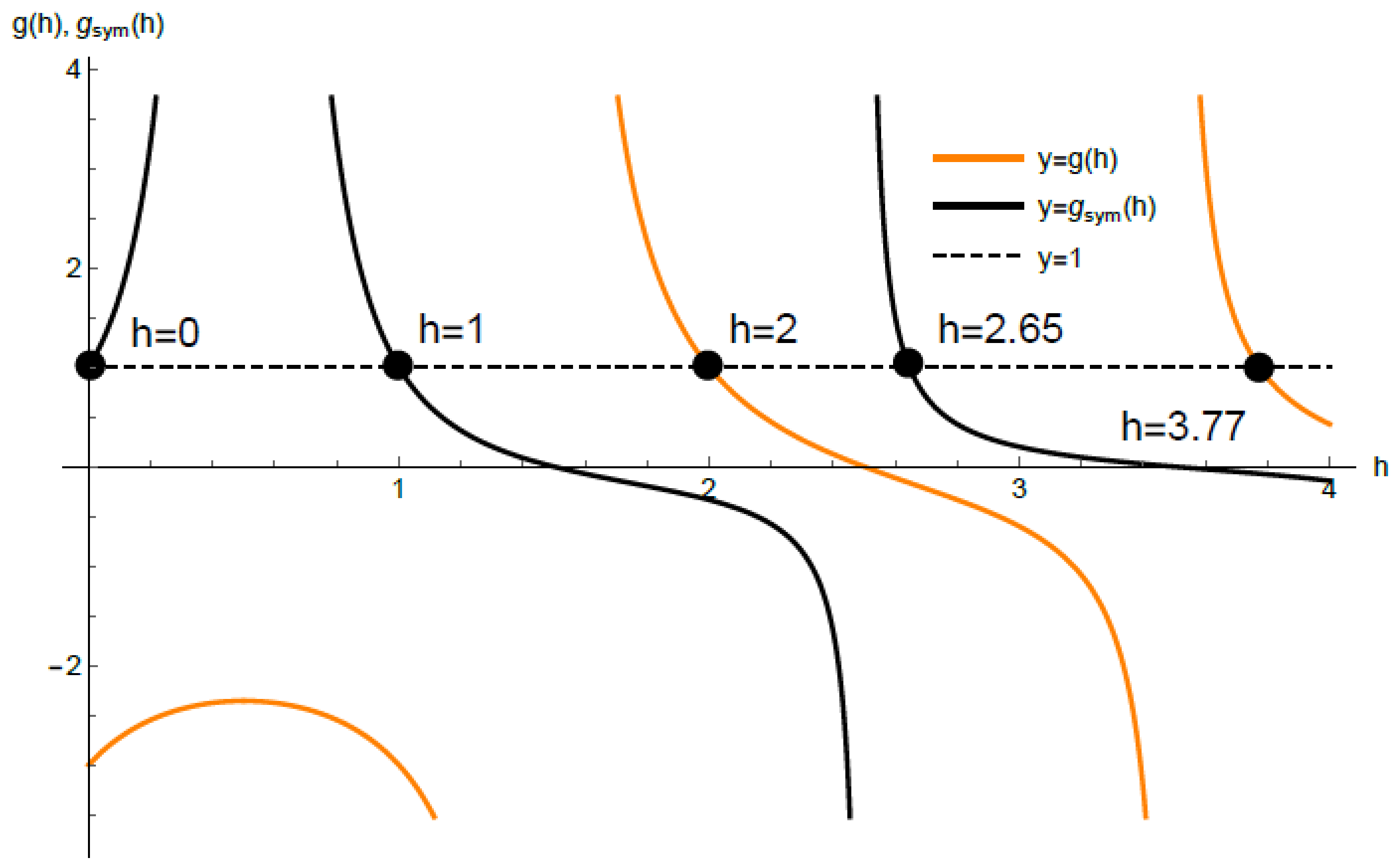
- Gauge invariant two-particle operators

$$\mathcal{O}_2^n = \bar{\psi}^{abc} (D_t^n \psi)^{abc} \qquad n = 0, 1, \ldots$$

including $\bar{\psi}^{abc}\psi^{abc}$

Spectrum of two-particle operators

- The integral equation also admits symmetric solutions
$$v(t_1,t_2)=\frac{1}{|t_1-t_2|^{1/2-h}}$$
- Calculating the integrals we get
$$g_{\rm sym}(h)=-\frac{1}{4\pi}l^-_{\frac{3}{2}-h,\frac{1}{2}}l^+_{1-h,\frac{1}{2}}=-\frac{1}{2}\frac{\tan(\frac{\pi}{2}(h+\frac{1}{2}))}{h-1/2}$$
- The first solution is h=1 corresponding to U(1) charge $\bar{\psi}^{abc}\psi^{abc}$



- The additional scaling dimensions

$$h \approx 2.65,\ 4.58,\ 6.55,\ 8.54$$

approach $h_n = n + \frac{1}{2} + \frac{1}{\pi n} + \mathcal{O}(n^{-3})$

Bosonic Tensor Model in General d

- Action with a potential that is not positive definite

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} \right)$$

- Schwinger-Dyson equation for 2pt function
Patashinsky, Pokrovsky

$$G^{-1}(p) = -\lambda^2 \int \frac{d^d k d^d q}{(2\pi)^{2d}} G(q) G(k) G(p+q+k)$$

- Has solution

$$G(p) = \lambda^{-1/2} \left(\frac{(4\pi)^d d\Gamma(\frac{3d}{4})}{4\Gamma(1-\frac{d}{4})} \right)^{1/4} \frac{1}{(p^2)^{\frac{d}{4}}}$$

Spectrum of two-particle spin zero operators

- Schwinger-Dyson equation

$$\int d^d x_3 d^d x_4 K(x_1, x_2; x_3, x_4) v_h(x_3, x_4) = g(h) v_h(x_1, x_2)$$

$$K(x_1, x_2; x_3, x_4) = 3\lambda^2 G(x_{13}) G(x_{24}) G(x_{34})^2$$

$$v_h(x_1, x_2) = \frac{1}{[(x_1 - x_2)^2]^{\frac{1}{2}(\frac{d}{2} - h)}}$$

$$g_{\text{bos}}(h) = -\frac{3\Gamma\left(\frac{3d}{4}\right)\Gamma\left(\frac{d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{h}{2} - \frac{d}{4}\right)}{\Gamma\left(-\frac{d}{4}\right)\Gamma\left(\frac{3d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{d}{4} + \frac{h}{2}\right)}$$

- Spectrum in d=1 again includes scaling dimension h=2, suggesting the existence of a gravity dual.
- However, the leading solution is complex, which suggests that the large N CFT is unstable Giombi, IK, Tarnopolsky $h_0 = \frac{1}{2} + 1.525i$
- It corresponds to the operator $\phi^{abc}\phi^{abc}$
- In d=4-ε

$$h_0 = 2 \pm i\sqrt{6\epsilon} - \frac{1}{2}\epsilon + \mathcal{O}(\epsilon^{3/2})$$

- The dual scalar field in AdS violates the Breitenlohner-Freedman bound.

Fixed Point in 4-ε Dimensions

- The tetrahedron operator

$$O_t(x) = \phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_2b_2c_1}$$

mixes with the pillow and double-sum operators

$$O_p(x) = \frac{1}{3}\left(\phi^{a_1b_1c_1}\phi^{a_1b_1c_2}\phi^{a_2b_2c_2}\phi^{a_2b_2c_1} + \phi^{a_1b_1c_1}\phi^{a_2b_1c_1}\phi^{a_2b_2c_2}\phi^{a_1b_2c_2} + \phi^{a_1b_1c_1}\phi^{a_1b_2c_1}\phi^{a_2b_1c_2}\phi^{a_2b_2c_2}\right),$$

$$O_{ds}(x) = \phi^{a_1b_1c_1}\phi^{a_1b_1c_1}\phi^{a_2b_2c_2}\phi^{a_2b_2c_2}$$

- The renormalizable action is

$$S = \int d^dx \left(\frac{1}{2}\partial_\mu\phi^{abc}\partial^\mu\phi^{abc} + \frac{1}{4}\left(g_1O_t(x) + g_2O_p(x) + g_3O_{ds}(x)\right)\right)$$

- The large N scaling is

$$g_1 = \frac{(4\pi)^2 \tilde{g}_1}{N^{3/2}}, \quad g_2 = \frac{(4\pi)^2 \tilde{g}_2}{N^2}, \quad g_3 = \frac{(4\pi)^2 \tilde{g}_3}{N^3}$$

- The 2-loop beta functions and fixed points:

$$\tilde{\beta}_t = -\epsilon \tilde{g}_1 + 2\tilde{g}_1^3 ,$$

$$\tilde{\beta}_p = -\epsilon \tilde{g}_2 + \left(6\tilde{g}_1^2 + \frac{2}{3}\tilde{g}_2^2\right) - 2\tilde{g}_1^2 \tilde{g}_2 ,$$

$$\tilde{\beta}_{ds} = -\epsilon \tilde{g}_3 + \left(\frac{4}{3}\tilde{g}_2^2 + 4\tilde{g}_2\tilde{g}_3 + 2\tilde{g}_3^2\right) - 2\tilde{g}_1^2(4\tilde{g}_2 + 5\tilde{g}_3)$$

$$\tilde{g}_1^* = (\epsilon/2)^{1/2}, \quad \tilde{g}_2^* = \pm 3i(\epsilon/2)^{1/2}, \quad \tilde{g}_3^* = \mp i(3 \pm \sqrt{3})(\epsilon/2)^{1/2}$$

- The scaling dimension of $\phi^{abc}\phi^{abc}$ is

$$\Delta_O = d - 2 + 2(\tilde{g}_2^* + \tilde{g}_3^*) = 2 \pm i\sqrt{6\epsilon} + \mathcal{O}(\epsilon)$$

Super Melons

- May consider a supersymmetric model with "tetrahedron superpotential"

$$W = \frac{1}{4} g \Phi^{a_1 b_1 c_1} \Phi^{a_1 b_2 c_2} \Phi^{a_2 b_1 c_2} \Phi^{a_2 b_2 c_1}$$

- In d=3 such a theory is renormalizable, so for d<3 it may flow to an interacting superconformal theory.
- Includes the positive sextic scalar potential.

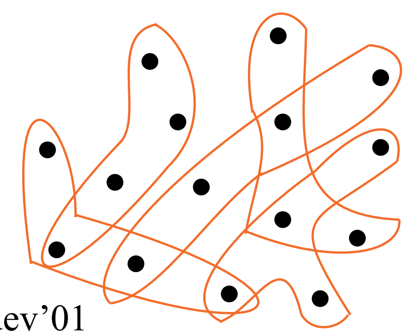
Sachdev-Ye-Kitaev Model

$$H = \frac{1}{4!}\sum_{i_1,i_2,i_3,i_4=1}^{N} J_{i_1 i_2 i_3 i_4}\chi_{i_1}\chi_{i_2}\chi_{i_3}\chi_{i_4}$$

- Majorana fermions $\{\chi_i, \chi_j\} = \delta_{ij}$
- $J_{i_1 i_2 i_3 i_4}$ are Gaussian random

$$\langle J^2_{i_1 i_2 i_3 i_4}\rangle = 3!\frac{J^2}{N^3} \quad \langle J_{i_1 i_2 i_3 i_4}\rangle = 0$$

- Has $O(N_{SYK})$ symmetry after averaging over disorder



Sachdev, Ye ‘93,
Georges, Parcollet, Sachdev’01
Kitaev ‘15

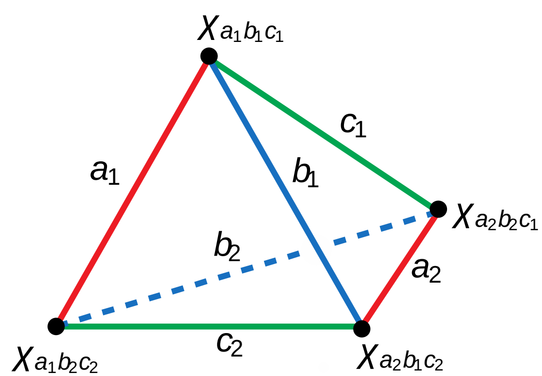
O(N)^3 Tensor Model

$$H = \frac{1}{4}\sum_{a_1,\dots,c_2=1}^{N} \frac{J}{N^{3/2}}\chi_{a_1 b_1 c_1}\chi_{a_1 b_2 c_2}\chi_{a_2 b_1 c_2}\chi_{a_2 b_2 c_1}$$

- Majorana fermions

$$\{\chi_{abc}, \chi_{a'b'c'}\} = \delta_{aa'}\delta_{bb'}\delta_{cc'}$$

- No disorder
- Has $O(N)_a$ x $O(N)_b$ x $O(N)_c$ symmetry



IK, Tarnopolsky’16

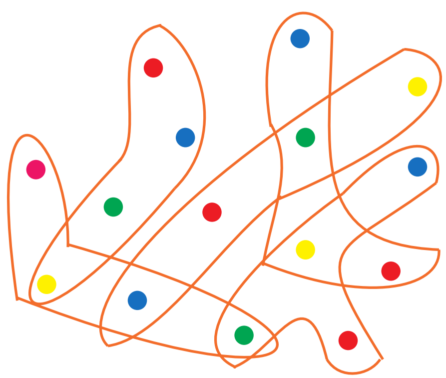
Gross-Rosenhaus Model q=4, f=4

$$H = \sum_{i_1,i_2,i_3,i_4=1}^{N} J_{i_1 i_2 i_3 i_4} \chi^0_{i_1} \chi^1_{i_2} \chi^2_{i_3} \chi^3_{i_4}$$

- Majorana fermions $\{\chi_i^a, \chi_j^b\} = \delta_{ij}\delta^{ab}$
- $J_{i_1 i_2 i_3 i_4}$ are Gaussian random

$$\langle J^2_{i_1 i_2 i_3 i_4} \rangle = 4^4 \frac{J^2}{N^3} \quad \langle J_{i_1 i_2 i_3 i_4} \rangle = 0$$

- Has $O(N_{SYK})$ x $O(N_{SYK})$ x
- $O(N_{SYK})$ x $O(N_{SYK})$ symmetry



Gross, Rosenhaus' 16

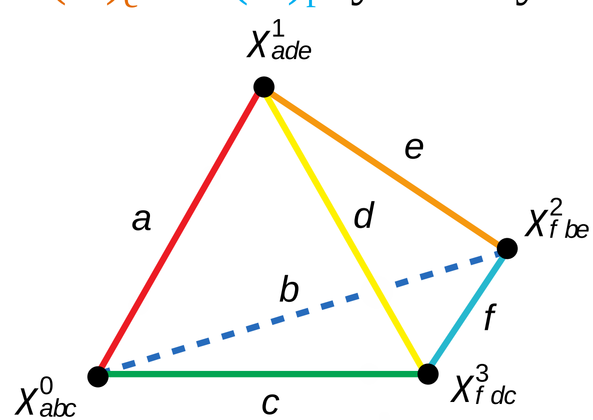
Gurau-Witten Model

$$H = \sum_{a,\dots,f=1}^{N} \frac{J}{N^{3/2}} \chi^0_{abc} \chi^1_{ade} \chi^2_{fbe} \chi^3_{fdc}$$

- Majorana fermions

$$\{\chi^A_{abc}, \chi^B_{a'b'c'}\} = \delta_{aa'}\delta_{bb'}\delta_{cc'}\delta^{AB}$$

- No disorder
- Has $O(N)_a$ x $O(N)_b$ x $O(N)_c$ x $O(N)_d$ x $O(N)_e$ x $O(N)_f$ symmetry



Gurau '10
Witten'16

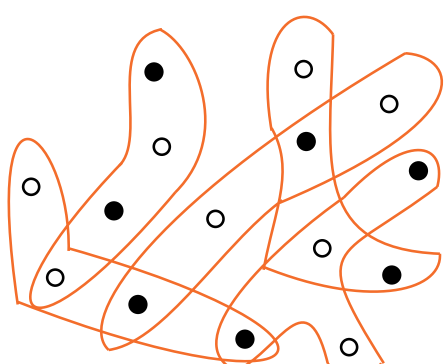
Complex SYK Model

$$H = \frac{1}{4!} \sum_{i_1,i_2,i_3,i_4=1}^{N} J_{i_1 i_2 i_3 i_4} \chi^\dagger_{i_1} \chi^\dagger_{i_2} \chi_{i_3} \chi_{i_4}$$

- Complex fermions $\{\chi_i, \chi_j^\dagger\} = \delta_{ij}$
- $J_{i_1 i_2 i_3 i_4}$ are Gaussian random

$$\langle J^2_{i_1 i_2 i_3 i_4} \rangle = 3! \frac{J^2}{N^3} \quad \langle J_{i_1 i_2 i_3 i_4} \rangle = 0$$

- Has U(N_{SYK}) symmetry after averaging over disorder



Sachdev '15
Davison, Fu, Gu, Georges, Jensen, Sachdev '16

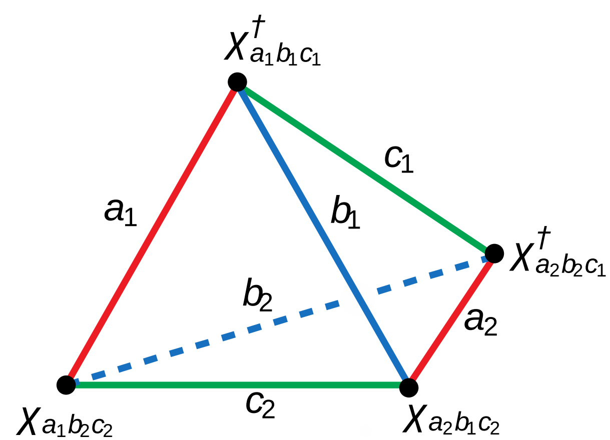
Complex Tensor Model

$$H = \frac{1}{4} \sum_{a_1,\ldots,c_2=1}^{N} \frac{J}{N^{3/2}} \chi^\dagger_{a_1 b_1 c_1} \chi^\dagger_{a_2 b_2 c_1} \chi_{a_1 b_2 c_2} \chi_{a_2 b_1 c_2}$$

- Complex fermions

$$\{\chi_{abc}, \chi^\dagger_{a'b'c'}\} = \delta_{aa'}\delta_{bb'}\delta_{cc'}$$

- Has $SU(N)_a$ x $SU(N)_b$ x $O(N)_c$ x U(1) symmetry and no disorder



IK, GT'16