Decoherence in Neutrino Propagation through matter and bounds from IceCube/DeepCore

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Based on Collaboration with Pilar Coloma, Jacobo Lopez-Pavan, Ivan Martinez-Soler,

arXiv: 1803.04438 [hep-ph]

PANE2018, ICTP, May 31, 2018

Outline

Introduction

Neutrino oscillation in matter with decoherence effect

Analysis Procedure

Analysis Results

Conclusions

Introduction

- So far, no convincing evidence/confirmation of physics beyond the standard three neutrino flavor framework

- It is interesting to probe/constrain some new physics beyond the standard framework

- As one of the examples of such new physics, in this talk, we consider (non-standard) quantum decoherence in neutrino oscillation

- It is important to test the paradigm of the standard three flavor framework

Introduction

- Neutrino oscillations occur due to quantum interference
- "Coherence" is needed to that happen
- Oscillation is suppressed due to "decoherence" effect, for example, by

  separation of the wave packets

  matter density fluctuation

  finite energy and/or spatial resolution (uncertainty)

- we consider non-standard decoherence effect which may come from some new physics (which may be related to quantum gravity) assuming a phenomenological model

Introduction

Incomplete list of previous works on decoherence:

Benatti, Floreanini, JHEP 02, 032 (2000)

Lisi, Marrone, Montanitno, PRL 85, 1166 (2000)

Gago et al., PRD63, 073001 (2001)

Morgan et al., Astropart. Phys.25, 311 (2006)

Fogli et. al., PRD 76, 0330066 (2007)

Farzan, Schwetz, Smirnov, JHEP 07, 067 (2008)

Bakhti, Farzan, Schwetz, JHEP 05, 007 (2015)

Oliveira, Eur. Phys. J. C76, 417 (2016)

Guzzo, Holanda, Oliveira, NPB908, 408 (2016)

Coelho, Mann, Bashar, PRL118, 221801 (2017)

Carpio, Massoni, Gago, arXiv:1711.03690 [hep-ph]

Gomes et al., arXiv:1805.09818 [hep-ph]

Quantum Decoherence: density matrix formalism

$$\frac{d\rho}{dt} = -i\,[H, \rho] - \mathcal{D}\,[\rho]$$

H: Hamiltonian

describes decoherence

assuming positivity

$$\mathcal{D}\,[\rho] = \sum_m \left[\{\rho, D_m D_m^\dagger\} - 2 D_m \rho D_m^\dagger \right]$$

(Lindbald form)

Hermicity $D_m = D_m^\dagger$ (to avoid unitarity violation)

$\boxed{\text{Energy Conservation } [H, D_m] = 0}$

(or small enough to be neglected)

Neutrino propagation in uniform matter

$$H = \tilde{U}\,\mathrm{diag}\left\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\right\}\tilde{U}^\dagger \equiv \tilde{U} H_d \tilde{U}^\dagger$$

$$D_m = \tilde{U}\,\mathrm{diag}\left\{d_m^1, d_m^2, d_m^3\right\}\tilde{U}^\dagger \equiv \tilde{U} D_m^d \tilde{U}^\dagger$$

$$\frac{d\tilde{\rho}_{ij}}{dt} = -\left[\gamma_{ij} - i\Delta\tilde{h}_{ij}\right]\tilde{\rho}_{ij},$$

where

$$\tilde{\rho} = \tilde{U}^\dagger \rho \tilde{U}$$

$\tilde{U}$: mixing matrix in matter

$$\gamma_{ij} \equiv \sum_m \left(d_m^i - d_m^j\right)^2 = \gamma_{ji} > 0$$

$$\tilde{\rho}_{ij}(t) = \tilde{\rho}_{ij}(0)\, e^{-\left[\gamma_{ij} - i\Delta\tilde{h}_{ij}\right]t}$$

For constant matter density

$$P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) = \sum_{i,j} \tilde{U}_{\beta i} \tilde{U}^*_{\beta j}\, \tilde{\rho}_{ij}(t)$$

$$= \sum_{i,j} \tilde{U}^*_{\alpha i} \tilde{U}_{\beta i} \tilde{U}_{\alpha j} \tilde{U}^*_{\beta j} e^{-\left[\gamma_{ij} - i\Delta\tilde{h}_{ij}\right]t}$$

In a more familiar way,

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 2\sum_{i<j} \mathrm{Re}\left[\tilde{U}^*_{\alpha i} \tilde{U}_{\beta i} \tilde{U}_{\alpha j} \tilde{U}^*_{\beta j}\right]\left(1 - e^{-\gamma_{ij} L} \cos\tilde{\Delta}_{ij}\right)$$
$$-2\sum_{i<j} \mathrm{Im}\left[\tilde{U}^*_{\alpha i} \tilde{U}_{\beta i} \tilde{U}_{\alpha j} \tilde{U}^*_{\beta j}\right] e^{-\gamma_{ij} L} \sin\tilde{\Delta}_{ij},$$

$$\tilde{\Delta}_{ij} \equiv \frac{\Delta\tilde{m}^2_{ij} L}{2E}$$

$\Delta\tilde{m}^2_{ij} \equiv \tilde{m}^2_i - \tilde{m}^2_j$: effective mass squared difference in matter

γ_{ij} : decoherence parameters (see next slide)

Assumptions on the Decoherence Parameters

Following the previous works, we assume power-law energy dependence

$$\boxed{\gamma_{ij} = \gamma_{ji} \equiv \gamma_{ij}^{0} \left(\frac{E}{\mathrm{GeV}} \right)^{n}}$$

to have sizable effect of decoherence, we can roughly estimate that

$$\gamma_{ij}^{0} \sim 1.7 \cdot 10^{-19} \left(\frac{L}{\mathrm{km}} \right)^{-1} \left(\frac{E}{\mathrm{GeV}} \right)^{-n} \mathrm{GeV}$$

(but not enough condition)

For 1 layer of constant matter density

$$P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) = \sum_{i,j} \tilde{U}_{\beta i} \tilde{U}^*_{\beta j}\, \tilde{\rho}_{ij}(t)$$
$$= \sum_{i,j} \tilde{U}^*_{\alpha i} \tilde{U}_{\beta i} \tilde{U}_{\alpha j} \tilde{U}^*_{\beta j} e^{-\left[\gamma_{ij} - i\Delta\tilde{h}_{ij}\right]t}$$

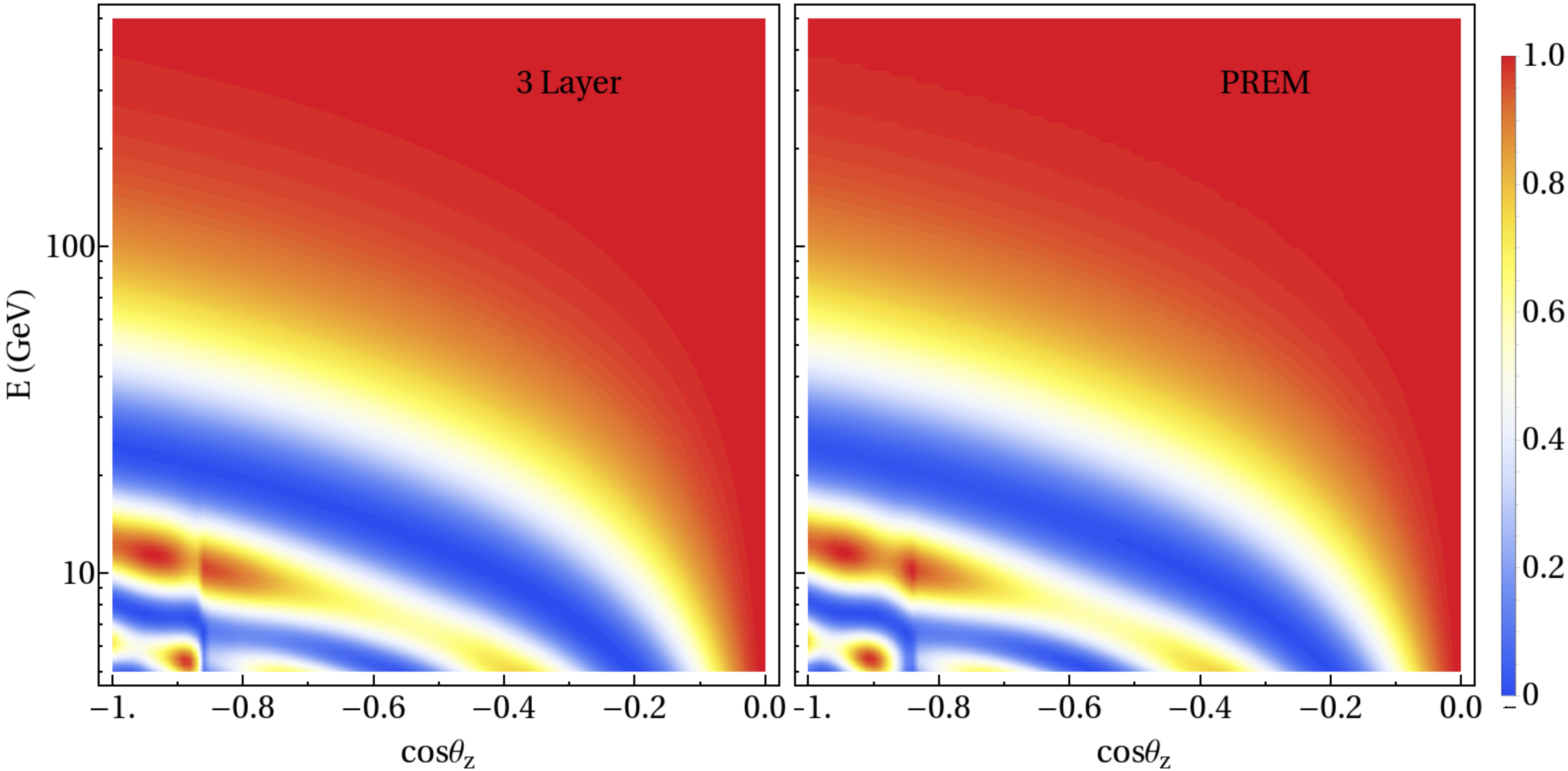
For multi layers of constant matter densities

$$P_{\alpha\beta} = \sum_{\delta,\gamma} \tilde{\mathcal{A}}^B_{\beta\delta\gamma\beta} \tilde{\mathcal{A}}^A_{\delta\alpha\alpha\gamma}$$ for 2 layers

$$P_{\alpha\beta} = \sum_{\delta,\gamma,\theta,\phi} \tilde{\mathcal{A}}^C_{\beta\delta\gamma\beta} \tilde{\mathcal{A}}^B_{\delta\theta\phi\gamma} \tilde{\mathcal{A}}^A_{\theta\alpha\alpha\phi}$$ for 3 layers

$$\tilde{\mathcal{A}}^M_{\alpha\beta\gamma\delta} \equiv \sum_{i,j} \tilde{U}^M_{\alpha i} \tilde{U}^{M*}_{\beta i} \tilde{U}^M_{\gamma j} \tilde{U}^{M*}_{\delta j} e^{-\left[\gamma_{ij} - i(\Delta\tilde{m}^M_{ij})^2/2E\right]\Delta L_M}$$

3 layer approximation of the Earth matter density profile (mantle - core - mantle)



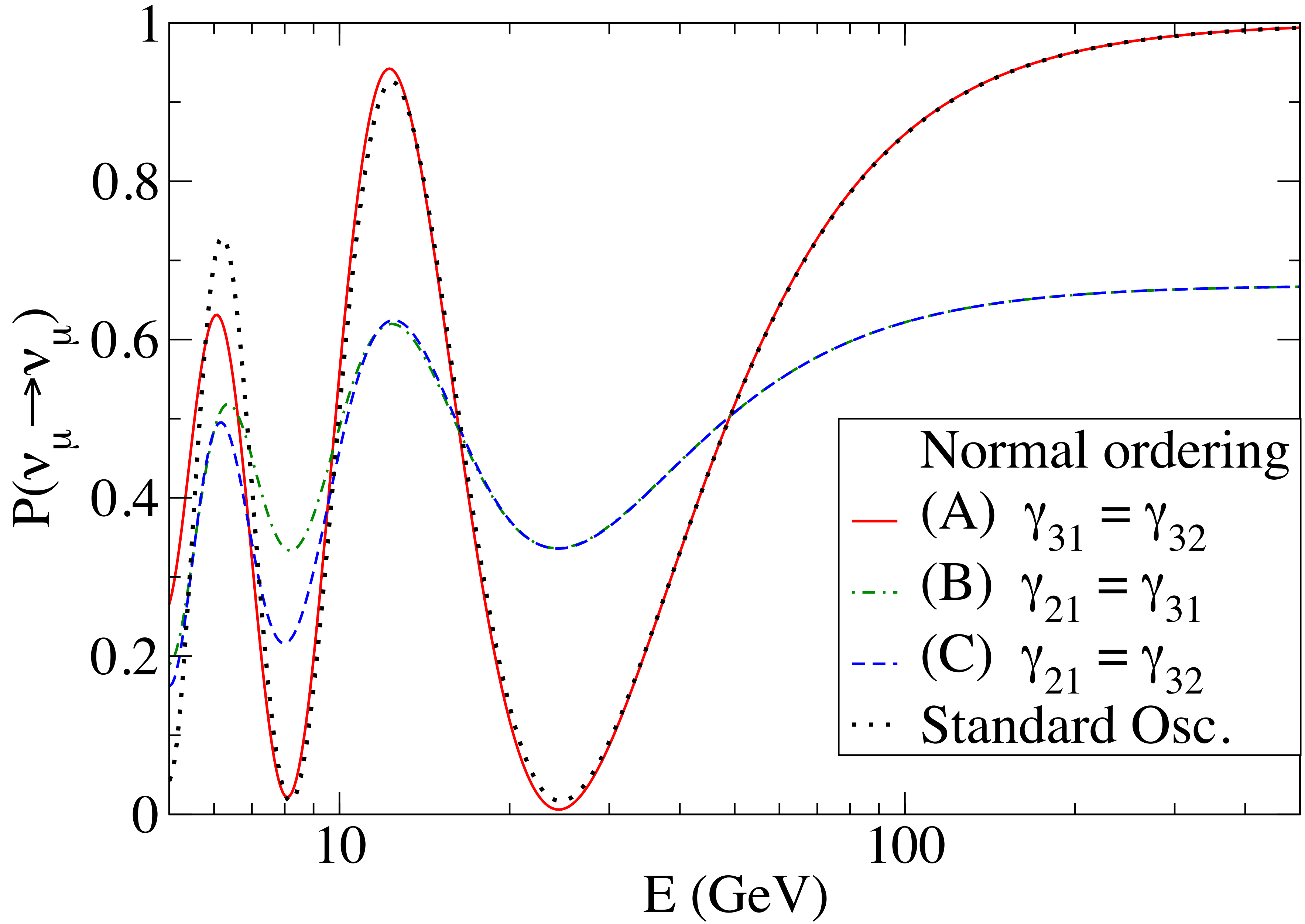
We consider the following 3 distinct cases

(A) Atmospheric limit: $\Upsilon_{21} = 0$ ($\Upsilon_{32} = \Upsilon_{31}$)

(B) Solar limit I: $\Upsilon_{32} = 0$ ($\Upsilon_{21} = \Upsilon_{31}$)

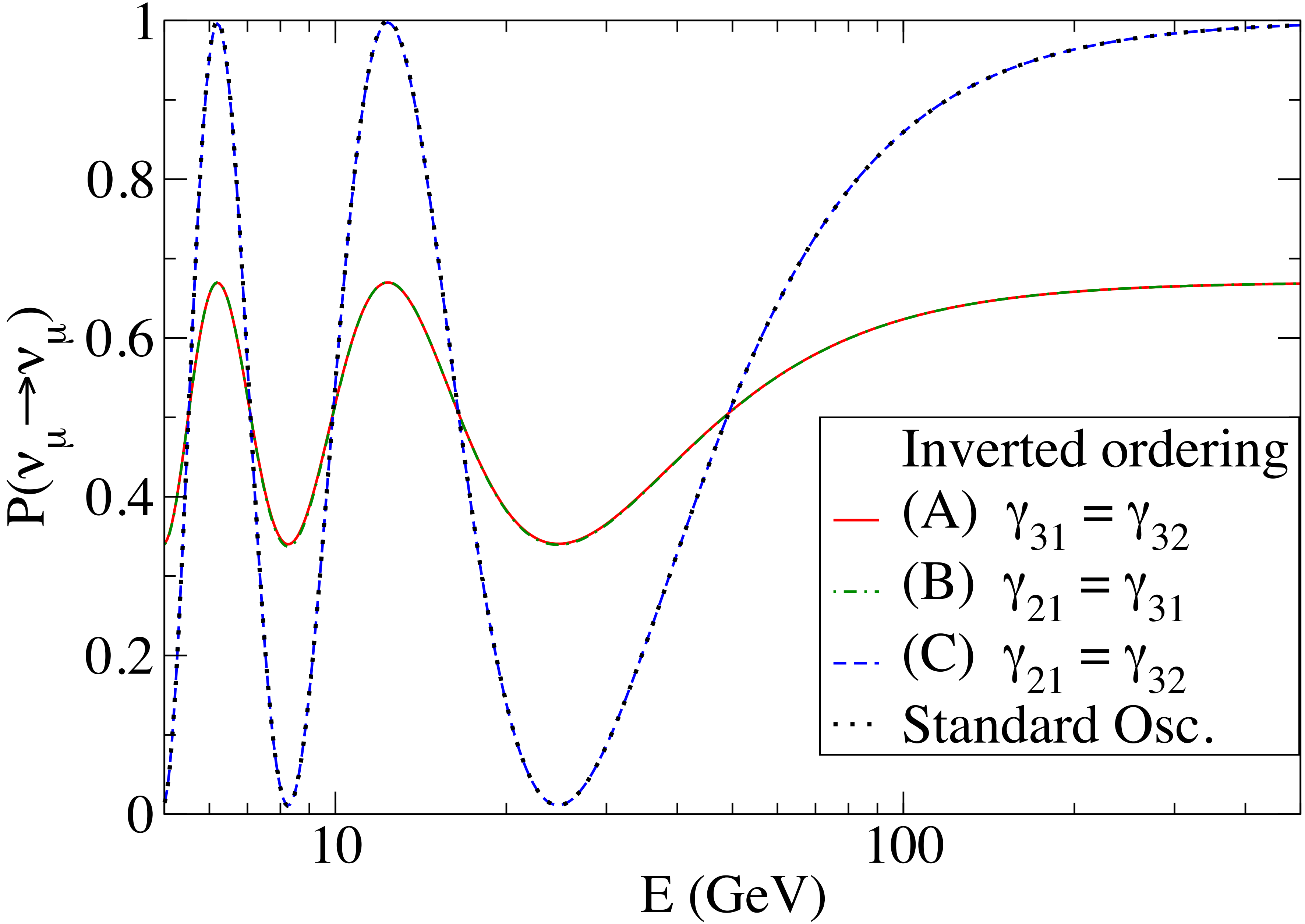
(C) Solar limit II: $\Upsilon_{31} = 0$ ($\Upsilon_{21} = \Upsilon_{32}$)

Some examples of oscillation probabilities



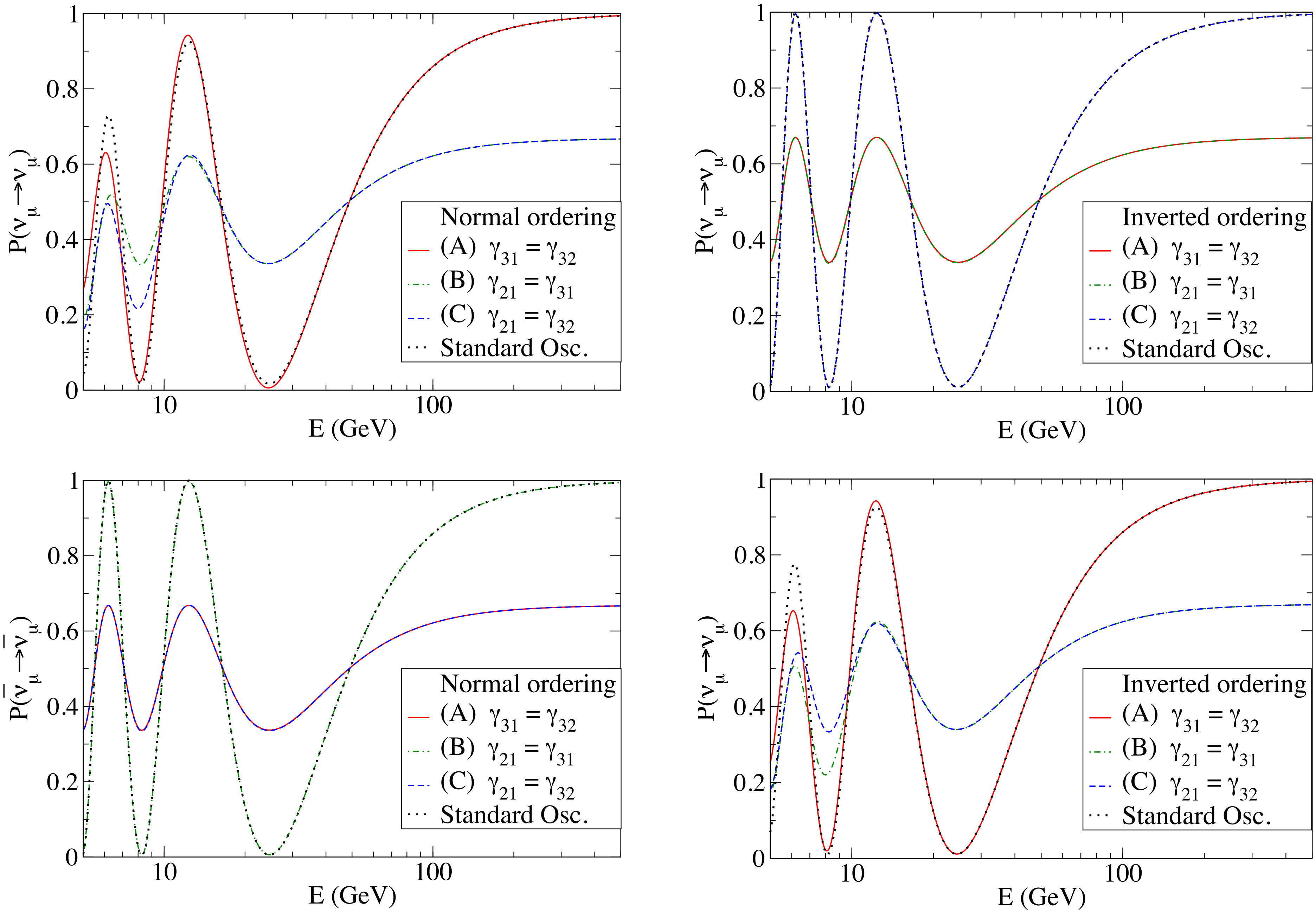
$\cos\theta_z = -1 \qquad \gamma = 2.3 \times 10^{-23}\ \mathrm{GeV}$

Some examples of oscillation probabilities



$\cos\theta_z = -1 \qquad \gamma = 2.3\times 10^{-23}\ \mathrm{GeV}$

Some examples of oscillation probabilities



$$\cos\theta_z = -1 \qquad \gamma = 2.3 \times 10^{-23}\ \mathrm{GeV}$$

We consider the following 3 distinct cases

(A) Atmospheric limit: $\Upsilon_{21} = 0$ ($\Upsilon_{32} = \Upsilon_{31}$)

(B) Solar limit I: $\Upsilon_{32} = 0$ ($\Upsilon_{21} = \Upsilon_{31}$)

(C) Solar limit II: $\Upsilon_{31} = 0$ ($\Upsilon_{21} = \Upsilon_{32}$)

there are following approximated correspondences

NO of (A)	$\longleftrightarrow$	IO of (C)	$\gamma_{21}, \tilde{\Delta}_{21} \leftrightarrow \gamma_{31}, \tilde{\Delta}_{31}$
NO of (B)	$\longleftrightarrow$	IO of (A)	$\gamma_{32}, \tilde{\Delta}_{32} \leftrightarrow \gamma_{21}, \tilde{\Delta}_{21}$
NO of (C)	$\longleftrightarrow$	IO of (B)	$\gamma_{31}, \tilde{\Delta}_{31} \leftrightarrow \gamma_{32}, \tilde{\Delta}_{32}$

NO (IO): Normal (Inverted) mass Ordering

For higher energy (> 15 GeV) neutrinos,

for normal mass ordering

$$P_{\mu\mu}^{\mathrm{NO}} \approx 1 - \frac{1}{2}\sin^2 2\theta_{23}\left(1 - e^{-\gamma_{21}L}\cos\tilde{\Delta}_{21}\right) \quad \text{for } \nu$$

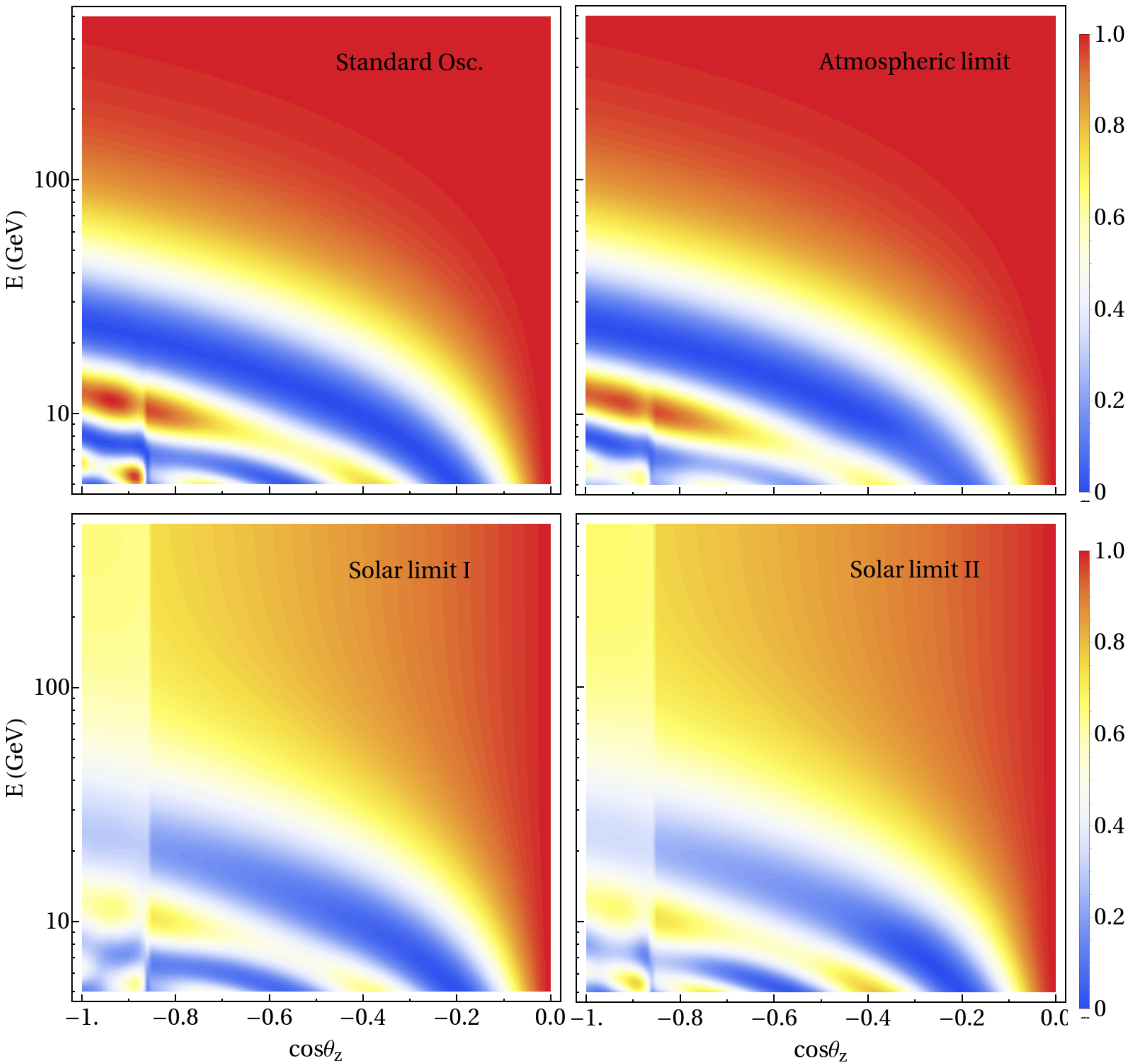
$$P_{\bar{\mu}\bar{\mu}}^{\mathrm{NO}} \approx 1 - \frac{1}{2}\sin^2 2\theta_{23}\left(1 - e^{-\gamma_{32}L}\cos\tilde{\Delta}_{32}\right) \quad \text{for } \bar{\nu}$$

for inverted mass ordering

$$P_{\mu\mu}^{\mathrm{IO}} \approx 1 - \frac{1}{2}\sin^2 2\theta_{23}\left(1 - e^{-\gamma_{31}L}\cos\tilde{\Delta}_{31}\right) \quad \text{for } \nu$$

$$P_{\bar{\mu}\bar{\mu}}^{\mathrm{IO}} \approx 1 - \frac{1}{2}\sin^2 2\theta_{23}\left(1 - e^{-\gamma_{21}L}\cos\tilde{\Delta}_{21}\right) \quad \text{for } \bar{\nu}$$

To see the effect more globally ν oscillogram is useful



Analysis Procedure

IceCube: Poissonian log-likelihood analysis

DeepCore: Gaussian Maximum likelihood analysis

For each analysis, simultaneous fit on the parameters of

$$\Delta m^2_{32}, \theta_{23} \text{ and } \gamma_{ij}$$

was performed

Analysis Procedure

For IceCube, we consider the data presented in PhD thesis by B.J.P. Jones, available at http://hdl.handle.net/1721.1/101327

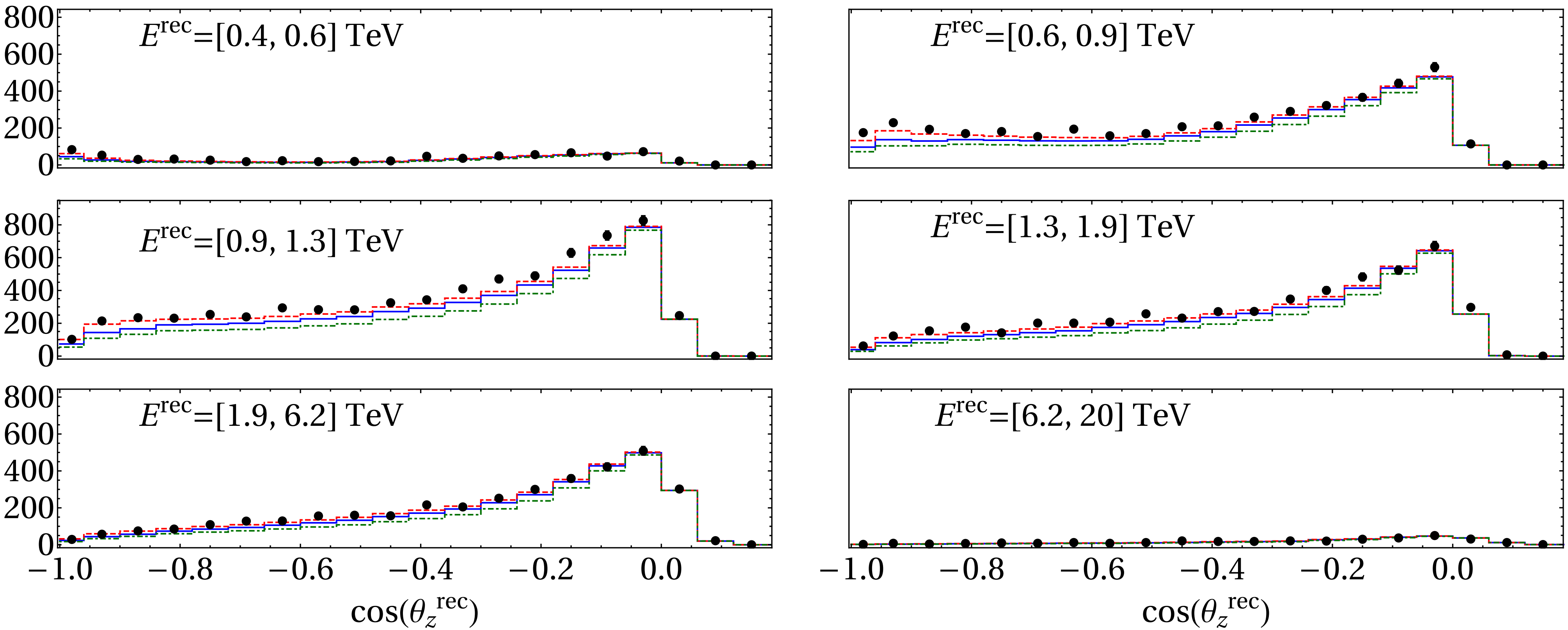
from 400 GeV to 20 TeV, from $\cos\theta_z$= -1.02 to 0.24

systematic errors for IceCube data

Source of uncertainty	Value
Flux - normalization	Free
Flux - π/K ratio	10%
Flux - energy dependence as $(E/E_0)^\eta$	$\Delta\eta = 0.05$
Flux - $\bar{\nu}/\nu$	2.5%
DOM efficiency	5%
Photon scattering	10%
Photon absorption	10%

Analysis Procedure

Expected event distribution for IceCube with and w/o decoherence



IceCube Data: B.J.P.Jones, PhD thesis http://hdl.handle.net/1721.1/101327

Analysis Procedure

For DeepCore we consider the data presented in Aartsen et al., PRL117, 071801 (2016)

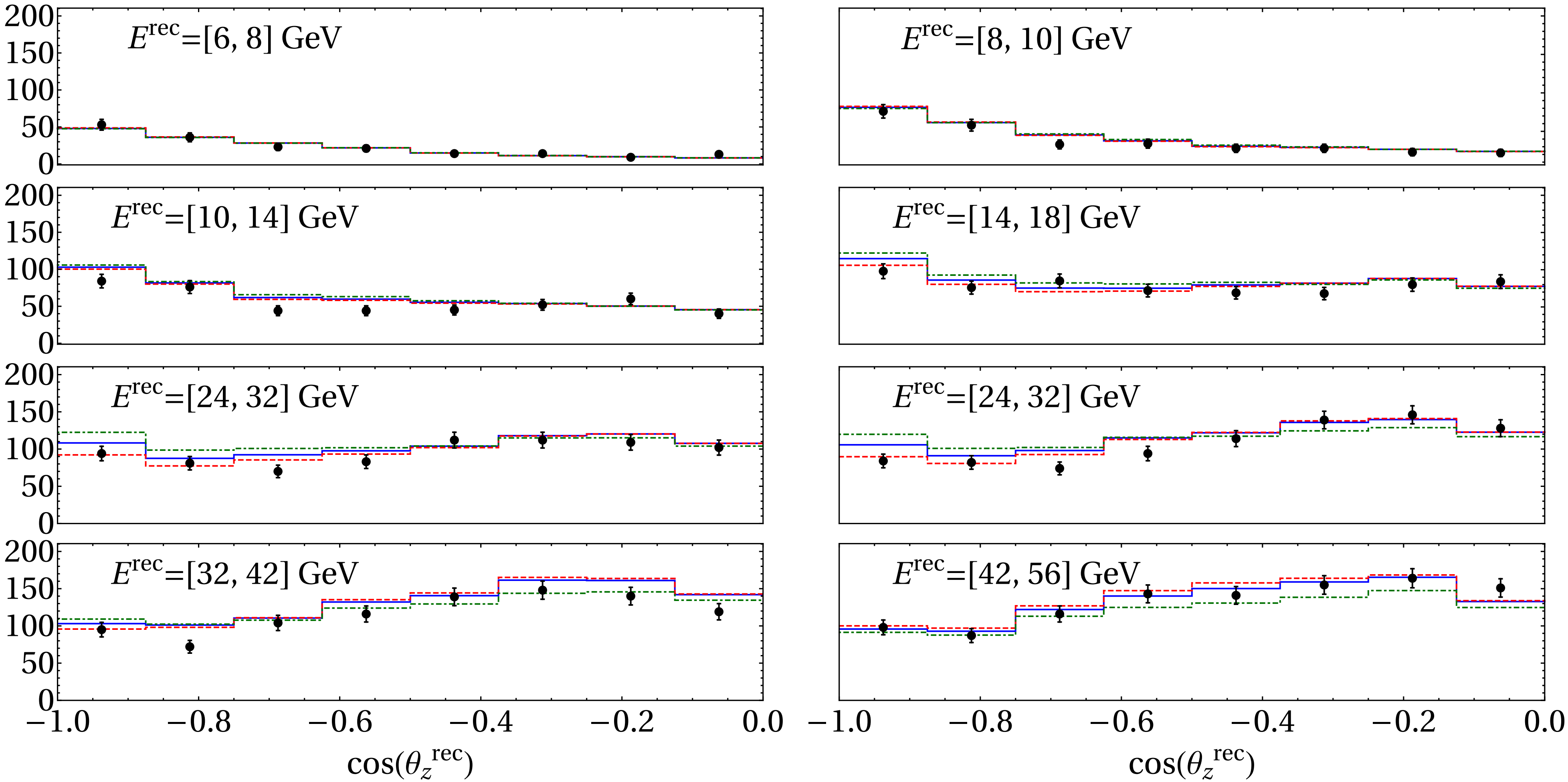
from ~10 GeV to ~1 TeV, 8 bins below $\cos\theta_z = 0$

systematic errors for DeepCore data

Source of uncertainty	Value
Flux - normalization	Free
Flux - energy dependence as $(E/E_0)^\eta$	$\Delta\eta = 0.05$
Flux - $(\nu_e + \bar{\nu}_e)/(\nu_\mu + \bar{\nu}_\mu)$ ratio	20%
Background - normalization	Free
DOM efficiency	10%
Optical properties of the ice	1%

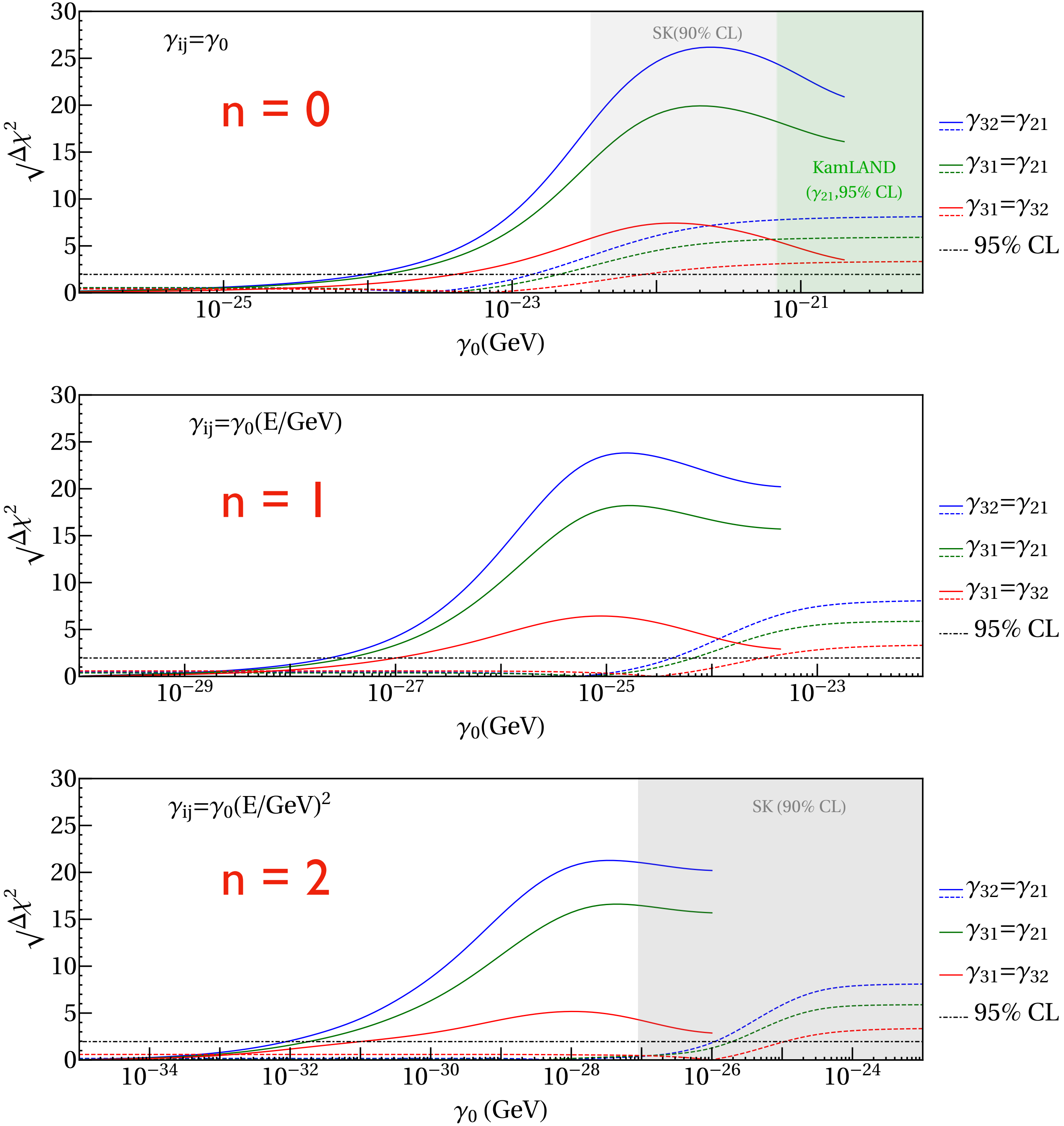
Analysis Procedure

Expected event distribution for DeepCore with and w/o decoherence



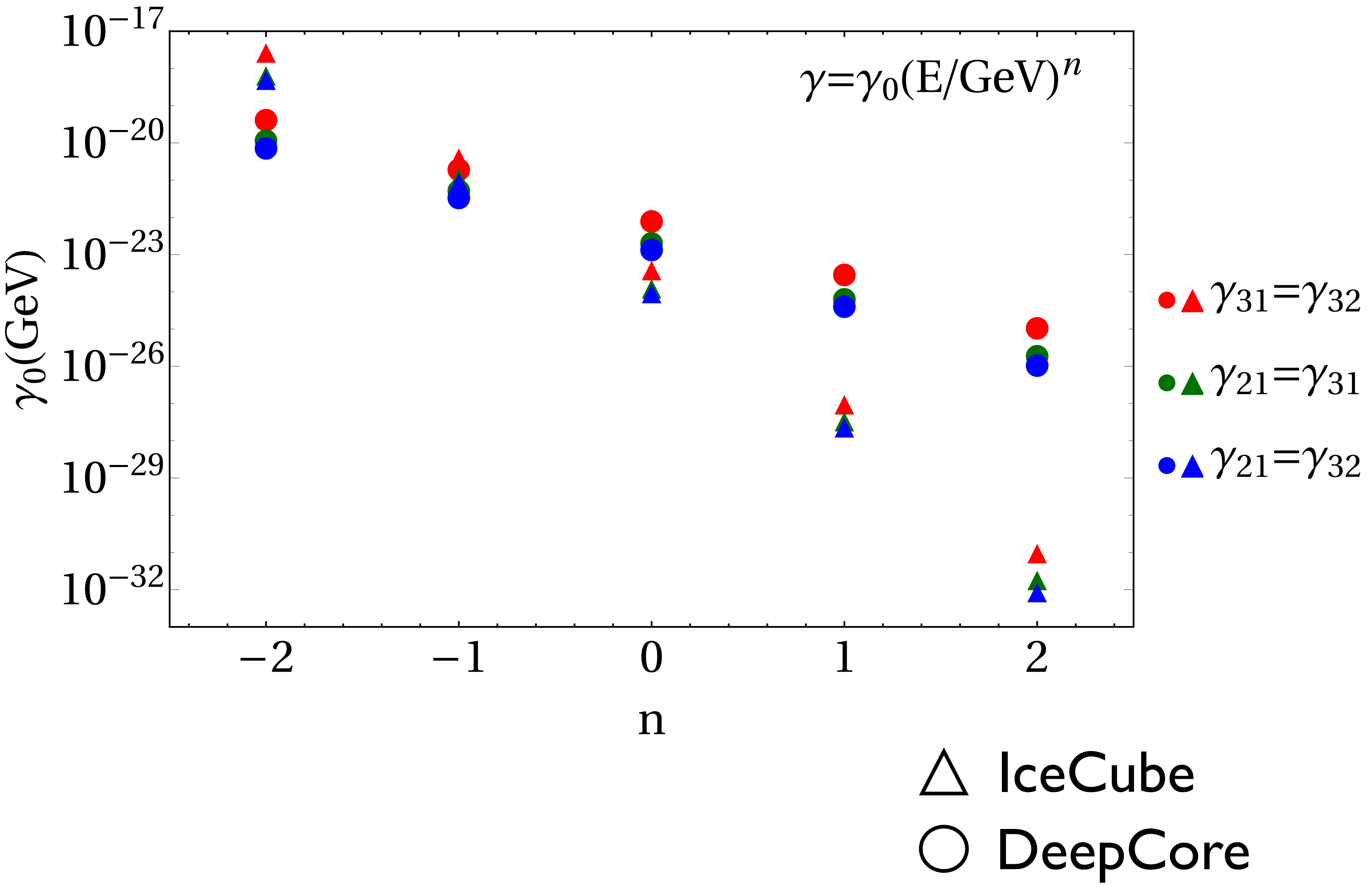
DeepCore Data: IceCube Collab. PRL117, 071701 (2016)

Bounds from IceCube and DeepCore

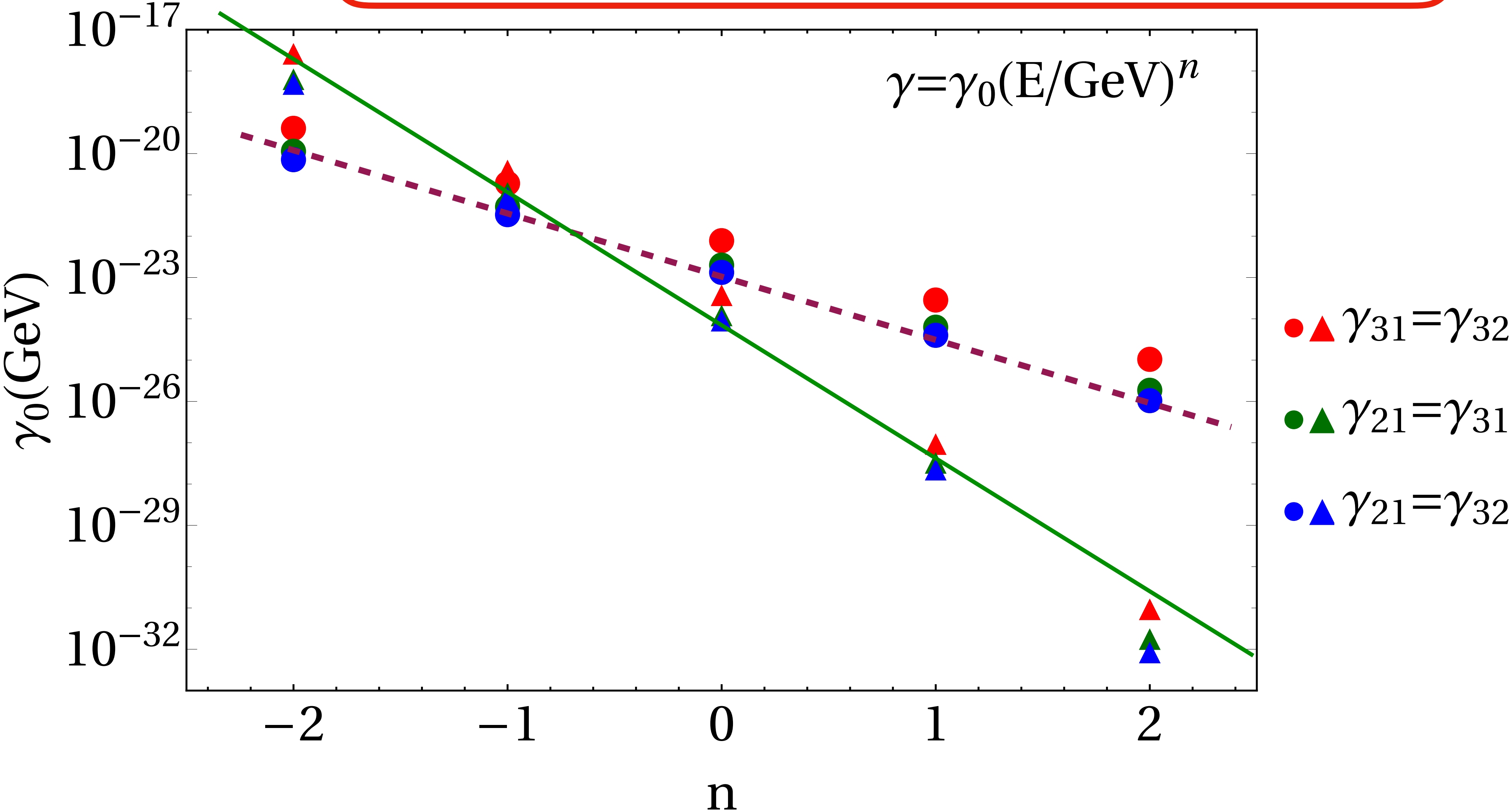


solid curves: IC

dashed curves: DC


10^{-17}
10^{-20}
10^{-23}
10^{-26}
10^{-29}
10^{-32}
γ_0(GeV)
γ=γ_0(E/GeV)^n
−2
−1
0
1
2
n
γ_{31}=γ_{32}
γ_{21}=γ_{31}
γ_{21}=γ_{32}
IceCube
DeepCore

roughly, $\boxed{\begin{array}{l}\ln(\gamma_0/\mathrm{GeV}) \sim \text{constant} - n\ln(E_0/\mathrm{GeV}) \\ \quad E_0 \sim 2.5\ \mathrm{TeV}(30\mathrm{GeV})\ \text{for IceCube (DeepCore)}\end{array}}$



△ IceCube

○ DeepCore

Summary of Bounds we obtained

		$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$
Normal Ordering	**IceCube (this work)**					
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$2.8\cdot10^{-18}$	$4.2\cdot10^{-21}$	$\mathbf{4.0\cdot10^{-24}}$	$\mathbf{1.0\cdot10^{-27}}$	$\mathbf{1.0\cdot10^{-31}}$
	Solar I ($\gamma_{31}=\gamma_{21}$)	$6.8\cdot10^{-19}$	$1.2\cdot10^{-21}$	$\mathbf{1.3\cdot10^{-24}}$	$\mathbf{3.5\cdot10^{-28}}$	$\mathbf{1.9\cdot10^{-32}}$
	Solar II ($\gamma_{32}=\gamma_{21}$)	$5.2\cdot10^{-19}$	$9.2\cdot10^{-22}$	$9.7\cdot10^{-25}$	$2.4\cdot10^{-28}$	$9.0\cdot10^{-33}$
	DeepCore (this work)					
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$\mathbf{4.3\cdot10^{-20}}$	$2.0\cdot10^{-21}$	$8.2\cdot10^{-23}$	$3.0\cdot10^{-24}$	$1.1\cdot10^{-25}$
	Solar I ($\gamma_{31}=\gamma_{21}$)	$\mathbf{1.2\cdot10^{-20}}$	$5.4\cdot10^{-22}$	$2.1\cdot10^{-23}$	$6.6\cdot10^{-25}$	$2.0\cdot10^{-26}$
	Solar II ($\gamma_{32}=\gamma_{21}$)	$7.5\cdot10^{-21}$	$3.5\cdot10^{-22}$	$1.4\cdot10^{-23}$	$4.2\cdot10^{-25}$	$1.1\cdot10^{-26}$
Inverted Ordering	**IceCube (this work)**					
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$6.8\cdot10^{-19}$	$1.2\cdot10^{-21}$	$\mathbf{1.3\cdot10^{-24}}$	$\mathbf{3.5\cdot10^{-28}}$	$\mathbf{1.9\cdot10^{-32}}$
	Solar I ($\gamma_{31}=\gamma_{21}$)	$5.2\cdot10^{-19}$	$9.2\cdot10^{-22}$	$9.8\cdot10^{-25}$	$2.4\cdot10^{-28}$	$9.0\cdot10^{-33}$
	Solar II ($\gamma_{32}=\gamma_{21}$)	$2.8\cdot10^{-18}$	$4.2\cdot10^{-21}$	$\mathbf{4.1\cdot10^{-24}}$	$\mathbf{1.0\cdot10^{-27}}$	$\mathbf{1.0\cdot10^{-31}}$
	DeepCore (this work)					
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$\mathbf{1.4\cdot10^{-20}}$	$5.8\cdot10^{-22}$	$2.2\cdot10^{-23}$	$7.5\cdot10^{-25}$	$2.3\cdot10^{-26}$
	Solar I ($\gamma_{31}=\gamma_{21}$)	$8.3\cdot10^{-21}$	$3.6\cdot10^{-22}$	$1.4\cdot10^{-23}$	$4.7\cdot10^{-25}$	$1.3\cdot10^{-26}$
	Solar II ($\gamma_{32}=\gamma_{21}$)	$\mathbf{5.0\cdot10^{-20}}$	$2.3\cdot10^{-21}$	$9.4\cdot10^{-23}$	$3.3\cdot10^{-24}$	$1.2\cdot10^{-25}$
	Previous Bounds					
	SK (two families) [7]		$2.4\cdot10^{-21}$	$4.2\cdot10^{-23}$		$1.1\cdot10^{-27}$
	MINOS (γ_{31},γ_{32}) [32]		$\mathbf{2.5\cdot10^{-22}}$	$1.1\cdot10^{-22}$	$2\cdot10^{-24}$	
	KamLAND (γ_{21}) [15]		$\mathbf{3.7\cdot10^{-24}}$	$6.8\cdot10^{-22}$	$1.5\cdot10^{-19}$	

Summary of Bounds we obtained

		$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
Normal Ordering	**IceCube (this work)**						
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$2.8\cdot10^{-18}$	$4.2\cdot10^{-21}$	$\mathbf{4.0\cdot10^{-24}}$	$\mathbf{1.0\cdot10^{-27}}$	$\mathbf{1.0\cdot10^{-31}}$	(1)
	Solar I ($\gamma_{31}=\gamma_{21}$)	$6.8\cdot10^{-19}$	$1.2\cdot10^{-21}$	$\mathbf{1.3\cdot10^{-24}}$	$\mathbf{3.5\cdot10^{-28}}$	$\mathbf{1.9\cdot10^{-32}}$	(2)
	Solar II ($\gamma_{32}=\gamma_{21}$)	$5.2\cdot10^{-19}$	$9.2\cdot10^{-22}$	$9.7\cdot10^{-25}$	$2.4\cdot10^{-28}$	$9.0\cdot10^{-33}$	(3)
	DeepCore (this work)						
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$\mathbf{4.3\cdot10^{-20}}$	$2.0\cdot10^{-21}$	$8.2\cdot10^{-23}$	$3.0\cdot10^{-24}$	$1.1\cdot10^{-25}$	(4)
	Solar I ($\gamma_{31}=\gamma_{21}$)	$\mathbf{1.2\cdot10^{-20}}$	$5.4\cdot10^{-22}$	$2.1\cdot10^{-23}$	$6.6\cdot10^{-25}$	$2.0\cdot10^{-26}$	(5)
	Solar II ($\gamma_{32}=\gamma_{21}$)	$7.5\cdot10^{-21}$	$3.5\cdot10^{-22}$	$1.4\cdot10^{-23}$	$4.2\cdot10^{-25}$	$1.1\cdot10^{-26}$	(6)
Inverted Ordering	**IceCube (this work)**						
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$6.8\cdot10^{-19}$	$1.2\cdot10^{-21}$	$\mathbf{1.3\cdot10^{-24}}$	$\mathbf{3.5\cdot10^{-28}}$	$\mathbf{1.9\cdot10^{-32}}$	~(2)
	Solar I ($\gamma_{31}=\gamma_{21}$)	$5.2\cdot10^{-19}$	$9.2\cdot10^{-22}$	$9.8\cdot10^{-25}$	$2.4\cdot10^{-28}$	$9.0\cdot10^{-33}$	~(3)
	Solar II ($\gamma_{32}=\gamma_{21}$)	$2.8\cdot10^{-18}$	$4.2\cdot10^{-21}$	$\mathbf{4.1\cdot10^{-24}}$	$\mathbf{1.0\cdot10^{-27}}$	$\mathbf{1.0\cdot10^{-31}}$	~(1)
	DeepCore (this work)						
	Atmospheric ($\gamma_{31}=\gamma_{32}$)	$\mathbf{1.4\cdot10^{-20}}$	$5.8\cdot10^{-22}$	$2.2\cdot10^{-23}$	$7.5\cdot10^{-25}$	$2.3\cdot10^{-26}$	~(5)
	Solar I ($\gamma_{31}=\gamma_{21}$)	$8.3\cdot10^{-21}$	$3.6\cdot10^{-22}$	$1.4\cdot10^{-23}$	$4.7\cdot10^{-25}$	$1.3\cdot10^{-26}$	~(6)
	Solar II ($\gamma_{32}=\gamma_{21}$)	$\mathbf{5.0\cdot10^{-20}}$	$2.3\cdot10^{-21}$	$9.4\cdot10^{-23}$	$3.3\cdot10^{-24}$	$1.2\cdot10^{-25}$	~(4)
	Previous Bounds						
	SK (two families) [7]		$2.4\cdot10^{-21}$	$4.2\cdot10^{-23}$		$1.1\cdot10^{-27}$	
	MINOS (γ_{31},γ_{32}) [32]		$\mathbf{2.5\cdot10^{-22}}$	$1.1\cdot10^{-22}$	$2\cdot10^{-24}$		
	KamLAND (γ_{21}) [15]		$\mathbf{3.7\cdot10^{-24}}$	$6.8\cdot10^{-22}$	$1.5\cdot10^{-19}$		

Conclusions

- We revisit the quantum decoherence in the context neutrino oscillation with full 3 flavor framework
- We found that the bounds and/or sensitivities depend strongly on the neutrino mass ordering and matter effect is important
- For neutrinos, the decoherence effect is mainly driven by Υ_{21} (Υ_{31}) for normal (inverted) mass ordering
- For antineutrinos, the decoherence effect is mainly driven by Υ_{32} (Υ_{21}) for normal (inverted) mass ordering
- 3 flavor analysis is required to interpret correctly the bounds on the decoherence parameters
- We obtained better (improved) bounds for most of the cases except for $n = -1$

Thank you very much for your attention!

backup slides

Neutrino propagation in non-uniform matter: adiabatic regime

relevant for solar ${}^{8}B$ neutrinos

$$P_{\alpha\beta} = \langle \nu_\beta | \hat{\rho}^{(\alpha)}(t) | \nu_\beta \rangle = \sum_{i,j} \tilde{\rho}^{(\alpha)}_{ij}(0) e^{-\left[\gamma_{ij} - i\Delta\tilde{h}_{ij}\right]t} \langle \nu_\beta | \tilde{\nu}^{eff}_i \rangle \langle \tilde{\nu}^{eff}_j | \nu_\beta \rangle,$$

$$\tilde{\rho}^{(e)}_{ij}(0) = \tilde{U}^{0*}_{ei} \tilde{U}^{0}_{ej}$$

$$\begin{aligned} P_{e\beta} &\approx \sum_{i,j} \tilde{U}^{0*}_{ei} U_{\beta i} \tilde{U}^{0}_{ej} U^{*}_{\beta j} e^{-\left[\gamma_{ij} - i\Delta\tilde{h}_{ij}\right]t} \\ &= \sum_i |\tilde{U}^0_{ei}|^2 |U_{\beta i}|^2 + 2\sum_{i<j} \mathrm{Re}\left[\tilde{U}^{0*}_{ei} U_{\beta i} \tilde{U}^{0}_{ej} U^{*}_{\beta j}\right] e^{-\gamma_{ij}t} \cos\tilde{\Delta}_{ij} \\ &\qquad -2\sum_{i<j} \mathrm{Im}\left[\tilde{U}^{0*}_{ei} U_{\beta i} \tilde{U}^{0}_{ej} U^{*}_{\beta j}\right] e^{-\gamma_{ij}t} \sin\tilde{\Delta}_{ij}. \end{aligned}$$

after averaging out, finally we have

$$P_{e\beta} \approx \sum_i |\tilde{U}^0_{ei}|^2 |U_{\beta i}|^2$$

$$
\cos 2\tilde{\theta}_{13} = \frac{\cos 2\theta_{13} - a/\Delta m^2_{ee}}{\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}},
$$

$$
\cos 2\tilde{\theta}_{12} = \frac{\cos 2\theta_{12} - a'/\Delta m^2_{21}}{\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12} \cos^2(\tilde{\theta}_{13} - \theta_{13})}},
$$

$$
\Delta\tilde{m}^2_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12} \cos^2(\tilde{\theta}_{13} - \theta_{13})},
$$

$$
\Delta\tilde{m}^2_{31} = \Delta m^2_{31} + (a - \frac{3}{2}a') + \frac{1}{2}(\Delta\tilde{m}^2_{21} - \Delta m^2_{21}),
$$

$$
\Delta\tilde{m}^2_{32} = \Delta\tilde{m}^2_{31} - \Delta\tilde{m}^2_{21}.
$$

where

$$
\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}
$$

$$
a' = a\cos^2\tilde{\theta}_{13} + \Delta m^2_{ee}\sin^2(\tilde{\theta}_{13} - \theta_{13})
$$

Denton, Minakata, Parke, arXiv:1801.06514

Results of More general analysis varying 2 decoherence parameters Υ_{31} and Υ_{21} freely

$$\gamma_{32} = \left(\sqrt{\gamma_{21}} \pm \sqrt{\gamma_{31}}\right)^2 .$$