Part 1
Degenerate Fast and Ultra-fast Diffusion

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Introduction

In this series of lectures we will discuss extrinsic geometric flows with emphasis on geometric and analytical aspects of degenerate (slow) and singular (fast) diffusion.

Examples degenerate (slow) diffusion:

- Porous medium equation
- Gauss curvature flow

Examples of singular (fast and ultra-fast) diffusion:

- Fast-diffusion equation
  (Ricci flow on $\mathbb{R}^2$ and Yamabe flow on $\mathbb{R}^n$, $n \geq 3$)
- Inverse mean curvature flow.

Emphasis will be given to:

- existence of entire graph solutions on $\mathbb{R}^n$ (non-compact case)
- a priori estimates of soutions
- optimal regularity of solutions

The Heat Equation

The simplest model of diffusion is the familiar heat equation:

$$u_t = \Delta u, \quad (x,t) \in \Omega \times [0,T], \quad \Omega \subset \mathbb{R}^n$$

(u is the density of heat, chemical concentration etc.)

Fundamental properties of the Heat equation:

- Smoothing Effect: Solutions become instantly smooth, at time $t > 0$.
- Infinite Speed of Propagation: Solutions with non-negative compactly supported initial data $u(\cdot, 0)$, become instantly strictly positive, at time $t > 0$.
- The Fundamental Solution:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}}, \qquad t > 0.$$

A basic model of non-linear diffusion

We consider the simplest model of quasilinear diffusion:

$$(*) \qquad u_t = \Delta u^m = \mathrm{div}\,(m\,u^{m-1}\,\nabla u), \qquad u \geq 0$$

for various values exponents $m \in \mathbb{R}$.

- Porous medium equation (Slow Diffusion) $m > 1$:
  The diffusivity $D(u) = m\,u^{m-1} \downarrow 0$, as $u \downarrow 0$. $(*)$ becomes degenerate at $u = 0$, resulting to finite speed of propagation.
- Fast Diffusion $0 \leq m < 1$:
  The diffusivity $D(u) = m\,u^{m-1} \uparrow +\infty$, as $u \downarrow 0$. $(*)$ becomes singular at $u = 0$, resulting to fast diffusion.
- Ultra-Fast Diffusion $m < 0$:
  When $m < 0$ we have ultra-fast diffusion with new interesting phenomena for example instant vanishing in some cases.
- Equation $(*)$ appears in many physical applications and in geometry (2 - dim Ricci flow and $n \geq 3$ - dim Yamabe flow).

Contraction of hyper-surfaces by functions of their principal curvatures

An extrinsic geometric flow of co-dim one is typically the evolution of an n-dimensional hyper-surface M_t^n embedded in $\mathbb{R}^{n+1}$ by:

$$\frac{\partial P}{\partial t} = \sigma \, \nu$$

with speed $\sigma = \sigma(\lambda_1, \cdots, \lambda_n)$ a smooth function of the principal curvatures λ_i of the surface M_t.

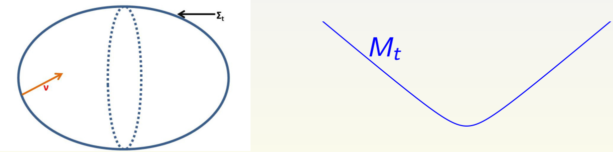


Figure: Hypersurface M_t^n compact in $\mathbb{R}^{n+1}$ or a graph over $\mathbb{R}^n$

Examples of Extrinsic Geometric flows:

- Mean curvature Flow (MCF): $\sigma = H = \lambda_1 + \cdots + \lambda_n$
- Mean curvature Flow (IMCF): $\sigma = -\frac{1}{H} = -\frac{1}{\lambda_1 + \cdots + \lambda_n}$
- Gauss curvature flow (GCF): $\sigma = K = \lambda_1 \cdots \lambda_n$
- GCF^α: $\sigma = K^\alpha = (\lambda_1 \cdots \lambda_n)^\alpha$, $0 < \alpha < \infty$.
- Harmonic mean curvature flow (HMCF): $\sigma = \frac{1}{\lambda_1^{-1} + \ldots + \lambda_n^{-1}}$.

Evolution Equations for Curvature flows

- CSF: Motion of a plane curve $y = u(x,t)$ by its Curvature

$$u_t = \frac{u_{xx}}{1+u_x^2}.$$

- MCF: Motion of a surface $z = u(x,y,t)$ in $\mathbb{R}^3$ by its Mean Curvature

$$u_t = \frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{1+|Du|^2}.$$

- GCF: Motion of a surface $y = u(x,y,t)$ in $\mathbb{R}^3$ by its Gaussian Curvature

$$u_t = \frac{\det D^2 u}{(1+|Du|^2)^{3/2}}.$$

It resembles the evolution Monge-Ampére equation.

Evolution Equations for Curvature flows

- IMCF: Motion of a surface $y = u(x,y,t)$ in $\mathbb{R}^3$ by its Inverse Mean Curvature

$$u_t = -\frac{(1+|Du|^2)^2}{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}.$$

Remarks:

- The CSF and MCF are strictly parabolic and quasi-linear.
- The GCF and GCF^α are fully-nonlinear. They become degenerate (slow-diffusion) when the Gauss curvature $K = 0$.
- The IMCF becomes singular (fast-diffusion) as the Mean curvature $H \to 0$.

Typical Questions

- Short and long time existence of solutions
- Regularity of solutions (classical or weak solutions)
- Free-boundaries
- Formation of singularities and convergence
- Final shape of the hyper-surface

Outline of lectures

1. Introduction to linear and nonlinear diffusion
   - Widder theory for the Heat equation
   - The Cauchy problem for the Porous medium equation
   - The Cauchy problem for the Fast and Ultra-fast diffusion equation
2. The Mean curvature flow on entire graphs
3. The Inverse mean curvature flow on entire graphs
4. The Gauss Curvature flow
   - The Gauss Curvature flow on complete on compact surfaces
   - Regularity in Gauss curvature flow
   - Firey's conjecture

The Heat Equation

The simplest model of diffusion is the heat equation:

$$u_t = \Delta u.$$

The fundamental properties of the heat equation are:

- Smoothing Effect: Solutions become instantly smooth, at time $t > 0$.
- Infinite Speed of Propagation: Solutions with non-negative compactly supported initial data $u(\cdot, 0)$, become instantly strictly positive, at time $t > 0$.

Both properties are shown in the Fundamental solution:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}\, e^{-\frac{|x|^2}{4t}}, \qquad t > 0$$

which has initial data the dirac mass δ_0.

The Widder theory for the Heat equation

- In the 1940s D. Widder studied the characterization of the class of all nonnegative weak solutions of the heat equation

$$(HE) \quad u_t = \Delta u \qquad \text{in } S_T = \mathbb{R}^n \times (0, T]$$

- Definition of weak solution: $u \in L^1_{\text{loc}}(S_T)$ and the equation holds in the distributional sense.
- Regularity: It follows by classical regularity theorems that the solution u instantly C^∞ smooth i.e. $u \in C^\infty(S_T)$.
- Scaling: u solves (HE) $\iff$ $v(x,t) := \frac{u(\alpha x, \alpha^2 t)}{\gamma}$ solves (HE)

The Widder theory for the Heat equation

Let u be a nonnegative weak solution of the (HE) on S_T. Then:

- There exists an absolute constant $C > 0$ such that:

$$\sup_{0<t<T/2} \int u(x,t)\, e^{-C\,|x|^2}\, dx \;< \infty.$$

- Existence of initial trace: there exists a nonnegative Borel measure μ on $\mathbb{R}^n$ such that

$$\lim_{t \downarrow 0} u(\cdot, t) = d\mu \qquad \text{in } D^1(\mathbb{R}^n).$$

  and satisfies satisfies the growth condition

$$(*) \qquad \int e^{-C\,\frac{|x|^2}{T}}\, d\mu \;< \infty.$$

- The solution is uniquely determined from its initial trace μ.

The Widder theory for the Heat equation

- For each nonnegative Borel measure μ on $\mathbb{R}^n$ satisfying

$$(*) \qquad \int e^{-C\,\frac{|x|^2}{T}}\,d\mu \,<\infty.$$

there exists a nonnegative continuous weak solution u of (HE) in S_T with trace μ.

- The solution u satisfies the pointwise estimate

$$(B) \qquad u(x,t) \leq C_t(u)\, e^{C\,|x|^2}$$

where C is an absolute constant and $C_t(u)$ depends on u.

- Important property: every $u \geq 0$ solution of (HE) satisfies the parabolic Harnack inequality from which (B) follows.
- Non-uniqueness: for changing sign solutions which do not satisfy (B).

The Porous Medium Equation

The simplest model of non-linear degenerate diffusion is the porous medium equation:

$$u_t = \Delta u^m = \mathrm{div}\,(m\,u^{m-1}\,\nabla u), \qquad m > 1.$$

- It describes various diffusion processes, for example the flow of gas through a porous medium, where u is the density of the gas and $v := u^{m-1}$ is the pressure of the gas.
- Since, the diffusivity $D(u) = m\,u^{m-1} \downarrow 0$, as $u \downarrow 0$ the equation becomes degenerate at $u = 0$, resulting to the phenomenon of finite speed of propagation.

Scaling

- Because it is nonlinear, the equation

$$(*) \qquad u_t = \Delta u^m, \qquad m \neq 1$$

  has rich scaling properties.

- If u is a solution of $(*)$, then

$$v(x,t) := \frac{u(\alpha x, \beta t)}{\gamma}$$

  is also a solution of (PM) if and only if

$$\gamma = \left(\frac{\alpha^2}{\beta}\right)^{1/(m-1)}.$$

The Aronson-Bénilan inequality

- Aronson-Bénilan Inequality: Every solution u to the p.m.e. satisfies the differential inequality

$$(*_1) \qquad u_t \geq -\frac{k\,u}{t}, \qquad \lambda = \frac{1}{(m-1)+\frac{2}{n}}.$$

- The *pressure* $v := \frac{m}{m-1}\,u^{m-1}$ which evolves by the equation

$$v_t = (m-1)\,v\,\Delta v + |\nabla v|^2$$

satisfies the differential inequality

$$(**) \qquad \Delta v \geq -\frac{\lambda}{t}.$$

- Remark: The Aronson-Bénilan $(**)$ is sharp and becomes equality when v is the self-similar Barenblatt solution: $v = t^{\mu}\,(C - k\,\frac{|x|^2}{t^{2\mu}})$.

The Li-Yau type Harnack inequality

- The Aronson-Bénilan inequality $\Delta v \geq -\frac{\lambda}{t}$ and the equation for v imply the Li-Yau type differential inequality:

$$v_t + (m-1)\,\lambda\,\frac{v}{t} \geq |\nabla v|^2.$$

- Integrating this inequality on optimal paths gives the following Harnack Inequality due to Auchmuty-Bao and Hamilton:

$$v(x_1,t_1) \leq \left(\frac{t_2}{t_1}\right)^{\mu} \left[v(x_2,t_2) + \frac{\delta}{4}\,\frac{|x_2-x_1|^2}{t_2^{\delta}-t_1^{\delta}}\,t_2^{-\mu} \right]$$

  if $0 < t_1 < t_2$, with $0 < \mu, \lambda < 1$ and $\delta > 0$.

- Application: If $v(0,T) < \infty$, then for all $0 < t < T - \epsilon$ we have:

$$v(x,t) \leq \; t^{-\mu}\,(T^{\mu}\,v(0,T) + C(n,m,\epsilon)\,|x|^2)$$

  i.e. the pressure v grows at most quadratically as $|x| \to \infty$.

"Easy" Aronson-Bénilan inequality

- Using simply the scaling of the equation one may show the easy (weaker) Aronson-Bénilan Inequality:

$$(*_2) \qquad u_t \geq -\frac{u}{(m-1)t}.$$

- Proof: If $u_t = \Delta u^m$, then $u_\lambda := \lambda u(x, \lambda^{m-1} t), \lambda > 1$ also satisfies the same equation. Moreover

$$u_\lambda(x,0) \geq u(x,0) \overset{CP}{\Longrightarrow} u_\lambda(x,t) \geq u(x,t), \;\; t > 0$$

Thus,

$$\frac{d}{d\lambda}|_{\lambda=1} u_\lambda \geq 0.$$

But

$$\frac{d}{d\lambda} u_\lambda = (m-1)\, \lambda^{m-2} t\, u_t + u$$

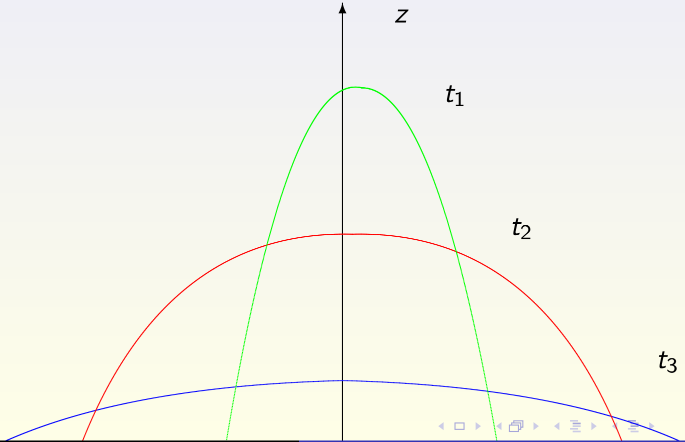
so we conclude for $\lambda = 1$ that

$$(m-1)\, t\, u_t + u \geq 0 \qquad \text{QED !!}.$$

The Barenblatt Solution

The Barenblatt solution: $U(x,t) = t^{-\lambda} \left(C - k \, \frac{|x|^2}{t^{2\mu}} \right)_+^{\frac{1}{m-1}}$ with $\lambda, \mu, k > 0$. It plays the role of the fundamental solution.

For $0 < t_1 < t_2 < t_3$ we have:



Finite Speed of propagation

The Barenblatt solution shows that solutions to the p.m.e have the following properties:

- Finite speed of propagation: If the initial data u_0 has compact support, then the solution $u(\cdot, t)$ will have compact support at all times t.
- Free-boundaries: The interface $\Gamma = \partial(\overline{\text{supp}\, u})$ behaves like a free-boundary propagating with finite speed.
- Solutions are not smooth: Solutions with compact support are only of class C^α near the interface.
- Weak solutions: We say that $u \geq 0$ is a weak solution of the equation $u_t = \Delta u^m$ in $Q_T := \Omega \times (0, T)$, if it is continuous on Q_T and satisfies the equations in the distributional sense.

The Cauchy problem with general initial data

Let $u \geq 0$ be a weak solution of $u_t = \Delta u^m$ on $\mathbb{R}^n \times (0, T]$.

- The initial trace μ_0 exists; there exists a Borel measure μ such that
$$\lim_{t \downarrow 0} u(\cdot, t) = \mu_0 \qquad \text{in} \quad D'(\mathbb{R}^n)$$
and satisfies the growth condition
$$(*) \qquad \sup_{R>1} \frac{1}{R^{n+2/(m-1)}} \int_{|x|<R} d\mu_0 < \infty.$$
- The trace μ_0 determines the solution uniquely.
- For every measure μ_0 on $\mathbb{R}^n$ satisfying $(*)$ there exists a continuous weak solution u of the p.m.e. with trace μ_0.
- All solutions satisfy the estimate $u(x, t) \leq C_t(u)\, |x|^{2/(m-1)}$, as $|x| \to \infty$.

Regularity of solutions -Two important estimates from linear theory

Let $u \in C^2(Q_\rho)$, $Q_\rho := B_\rho(0) \times (-\rho^2, 0]$, be a solution of:

$$u_t = a_{ij}\, D_{ij} u + b_i\, D_i u + c\, u$$

where

$$(*) \quad \lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \quad |b_i| + |c| \le \Lambda$$

- Schauder $C^{2+\alpha}$ estimate: If $a_{ij}, b_i, c \in C^\alpha(Q_\rho)$, then

$$\|u\|_{C^{2+\alpha}(Q_{\frac{\rho}{2}})} \le C\, \|u\|_{L^\infty(Q_\rho)}.$$

- Krylov-Safonov estimate: Under $(*)$, there exists $\gamma > 0$ such that

$$\|u\|_{C^\gamma(Q_{\frac{\rho}{2}})} \le C\, \|u\|_{L^\infty(Q_\rho)}.$$

Regularity of solutions

- Assume that u is a continuous weak solution of equation

$$u_t = \Delta u^m, \;\; m > 1 \quad \text{on } Q_\rho := B_\rho(0) \times (-\rho^2, 0].$$

- Remark: It follows from parabolic regularity theory that if $0 < \lambda \le u \le \Lambda$ in a parabolic cylinder Q_ρ, then $u \in C^\infty(Q_{\rho/2})$.

  Proof: If $0 < \lambda \le u \le \Lambda$ in Q, then $u_t = \operatorname{div}\left(m\, u^{m-1}\, \nabla u\right)$ is strictly parabolic with bounded measurable coefficients.

  It follows from the *Krylov-Safonov* estimate that $u \in C^\gamma$, for some $\gamma > 0$, hence $D(u) := m\, u^{m-1} \in C^\alpha$.

  We conclude that from the *Schauder estimate* that $u \in C^{2+\alpha}$ and by repeating then same estimate we obtain that $u \in C^\infty$.

Regularity of solutions

- Question: What is the optimal regularity of the solution u ?
- Caffarelli and Friedman: The solution u is of class C^α, for some $\alpha > 0$.
- This result is, in some sense, optimal: The Barenblatt solution $U(x,t) = t^{-\lambda} \left(C - k \frac{|x|^2}{t^{2\mu}} \right)_+^{\frac{1}{m-1}}$ with $\lambda, \mu, k > 0$ is only of class C^α near the interface $u = 0$.
- Question: Is it true that $u^{m-1} \in C^{0,1}$?

Fast Diffusion Equations

- Consider the non-linear of fast diffusion equation

$$u_t = \Delta u^m = \operatorname{div}\left(m\, u^{m-1}\, \nabla u\right), \qquad m < 1.$$

- It appears in physical applications such as diffusion in plasma and thin liquid film dynamics among other.

- Nonnegative weak solution: a continuous function $u \geq 0$ which satisfies the equation in distributional sense.

- Since, the diffusivity $D(u) = m\, u^{m-1} \uparrow +\infty$, as $u \downarrow 0$ the eq becomes singular at $u = 0$, resulting to fast-diffusion.

Scaling and the Barenblatt solution

- Scaling: If u solves the fast diffusion equation $u_t = \Delta u^m$, then

$$\tilde{u}(x,t) = \gamma^{-1}\, u(\alpha\, x, \beta\, t), \qquad \gamma = \left(\frac{\beta}{\alpha^2}\right)^{\frac{1}{1-m}}$$

also solves the same equation.

- Self-Similar solution: There exists a self-similar solution

$$U(x,t) = t^{-\lambda}\left(C + k\,\frac{|x|^2}{t^{2\mu}}\right)^{-\frac{1}{1-m}}$$

with

$$\lambda^{-1} = \frac{2}{n} - (1-m), \quad \mu = \frac{\lambda}{n}, \quad k = \frac{\lambda\,(1-m)}{2mn}.$$

The above is a solution if $\frac{2}{n} - (1-m) > 0$, i.e. $m > \frac{n-2}{n}$.

- The exponent $m = \frac{n-2}{n}$ is **critical**.

The Aronson-Bénilan inequality for $\frac{(n-2)_+}{n} < m < 1$

- If u is a solution to the fast-diffusion equation $u_t = \Delta u^m$, $m < 1$, then the pressure $v := \frac{m}{1-m}\, u^{-(1-m)}$ evolves by:

$$v_t = (1-m)\, v\, \Delta v - |\nabla v|^2.$$

- In the range of exponents $\frac{(n-2)_+}{n} < m < 1$, the pressure v satisfies the sharp Aronson-Bénilan inequality

$$(*_1) \qquad \Delta v \leq \frac{\lambda}{t}, \qquad \lambda = \lambda(m,n) > 0.$$

which implies the following Li-Yau type differential inequality

$$(*_2) \qquad -v_t + (1-m)\,\lambda\,\frac{v}{t} \geq |\nabla v|^2.$$

- The differential inequality $(*_2)$ becomes an equality when v is the self-similar solution $U(x,t) = t^{-\lambda} \left(C + k\, \frac{|x|^2}{t^{2\mu}} \right)^{-\frac{1}{1-m}}$.

The Harnack Inequality

- Integrating the inequality $(*_2)$ on optimal paths gives the Harnack Inequality due to Auchmuty-Bao and Hamilton:

$$v(x_2, t_2) \le \left(\frac{t_2}{t_1}\right)^{\mu} \left[v(x_1, t_1) + \frac{\delta}{4} \frac{|x_2 - x_1|^2}{t_2^{\delta} - t_1^{\delta}} \, t_1^{\mu} \right]$$

for $0 < t_1 < t_2$, with $\mu = \mu(m, n) > 0$ and $\delta = \delta(m, n) > 0$.

- Application: Solutions of $u_t = \Delta u^m$, with $\frac{(n-2)_+}{n} < m < 1$ satisfy the lower bound $u(x,t) \ge c(t)\,(1 + |x|^2)^{-\frac{1}{1-m}}$.
- Conclusion: Solutions become instantly strictly positive and remain so for all time.
- Remark: This is not true in the sub-critical range of exponents $m < \frac{(n-2)_+}{n}$, where solutions may vanish in finite time.

The other Aronson-Bénilan inequality

- A simple scaling argument shows that every solution u to the fast-diffusion equation $u_t = \Delta u^m$, $0 \leq m < 1$ satisfies the differential inequality

$$(*_3) \qquad u_t \leq \frac{u}{(1-m)\,t}.$$

- Integrating $(*_3)$ in time implies: $u(x,t_2) \leq u(x,t_1)\left(\frac{t_2}{t_1}\right)^{\frac{1}{1-m}}$, $\forall x \in \mathbb{R}^n$ i.e. the L^∞ norm of a solution doesn't blow up, if it is initially finite.
- Remark: In the range of exponents $\frac{(n-2)_+}{n} < m < 1$, solutions u exhibit a regularizing effect from L^1_{loc} to L^∞_{loc}:

$$\sup_{|x|\leq R} u(x,t) \leq F\left(t, R, \int_{B_{2R}} u_0(x)\,dx\right).$$

- This is not true when $m < \frac{(n-2)_+}{n}$.

The Cauchy problem in the super-critical case

Consider the fast-diffusion equation

$$(*) \qquad u_t = \Delta u^m, \qquad \frac{(n-2)_+}{n} < m < 1.$$

- For any nonnegative continuous weak solution u of $(*)$, there exists the initial trace i.e. a unique locally finite Borel measure μ_0 on $\mathbb{R}^n$ such that

$$\lim_{t \downarrow 0} u(\cdot, t) = \mu_0 \qquad \text{in} \quad D'(\mathbb{R}^n).$$

- The trace μ_0 determines the solution uniquely.
- For any locally finite Borel measure μ_0 on $\mathbb{R}^n$ there exists a continuous weak solution u of $(*)$ in $S_\infty = \mathbb{R}^n \times (0, \infty)$ with initial trace μ_0.
- Regularity: Solutions are C^∞ smooth !!!
- Remark: No growth condition needs to be imposed on the initial data for existence !!!

The sub-critical case $m < (n-2)_+/n$

- In the sub-critical case $m < \frac{(n-2)_+}{n}$ the analogues of the above results do not hold true. In particular, there exists no solution with initial data the Dirac mass.
- This makes the problem of the existence of solutions with initial data a measure a very delicate one.
- The Sobolev critical case of exponents $m = \frac{n-2}{n+2}$ is of particular geometric interest as it corresponds to the Ricci flow for $n = 2$ and the Yamabe flow for $n \geq 3$.

Ultra-fast diffusion on $\mathbb{R}^n$

- Consider the Cauchy problem for the ultra-fast diffusion eq. on $\mathbb{R}^n$

$$(*) \qquad \frac{\partial}{\partial t} u = -\Delta u^m, \qquad m < 0.$$

- Instant vanishing: There exists no solution of $(*)$ with initial data $u_0 \in L^1(\mathbb{R}^n)$.
- Necessary and sufficient condition for existence: The condition

$$u_0 \geq c\,|x|^{-2/(1-m)}, \qquad |x| >> 1$$

in an average sense is necessary and sufficient for existence.

- However, a radial structure near infinity of the initial data u_0 for $|x| >> 1$ is necessary for existence.
- We will see that in the geometric case of the IMCF this is replaced by the star shaped condition.

Ultra-fast diffusion on $\mathbb{R}^n$ - An example

- Assume $n = 2$ and for each $\phi \in (0, 2\pi]$, let W_ϕ denote the wedge

$$W_\phi := \{(r, \theta) \,:\, 0 \leq r < +\infty,\ 0 < \theta < \phi\}.$$

- Example: For each $m < 0$ there exists $\phi_m \in (0, 2\pi)$ such that for $u_0 = \chi_{W_\phi}$: there exists a solution of $(*)$ iff $\phi < \phi_m$.

- The (IMCF) flow

$$\frac{\partial}{\partial t} H = -\Delta H^{-1} - \frac{|A|^2}{H}.$$

corresponds to $m = -1$ and in this case $\phi_m = \pi$.

Future Lectures

- We have seen some basic properties for degenerate, fast and ultra-fast diffusion.
- Also classical results the solvability of the Cauchy problem on solutions to these equations $\mathbb{R}^n$.
- In our future lectures we will see how these properties and results relate to more recent works on geometric flows.