Part 2
Nonlinear extrinsic flows on entire graphs
MCF and IMCF

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Future Lectures

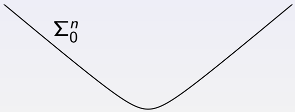
- We have seen in Lecture 1 some basic properties for degenerate and fast diffusion.

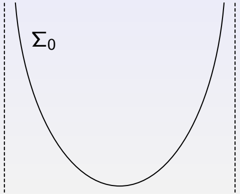
- Also classical results the solvability for the Cauchy problem for these equations on $\mathbb{R}^n$.

- In our future lectures we will see how these properties and results relate to more recent works on extrinsic geometric flows on complete non-compact graphs.

The evolution of complete graphs

- Assume that M_t is a complete non-compact graph over a domain $\Omega \subset \mathbb{R}^n$.





- Let $F_t : N^n \to \mathbb{R}^{n+1}$ be a family of immersions of our graph $M_t := F_t(N^n)$ in $\mathbb{R}^{n+1}$ evolving by the flow

$$\frac{\partial}{\partial t} F(p,t) = \sigma(p,t)\, \nu(p,t), \qquad p \in N^n$$

where $\nu(p,t)$ is a choice of normal vector and $\sigma(\lambda_1 \cdots \lambda_n)$ is the speed.

Examples of such flows

Examples of nonlinear extrinsic geometric flows

- Mean curvature flow: $\sigma = H = \lambda_1 + \cdots + \lambda_n$
- Inverse mean curvature flow: $\sigma = -\frac{1}{H} = -\frac{1}{\lambda_1 + \cdots + \lambda_n}$
- Gauss curvature flow: $\sigma = K = \lambda_1 \cdots \lambda_n$
- α-Gauss curvature flow:: $\sigma = K^\alpha = (\lambda_1 \cdots \lambda_n)^\alpha$, $0 < \alpha < \infty$.

Outline

We will discuss:

- The Mean curvature flow where on entire graphs.
  An example of quasilinear diffusion which resembles the heat equation.

- The Inverse mean curvature flow on entire convex graphs.
  An example for fully-nonlinear ultra-fast diffusion.

- The α-Gauss curvature flow on complete non-compact graphs.
  An example of fully-nonlinear slow diffusion.

The evolution of complete graphs - Graph Parametrization

- Let $x_{n+1} = \bar{u}(x,t)$, $\bar{u} : \Omega \times [0,T) \to \mathbb{R}$, be a graph over $\Omega \subset \mathbb{R}^n$ which evolves by a non-linear extrinsic flow.
- Graph Parametrization: $\bar{F}(x) := (x, \bar{u}(x))$, $x \in \mathbb{R}^n$.
- This is not the same as the geometric parametrization $F(\cdot,t) : N^n \to \mathbb{R}^{n+1}$ under which

$$\frac{\partial}{\partial t} F(p,t) = \sigma(p,t)\,\nu(p,t), \qquad p \in N^n$$

- In the graph parametrization the equation is:

$$\left(\frac{\partial}{\partial t}\bar{F}(x,t)\right)^{\perp} = \sigma\,\nu$$

- Then $\bar{u}$ satisfies the equation

$$\bar{u}_t = \sqrt{1+|D\bar{u}|)^2}\,\bar{\sigma}(x,\bar{u},D\bar{u},D^2\bar{u})$$

where $\bar{\sigma}$ is the speed as a function of $x, \bar{u}, D\bar{u}, D^2\bar{u}$.

MCF on entire graphs

- Let $M_t := F_t(N^n) \subset \mathbb{R}^{n+1}$, where $F_t : N^n \to \mathbb{R}^{n+1}$ immersions evolving by the Mean curvature flow

$$(\star_{MCF}) \qquad \frac{\partial}{\partial t} F(p,t) = H(p,t)\,\nu(p,t), \qquad p \in N^n.$$

- We assume that M_t is an entire graph over $\mathbb{R}^n$; i.e. there exists a unit vector $\omega \in \mathbb{R}^{n+1}$, such that

$$\langle \omega, \nu \rangle > 0, \qquad \text{on } M_t.$$

- From now on we take $\omega := e_{n+1}$.
- Graph parametrization: We may write M_t as $x_{n+1} = \bar{u}(x,t)$, for $\bar{u} : \mathbb{R}^n \times [0,T) \to \mathbb{R}$. Then the (MCF) is equivalent to:

$$(\star_{MCF_{\bar{u}}}) \qquad \bar{u}_t = \sqrt{1+|D\bar{u}|^2}\,\mathrm{div}\Big(\frac{D\bar{u}}{\sqrt{1+|D\bar{u}|^2}}\Big).$$

- Remark: We will use the geometric parametrization and not the graph parametrization.

Geometry on a graph

- Consider the graph $F = (x, u(x))$, $x \in \mathbb{R}^n$.
- Metric:

$$g_{ij} = \langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \rangle = \delta_{ij} + D_i \bar{u} D_j \bar{u}.$$

- Second fundamental form:

$$h_{ij} = -\langle \nu(x), \frac{\partial^2 F}{\partial x_i \partial x_j} \rangle$$

- Mean curvature H:

$$H = g^{ij} h_{ij} = \left(\delta_{ij} - \frac{D_i \bar{u} D_j \bar{u}}{1 + |D\bar{u}|^2} \right) \frac{D_{ij} \bar{u}}{\sqrt{1 + |D\bar{u}|^2}} = D_i \left(\frac{D_i \bar{u}}{\sqrt{1 + |D\bar{u}|^2}} \right)$$

MCF on entire graphs - Long time existence

The following is work by K. Ecker and G. Huisken (1989-1991):

- Theorem (Ecker-Huisken) Let M_0 be a locally Lipschitz entire graph over $\mathbb{R}^n$. Then, the (MCF) admits a C^∞ solution M_t with initial data M_0, for all $t > 0$. Moreover, M_t remains an entire graph over $\mathbb{R}^n$.
- Idea of the Proof:

  (i) Show a local bound on the gradient function $v := -\langle e_{n+1}, \nu\rangle^{-1} = \sqrt{1 + |Du|^2}$;

  (ii) Use this bound to obtain a local bound on the second fundamental form $|A|^2$ which is independent of the initial data.
- Remark 1: Note that no growth assumptions or smoothness need to be imposed on the initial graph.
- Remark 2: No uniqueness was shown.

MCF on entire graphs - Local gradient Estimate

- Assume that M_t is an entire graph over $\mathbb{R}^n$; i.e.

$$\langle e_{n+1}, \nu \rangle > 0, \qquad \text{on } M_t$$

for a choice of unit normal ν.

- Let M_t be given by $x_{n+1} = u(x,t)$, for $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$.
- Then $v := \langle e_{n+1}, \nu \rangle^{-1} = \sqrt{1 + |Du|^2}$ satisfies:

$$\frac{\partial}{\partial t} v = \Delta v - 2v^{-1} |\nabla v|^2 - |A|^2 \, v.$$

- For any $R > 0$, let $\eta(F,t) = (R^2 - |F|^2 - 2nt)_+$ a cut off function, where $F(\cdot, t) \in M_t$ denotes the position vector.
- Local gradient estimate: We have

$$v(F,t)\, \eta(F,t) \le \sup_{M_0} (v\, \eta).$$

MCF on entire graphs - Bound on $|A|^2$

- The second fundamental form $A = \{h_{ij}\}$ satisfies:

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4.$$

- The gradient $v := \langle e_{n+1}, \nu\rangle^{-1} = \sqrt{1+|Du|^2}$ satisfies:

$$\frac{\partial}{\partial t}v = \Delta v - 2v^{-1}|\nabla v|^2 - |A|^2\, v.$$

- Combine the evolutions of v and $|A|^2$ to obtain a local bound on $|A|^2$ which is independent of the initial data.
- Crucial bound on $|A|^2$: For $\rho > 0$, let $B_\rho(y_0) \subset \mathbb{R}^n$. Then, for any $\theta \in (0,1)$ and $0 \le t \le 1$:

$$\sup_{B_{\theta\rho}(y_0)} |A|^2(\cdot, t) \le C_n (1-\theta^2)^{-2}\left(\rho^{-2} + t^{-1}\right) \sup_{B_\rho(y_0)\times[0,t]} v^4.$$

MCF on entire graphs - Proof of the bound on $|A|^2$

- Proof: It uses the Caffarelli, Nirenberg and Spruck trick:
  Let $\varphi(v^2) = \frac{v^2}{1-kv^2}$. You derive that $g := |A|^2\phi(v^2)$ satisfies:

$$(*) \quad (\partial_t - \Delta) g \leq -2kg^2 - \frac{2k}{(1-kv^2)^2}|\nabla v|^2 g - 2\varphi v^{-3}\nabla v \nabla g.$$

- Let $m(t) := \sup_{M_t} g$ (if it is finite !). Then, formally if you could apply the maximum principle, $(*)$ would give

$$\frac{d}{dt} m(t) \leq -2k\, m(t)^2.$$

- Comparing with the solution of the ODE gives $m(t) \leq \frac{1}{2kt}$, i.e.

$$\sup_{M_t} |A|^2 \phi(v^2) \leq \frac{1}{2kt}.$$

- However this is not possible if $v := \sqrt{1+|Du|^2} \gg 1$, as $|F| \to +\infty$.

MCF on entire graphs - Proof of the bound on $|A|^2$

- To over come this difficulty we need to localize the equation of g by multiplying with the cut-off function

$$\eta(F,t) = (R^2 - r)_+, \quad r := |F|^2 + 2nt.$$

- Then, using $(*)$ we obtain that $G := g\eta t$ satisfies

$$(\partial_t - \Delta)G \le A \cdot \nabla G - 2kg^2\eta t + C\big((1 + \frac{1}{kv^2})r + R^2\big)gt + g\eta.$$

- Applying the maximum principle to $m(t) := \max_{r<R^2} G$, we conclude for any $\theta \in (0,1)$ the bound

$$\sup_{B_{\theta\rho}(y_0)} |A|^2(\cdot,t) \le C_n(1-\theta^2)^{-2}\big(\rho^{-2} + t^{-1}\big) \sup_{B_\rho(y_0)\times[0,t]} v^4.$$

Conclusion

- Theorem. If M_0 is a locally Lipschitz entire graph over $\mathbb{R}^n$, then the (MCF) admits a C^∞-smooth solution M_t with initial data M_0, for all $t > 0$. M_t is an entire graph over $\mathbb{R}^n$.
- At a maximal existence time T, if $T < +\infty$ the graph M_T is locally Lipschitz and also $|A|^2$ is locally bounded. Hence, the flow may be continued to show that $T = +\infty$.
- In addition, since the evolution equation is uniformly parabolic on compact sets, the solution M_t is C^∞ smooth by standard parabolic regularity.

- Let $F : N^n \times [0, T] \to \mathbb{R}^{n+1}$ be a smooth family of closed hypersurfaces in R^{n+1}. F defines a classical solution to the Inverse mean curvature flow in R^{n+1} if it satisfies

$$\frac{\partial}{\partial t} F(p,t) = \frac{1}{H(p,t)}\,\nu(p,t), \quad p \in N^n$$

where $H(\cdot, t) > 0$ and $\nu(p,t)$ denote the mean curvature and exterior unit normal of the surface M_t at the point $F(p,t)$.

Figure: Hypersurface M_t^n compact in $\mathbb{R}^{n+1}$ or a graph over $\mathbb{R}^n$

IMCF- An ultra-fast diffusion

- Under (IMCF) the Mean curvature satisfies the ultra-fast diffusion:

$$\frac{\partial}{\partial t} H = \frac{1}{H^2}\,\Delta H - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H}.$$

- The above equation can also be written as

$$\frac{\partial}{\partial t} H = -\Delta H^{-1} - \frac{|A|^2}{H}.$$

- This resembles the ultra fast diffusion equation on $\mathbb{R}^n$:

$$u_t = -\Delta u^m, \qquad m < 0.$$

Ultra-fast diffusion on $\mathbb{R}^n$

- Consider the Cauchy problem for the ultra-fast diffusion eq. on $\mathbb{R}^n$

$$(*) \qquad \frac{\partial}{\partial t} u = -\Delta u^m, \qquad m < 0.$$

- Instant vanishing: There exists no solution of $(*)$ with initial data $u_0 \in L^1(\mathbb{R}^n)$.
- Necessary and sufficient condition for existence: The condition

$$u_0 \geq c\,|x|^{-2/(1-m)}, \qquad |x| \gg 1$$

in an average sense is necessary and sufficient for existence.

- However, a radial structure near infinity of the initial data u_0 for $|x| \gg 1$ is necessary for existence.
- We will see that in the geometric case of the IMCF this is replaced by the star shaped condition.

Ultra-fast diffusion on $\mathbb{R}^n$ - An example

- Assume $n = 2$ and for each $\phi \in (0, 2\pi]$, let W_ϕ denote the wedge

$$W_\phi := \{(r, \theta) \,:\, 0 \leq r < +\infty,\ 0 < \theta < \phi\}.$$

- Example: For each $m < 0$ there exists $\phi_m \in (0, 2\pi)$ such that for $u_0 = \chi_{W_\phi}$: there exists a solution of $(*)$ iff $\phi < \phi_m$.

- The (IMCF) flow

$$\frac{\partial}{\partial t} H = -\Delta H^{-1} - \frac{|A|^2}{H}.$$

corresponds to $m = -1$ and in this case $\phi_m = \pi$.

- Lets go back to (IMCF)

$$\frac{\partial}{\partial t}F(p,t) = \frac{1}{H(p,t)}\,\nu(p,t), \quad p \in N^n$$

where $H(\cdot,t) > 0$ and $\nu(p,t)$ denote the mean curvature and exterior unit normal of the surface M_t at the point $F(p,t)$.

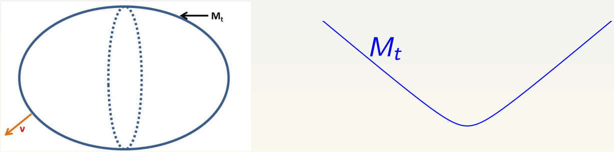


Figure: Hypersurface M_t^n compact in $\mathbb{R}^{n+1}$ or a graph over $\mathbb{R}^n$

IMCF on compact hypersurfaces - Background

- C. Gerhardt, J. Urbas: Existence for all $0 < t < +\infty$, for smooth star-shaped initial data with $H > 0$.
  Convergence as $t \to +\infty$ to a *homothetically expanding* spherical solution.

- For non star-shaped initial data, singularities may develop.

- K. Smoczyk : Singularities can only occur if the Mean curvature H becomes zero somewhere during the evolution.

- G. Huisken, T. Ilmanen: Developed a level set approach to weak solutions of the flow, allowing jumps of the surfaces and solutions of weakly positive mean curvature.

- G Huisken, T. Ilmanen used the weak solution formulation of the flow to derive energy estimates in General Relativity.

Under (IMCF) we have:

- $\dfrac{\partial}{\partial t} F = \dfrac{1}{H}\, \nu$
- $\dfrac{\partial}{\partial t} g_{ij} = \dfrac{2}{H}\, h_{ij}$
- $\dfrac{\partial}{\partial t} d\mu = d\mu$
- $\dfrac{\partial}{\partial t} \nu = -\nabla H^{-1} = \dfrac{1}{H^2}\, \nabla H$
- $\dfrac{\partial}{\partial t} h_{ij} = \dfrac{1}{H^2} \Delta h_{ij} - \dfrac{2}{H^3} \nabla_i H \nabla_j H + \dfrac{|A|^2}{H^2}\, h_{ij}$
- $\dfrac{\partial}{\partial t} H = \nabla_i \big(\dfrac{1}{H^2}\, \nabla_i H \big) - \dfrac{|A|^2}{H}$
- $\dfrac{\partial}{\partial t} \langle F - \bar{x}_0, \nu \rangle = \dfrac{1}{H^2}\, \Delta \langle F - \bar{x}_0, \nu \rangle + \dfrac{|A|^2}{H^2}\, \langle F - \bar{x}_0, \nu \rangle.$

IMCF on compact star-shaped hypersurfaces

- Let $M_t = F(\cdot, t)(N^n)$ be a solution to the (IMCF).
- M_t is called star-shaped if $\langle F, \nu \rangle > 0$ on M_t.
- Theorem (Huisken-Ilmanen) Let M_0 be a closed embedded C^1 hyper-surface satisfying:

$$0 \le H \le C \quad \text{and} \quad 0 < R_1 < \langle F, \nu \rangle < R_2.$$

Then, the (IMCF) admits a global solution M_t, $0 < t < +\infty$ with $H > 0$ for $t > 0$ and such that $M_t \to M_0$ in C^1 as $t \to 0$.

- Crucial Estimate: They establish an a priori bound

$$H \ge \theta_t := c_n \, R_1 \, R_2^{-1} \, |M_0|^{-1/n} \, \min(t^{1/2}, 1) \, e^{-t/n}$$

using integral estimates the ultra-fast character of the eq.

IMCF- Starshaped hyperfurfaces

- Assume that at $t = 0$, $0 < R_1 < \langle F, \nu \rangle < R_2$.

- Then, for all $t > 0$, we have

$$0 < e^{\frac{t}{n}} R_1 \leq \langle F, \nu \rangle \leq |F| \leq e^{\frac{t}{n}} R_2.$$

- Proof: it follows from the evolution equation

$$\frac{\partial}{\partial t} \langle F, \nu \rangle = \frac{1}{H^2} \Delta \langle F, \nu \rangle + \frac{|A|^2}{H^2} \langle F, \nu \rangle.$$

IMCF- An ultra-fast diffusion on H

- Main step to long time existence: Show that $H > 0$.
- H satisfies the ultra-fast diffusion:

$$\frac{\partial}{\partial t} H = \frac{1}{H^2}\,\Delta H - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H}$$
$$= -\Delta H^{-1} - \frac{|A|^2}{H}$$

- Combining it with:

$$\frac{\partial}{\partial t}\langle F, \nu\rangle = \frac{1}{H^2}\,\Delta\,\langle F, \nu\rangle + \frac{|A|^2}{H^2}\,\langle F, \nu\rangle$$

- $v := H\,\langle F, \nu\rangle$ satisfies the ultra-fast diffusion:

$$\frac{\partial}{\partial t} v = \nabla_i\big(\langle F, \nu\rangle^2\, v^{-2}\nabla_i v\big).$$

IMCF- Lower bound on H

- Theorem (Huisken-Ilmanen) If M_0 is star-shaped surface with $0 < R_1 < \langle F, \nu \rangle < R_2$, then $\exists c_n > 0$ such that

$$H \geq c_n\, R_1\, R_2^{-1}\, |M_0|^{-1/n} \min(t^{1/2}, 1)\, e^{-t/n}.$$

- Remark: The above bound does not depend on a lower bound on H at $t = 0$. Hence, H may vanish at $t = 0$.
- Sketch of Proof: If $u := (H \langle F, \nu \rangle)^{-1}$, then

$$\frac{\partial}{\partial t} u = \nabla_i \big(\langle F, \nu \rangle^2\, u^2 \nabla_i u \big) - 2 \langle F, \nu \rangle^2 u\, |\nabla u|^2.$$

Combined with the Michael-Simon Sobolev inequality

$$\Big(\int_{N^n} |f|^{\frac{n}{n-1}}\, d\mu \Big)^{\frac{n-1}{n}} \leq C(n) \int_{N^n} |\nabla f| + |H||f|\, d\mu.$$

gives L^∞-bound on u via de Giorgi - Stampacchia iteration.

IMCF- Bound on the second fundamental form

- Assume that $0 < \theta_0 < H < \theta_1$.
- The second fundamental form $A = \{h_{ij}\}_{1 \le i,j \le n}$ satisfies:

$$\frac{\partial}{\partial t} h_{ij} = \frac{1}{H^2} \Delta h_{ij} - \frac{2}{H^3} \nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij}.$$

- Recall $H_t = \frac{1}{H^2} \Delta H - \frac{2}{H^3} |\nabla H|^2 - \frac{|A|^2}{H^2} H$.
- Also $(g_{ij})_t = \frac{2}{H} h_{ij}$.
- Set $M_{ij} = H\, h_{ij}$ and $M^i_j = g^{il} M_{lj}$.
- By combining the evolution equations of h_{ij}, H and g_{ij}:

$$\frac{\partial}{\partial t} M^i_j = \frac{1}{H^2} \Delta M^i_j - \frac{2}{H^3} \nabla_k H \nabla_k M^j_i - \frac{2}{H^3} \nabla^i H \nabla_j H - \frac{2}{H^2} M^{ik} M_{kj}.$$

IMCF- Bound on the second fundamental form

- By the maximum principle on the matrix (M_i^j) one finds that its its maximum eigenvalue $\kappa_n = \lambda_n H$ satisfies:

$$\kappa_n \leq \frac{\theta_1^2}{2t}.$$

- Using the bound $H > \theta_0$, one concludes the bound

$$\lambda_n \leq \frac{\theta_1^2}{2\theta_0\, t}.$$

- Using also that $H \geq 0$, we finally conclude that

$$|A| \leq c_n \frac{\theta_1^2}{\theta_0\, t}.$$

IMCF-Long time existence of smooth solutions

- Theorem (Huisken-Ilmanen) Let M_0 be a closed embedded C^1 hyper-surface satisfying $0 \leq H \leq \theta_1$. Assume in addition that M_0 is strictly star-shaped, namely

$$0 < R_1 < \langle F, \nu \rangle < R_2.$$

Then, the (IMCF) admits a global solution M_t, $0 < t < +\infty$ with $H > 0$ for $t > 0$ and such that $M_t \to M_0$ in C^1 as $t \to 0$.

- Sketch of Proof: By combining the bounds:

$$H \geq \theta_t := c_n \, R_1 \, R_2^{-1} \, |M_0|^{-1/n} \min(t^{1/2}, 1) \, e^{-t/n}$$

and

$$|A| \leq c_n \, \frac{\theta_1^2}{\theta_t \, t}.$$

with classical higher regularity estimates.

Remarks and further work

- Recall:

$$H_t = -\Delta H^{-1} - \frac{|A|^2}{H}.$$

- $u := (H \langle F, \nu \rangle)^{-1}$ satisfies the porous medium type equation

$$u_t = \nabla_i \big(\langle F, \nu \rangle^2 u^2 \nabla_i u \big) - 2 \langle F, \nu \rangle^2 u \, |\nabla u|^2.$$

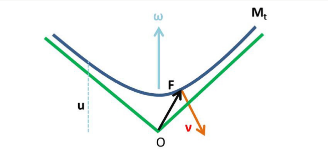
- The Huisken-Ilmanen bound is reminiscent of the L^∞ bound for solutions of the Dirichlet problem for the porous medium equation.
- B. Choi (2017) has recently shown the Huisken-Ilmanen bound by maximum principle argument.

IMCF on convex entire graphs

- The following is joint work with G. Huisken.
- Let $F_t : N^n \to \mathbb{R}^{n+1}$ a family of immersions of n-dimensional convex hypersurfaces $M_t := F_t(N^n)$ in $\mathbb{R}^{n+1}$ which evolve by Inverse mean curvature flow

$$\frac{\partial}{\partial t} F(p,t) = \frac{1}{H(p,t)}\, \nu(p,t), \qquad p \in M^n.$$

- ν is then the outer normal to the surface.



- Take $\omega = e_{n+1} \in \mathbb{R}^{n+1}$ and assume that M_0 lies above the cone given by $x_{n+1} = \alpha_0\, |x|$. The $\langle e_{n+1}, \nu \rangle < 0$.
- Goal: Establish the long time existence of the flow.

IMCF-Important evolution equations

Under (IMCF) we have:

- $F_t = \frac{1}{H}\,\nu$
- $(d\mu)_t = d\mu$
- $\nu_t = -\nabla H^{-1} = \frac{1}{H^2}\,\nabla H$
- $H_t = \frac{1}{H^2}\,\Delta H - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H}$
- $(H^{-1})_t = \frac{1}{H^2}\Delta H^{-1} + \frac{|A|^2}{H^2}\,H^{-1}$
- $(\langle e_{n+1}, \nu\rangle)_t = \frac{1}{H^2}\,\Delta\,\langle e_{n+1}, \nu\rangle + \frac{|A|^2}{H^2}\,\langle e_{n+1}, \nu\rangle$
- $(\langle F, e_{n+1}\rangle)_t = \frac{1}{H^2}\,\Delta\langle F, e_{n+1}\rangle + \frac{2}{H}\,\langle e_{n+1}, \nu\rangle$

Comparison Principle:

Assume that $f \in C^2(\mathbb{R}^n \times (0,\tau)) \cap C^0(\mathbb{R}^n \times [0,\tau))$ satisfies:

$$f_t \leq a_{ij}\, D_{ij} f + b_i\, D_i f + c\, f, \qquad \text{on } \mathbb{R}^n \times (0,\tau)$$

for some $\tau > 0$ with measurable coefficients such that:

$$\lambda |\xi|^2 \leq a_{ij}(x,t)\, \xi_i \xi_j \leq \Lambda |\xi|^2\, (|x|^2 + 1)$$

and

$$|b_i(x,t)| \leq \Lambda\, (|x|^2 + 1)^{1/2}, \qquad |c(x,t)| \leq \Lambda.$$

Assume in addition that the solution f has:

$$f(x,t) \leq C\, (|x|^2 + 1)^p, \qquad \text{on } \mathbb{R}^n \times [0,\tau], \qquad p > 0.$$

If $f(\cdot,0) \leq 0$ on $\mathbb{R}^n$, then $f \leq 0$ on $\mathbb{R}^n \times [0,\tau]$.

Bound from above on H

We will use the following a priori local and global bounds from above on H.

Assume that $M_t, t \in [0,\tau]$ is a C^2 graphical solution of (IMCF):

- Local bound from above: Let $\eta := (r^2 - |F - \bar{x}_0|^2)_+^2$. Then

$$\text{if} \quad \sup_{M_0} \eta\, H \le C_0 \quad \text{then} \quad \sup_{M_t} \eta\, H \le \max(C_0, 2n\, r^3).$$

- Global bound from above: If $M_t, t \in [0,\tau]$ is also convex, then

$$\sup_{t\in[0,\tau]} \sup_{M_t} \langle F, e_{n+1}\rangle H \le \sup_{M_0} \langle F, e_{n+1}\rangle H.$$

- Proofs Simply by the maximum principle !!!

Long-time Existence of solutions with super-linear growth

- Let M_0 be an entire graph $x_{n+1} = u_0(x)$ over $\mathbb{R}^n$ satisfying:

  (i) super-linear growth: $|Du_0(x)| \to \infty$, for $|x| \to \infty$.

  (ii) δ-starshaped: $H\langle F - \bar{x}_0, \nu\rangle \geq \delta > 0$, for $\bar{x}_0 \in \mathbb{R}^{n+1}$.

- Lemma: The condition $H\langle F - \bar{x}_0, \nu\rangle \geq \delta > 0$ is preserved under the flow.
- Proof: Simply follows from the comparison principle since $w := \langle F - \bar{x}_0, \nu\rangle$ satisfies the equation:

$$\Big(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta\Big) w = -\frac{2}{H^3}\nabla H \cdot \nabla w.$$

Theorem (D., Huisken-2017) Let $x_{n+1} = u_0(x)$ be an entire graph of class C^2 satisfying assumptions (i)-(ii). Then, there exists is a smooth entire graph solution $x_{n+1} = u(x,t)$ of the (IMCF) which is defined for all $0 < t < +\infty$.

If u_0 is convex, then the solution M_t is also convex.

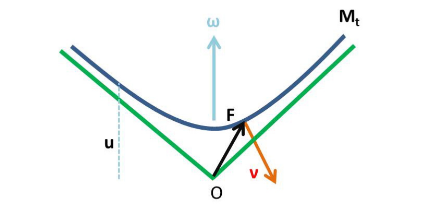
Proof: The proof follows the steps:

- We approximate the initial data M_0 by smooth and compact hypersurfaces which satisfy $H\langle F - \bar{x}_0, \nu\rangle \geq \tilde{\delta} > 0$.
- $H\langle F - \bar{x}_0, \nu\rangle \geq \tilde{\delta} > 0$ is preserved under (IMCF).
- To pass to the limit we use the following local bound on $|A|^2$ which was established by M.E. Heidusch:

$$\sup_{M_0 \cap B_R(0)} |A|^2 \leq C_2 \max\left(\max_{M_0 \cap B_R(0)} |A|^2, R^{-1} \max_{M_0 \cap B_R(0)} H + R^{-2}\right).$$

IMCF on convex entire graphs

- From now on we will restrict to convex entire graphs with conical behavior at infinity.
- The surface M_t may be expressed as the graph $x_{n+1} = u(x,t)$ of a function $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$.



- Then the (IMCF) is equivalent to the fully nonlinear PDE

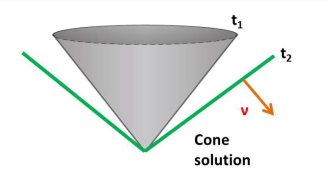
$$(\star_u) \qquad u_t = -\frac{\sqrt{1+|Du|^2}}{\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)}.$$

Conical solutions to IMCF

- On surfaces of revolution given by $x_{n+1} = u(r,t)$, $r = |x|$ the (IMCF) becomes

$$(\star_u) \qquad u_t = -\frac{(1+u_r^2)^2}{u_{rr} + (n-1)\,(1+u_r^2)\,u_r/r}.$$

- Separation of variables leads to the conical solutions $\mathcal{C}(x,t) = \alpha(t)\,|x| + \kappa$ where $\alpha'(t) = -\frac{1}{n-1}\left(\alpha(t) + \frac{1}{\alpha(t)}\right)$.



- These solutions become flat at some finite time $T < +\infty$ depending on the initial slope.

Self-similar entire graph solutions to IMCF

- In the (IMCF) one cannot scale the time variable t.
- Nevertheless, the (IMCF) admits radial self-similar solutions

$$u_\lambda(x,t) = e^{\lambda t}\, \bar{u}_\lambda(e^{-\lambda t}\,|x|)$$

where $x_{n+1} = \bar{u}_\lambda(x)$ are entire convex graphs over $\mathbb{R}^n$.

- Proposition: (D., Huisken) $\forall \lambda > 1/(n-1)$, $\exists!\ x_{n+1} = \bar{u}_\lambda(|x|)$ on $\mathbb{R}^n$ with $\bar{u}_\lambda(0) = -1$ with flux at infinity

$$\lim_{r\to\infty} \frac{r\,(\bar{u}_\lambda)_r(r)}{\bar{u}_\lambda(r)} = \frac{\lambda\,(n-1)}{(n-1)\,\lambda - 1} = q, \quad r = |x|.$$

- It follows that $u_\lambda(|x|) \sim |x|^q$ as $|x| \to \infty$ and $\lambda > 1/(n-1)$ iff $q \in (1, +\infty)$.
- $q = 1$ corresponds to the conical solution.

IMCF on asymptotically conical graphs

- Short time existence: (D., G. Huisken) Let M_0 be a C^2 convex entire graph $x_{n+1} = u_0(x)$, $x \in R^n$ satisfying

$$(*_1) \quad \alpha_0\, |x| < u_0(x) < \alpha_0\, |x| + \kappa,\ x \in \mathbb{R}^n,\ \alpha_0 > 0,\ \kappa > 0.$$

and

$$(*_2) \qquad 0 < c_0 < H\, \langle F, e_{n+1} \rangle < C_0.$$

Then, there exists a unique smooth convex solution M_t of the (IMCF) with initial data M_0, given by the entire graph $x_{n+1} = u(x,t)$, $x \in \mathbb{R}^n$, $t \in [0,\tau], \tau > 0$. Moreover:

$$(**_1) \qquad \alpha(t)\, |x| < u(x,t) < \alpha(t)\, |x| + \kappa.$$

and

$$(**_2) \qquad 0 < c_\tau < H\, \langle F, e_{n+1} \rangle < C.$$

- Remark. $\bar{u} := H\, \langle F, e_{n+1} \rangle$ is the height function.

IMCF on asymptotically conical graphs - Long time Existence

- Let M_t, $0 < t < \tau$ be the solution to (IMCF) satisfying

$$\alpha(t)\,|x| < u(x,t) < \alpha(t)\,|x| + \kappa.$$

- Let $T < +\infty$ s.t $\alpha(T) = 0$ (the cone at infinity becomes flat).
- Claim: The solution M_t will exist up to time T.
- Main Difficulty: to show that $H > 0$ on M_t for all $t < T$.
- H satisfies the ultra-fast diffusion:

$$\frac{\partial}{\partial t} H = \frac{1}{H^2}\,\Delta H - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H}.$$

- Moreover $H(\cdot,t) \approx \dfrac{C_t}{|F|}$ as $|F| \to +\infty$.

IMCF- Evolution of H

- Set $v := H \langle \hat{F}, \nu \rangle, \qquad \langle \hat{F}, \nu \rangle := -\langle F, e_{n+1} \rangle \langle e_{n+1}, \nu \rangle > 0.$
- Remark: Since $x_{n+1} = u(\cdot, t)$ we have $\langle F, e_{n+1} \rangle = u$ and $\langle e_{n+1}, \nu \rangle = -\frac{1}{\sqrt{1+|Du|^2}}$, hence $\langle \hat{F}, \nu \rangle \approx |F|$, for $|F| \gg 1$.
  Thus,

$$v := -H \langle F, e_{n+1} \rangle \langle e_{n+1}, \nu \rangle \approx C, \qquad \text{for } |F| \gg 1.$$

- Lemma: Let $v := H \langle \hat{F}, \nu \rangle$ evolves by the ultra fast diffusion

$$\left(\frac{\partial}{\partial t} - \frac{1}{H^2} \Delta\right) v = -\frac{2}{H^3} \nabla v \nabla H - 2 \langle e_{n+1}, \nu \rangle^2 + \frac{h_{ij}}{H} \langle \mathbf{e_i}, e_{n+1} \rangle \langle \mathbf{e_j}, e_{n+1} \rangle$$

- Proof: By combining the evolutions equations of H, $\langle F, e_{n+1} \rangle$ and $\langle e_{n+1}, \nu \rangle$.

IMCF- Basic step

- On a convex surface we have: $\frac{h_{ij}}{H}\langle \mathbf{e_i}, e_{n+1}\rangle \langle \mathbf{e_j}, e_{n+1}\rangle \geq 0$. Hence

$$(\frac{\partial}{\partial t} - \frac{1}{H^2}\Delta)v \geq -\frac{2}{H^3}\nabla v \nabla H - 2\langle e_{n+1}, \nu\rangle^2.$$

- Basic Step: To show that

$$v(\cdot, t) > c_\delta > 0, \qquad \text{for } 0 < t < T - \delta$$

where T is the time at which the cone at infinity disappears !

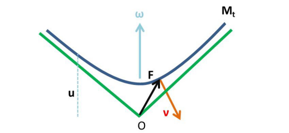
- No barriers can be constructed. No time scaling.

- The exact behavior of $v(\cdot, t)$ as $|F| \to +\infty$ needs to be used !!

IMCF- The evolution of H^{-1}

- Recall that $v := H\,\langle \hat{F}, \nu\rangle$, $\langle \hat{F}, \nu\rangle := -\langle F, e_{n+1}\rangle\,\langle e_{n+1}, \nu\rangle > 0$.
- By Short time existence on $0 < t < \tau$:

$$\lim_{|F(p,t)|\to\infty} v(p,t) = \gamma(t), \qquad \gamma(t) := \frac{(n-1)\alpha(t)^2}{1+\alpha(t)^2}.$$



- If $w := (\langle \hat{F}, \nu\rangle\, H)^{-1}$, then $\lim_{|F|\to\infty} w(p,t) = \gamma(t)^{-1}$ and

$$\frac{\partial w}{\partial t} - D_i\Big(\frac{1}{H^2} D_i w\Big) \le -\frac{2}{H^2 w}|\nabla w|^2 + 2\langle e_{n+1}, \nu\rangle^2 w^2.$$

- One needs to obtain a global L^∞ bound on $w := (\langle \hat{F}, \nu\rangle\, H)^{-1}$ for all $0 < t < T - \delta$, $\delta > 0$.

IMCF- L^p bounds on H^{-1}

- Since $\lim_{|F|\to\infty} w(\cdot,t) = \gamma(t)^{-1}$, $\hat{w}(\cdot,t) := (w(\cdot,t)\hat{\gamma}(t) - 1)_+$ is compactly supported if $\hat{\gamma}(t) < \gamma(t)$.
- L^p-Estimate: Assume that M_t is a solution to (IMCF) on $0 < t \le \tau$, $\tau < T - \delta$. Then, $\forall p \ge 1$, $\exists C = C(p, T, \delta)$ s.t.

$$\sup_{t\in[0,\tau]} \int_{M_t} \hat{w}^p(\cdot,t)\, d\mu \le C \left(1 + \int_{M_0} \hat{w}^p(\cdot,0)\, d\mu\right).$$

- Proof: By combining energy estimates on $\hat{w}$ and a suitable Hardy inequality.
- Hardy Inequality: For any function g that is compactly supported on M_t, we have

$$\int_{M_t} g^2\, d\mu \le C(n) \left(\int_{M_t} |\nabla g|^2 |F|^2\, d\mu + \int_{M_t} g^2 |H|\,|F|\, d\mu \right).$$

IMCF- L^∞-bounds on H^{-1} and H

- Let M_t be a solution to (IMCF) on $[0,\tau]$, $\tau < T - \delta$.
- L^∞ bound on H^{-1}: If $w := (\langle \hat{F}, \nu \rangle\, H)^{-1}$, then $\exists \mu > 0, \sigma > 0$ s.t. for any $0 < t_0 < \tau < T - \delta$:

$$\sup_{t \in (t_0,\tau]} \|w\|_{L^\infty(M_t)} \leq C_\delta\, t_0^{-\mu} \left(1 + \sup_{R \geq 1} R^{-n} \int_{M_0 \cap \{|F| \leq R\}} w\, d\mu \right)^\sigma .$$

- Proof: By the L^p bounds on w, a suitable Hardy inequality and a Moser iteration argument adopted to our situation.
- L^∞ bound on H: If $\langle F, e_{n+1} \rangle\, H \leq C_0$ at time $t = 0$, then

$$(\star_2) \qquad \sup_{t \in [0,\tau]} \sup_{M_t} \langle F, e_{n+1} \rangle H(\cdot, t) \leq \max\left(C_0, 2n\right).$$

- Proof: By the maximum principle.

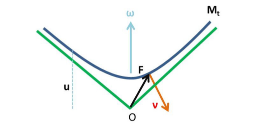
IMCF- Long time existence for $H > 0$

- Theorem (D., G. Huisken) Assume that M_0 is a C^2 convex entire graph $x_{n+1} = u_0(x)$, $x \in R^n$ with $H > 0$ satisfying:

  (i) $\alpha_0 |x| < u_0(x) < \alpha_0 |x| + \kappa, \alpha_0 > 0$, $\kappa > 0$, and
  (ii) $c_0 < H \langle F, e_{n+1} \rangle < C_0$.

  Let T be the time at which the cone at infinity becomes flat.

  Then, there exists a C^∞-smooth entire graph solution M_t of the (IMCF) on $0 < t < T$ with initial data M_0. The solution M_t becomes flat at $t = T$.



- Proof: By combining the L^∞-bounds on H and H^{-1} with $C^{2,\alpha}$ apriori estimates for fully-nonlinear parabolic PDE shown by Guji Tian and Xu-Jia Wang.

Comparison with ultra-fast diffusion on $\mathbb{R}^n$

- Let u be a solution of ultra-fast diffusion $u_t = -\Delta u^m$ on $\mathbb{R}^n \times (0, T)$ with $m < 0$.
- Then u satisfies the Aronson-Bénilan inequality

$$u_t \leq \frac{1}{1-m}\,\frac{u}{t}$$

  which acts as a substitute of the Harnack inequality and plays an important role in proving existence.
- The Aronson-Bénilan inequality is a simple consequence of the rich scaling of the equation.
- In the (IMCF) there is no time-scaling and there is no analogue of the Aronson-Bénilan inequality.

Open Problems

Open questions on (IMCF):

- IMCF on entire conical graphs without the assumption of convexity at infinity.
- IMCF on any entire graph with linear growth at infinity.
- Is there a Harnack inequality on H ?