Part 4
Ancient Solutions to Geometric Flows

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Summer School on Extrinsic flows
ICTP - Trieste

June 4-8, 2018

Ancient and Eternal Solutions

- Some of the most important problems in geometric PDE are related to the understanding of singularities.
- This usually happens through a blow up procedure which allows us to focus near a singularity.
- In the case of a time dependent equation, after passing to the limit, this leads to an ancient or eternal solution of the flow.
- These are special solutions which exist for all time

$$-\infty < t < T \qquad \text{where} \;\; T \in (-\infty, +\infty].$$

- Understanding ancient and eternal solutions often sheds new insight to the singularity analysis

Outline

In this talk we will address:

- ancient solutions to parabolic partial differential equations with emphasis to geometric flows:
  Mean Curvature flow, Ricci flow and Yamabe flow.

  1. uniqueness results for ancient or eternal solutions
  2. methods of constructing new ancient solutions from the gluing of two or more solitons (self-similar solutions).

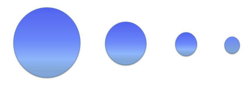
- new techniques and future research directions.

Ancient and Eternal solutions

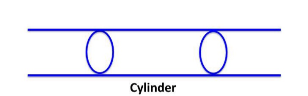
- Definition: A solution $u(\cdot, t)$ to a parabolic equation is called ancient if it is defined for all time $-\infty < t < T$, $T < +\infty$.
- Ancient solutions typically arise as blow up limits at a type I singularity.
- Definition: A solution $u(\cdot, t)$ to a parabolic equation is called eternal if it is defined for all $-\infty < t < +\infty$.
- Eternal solutions as blow up limits at a type II singularity.

Solitons

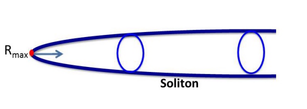
- Solitons (self-similar solutions) are typical examples of ancient or eternal solutions and often models of singularities.
- Some typical examples of solitons to geometric PDE are:
- Spheres:

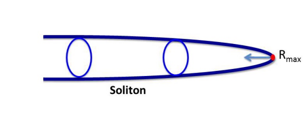


- Cylinders:



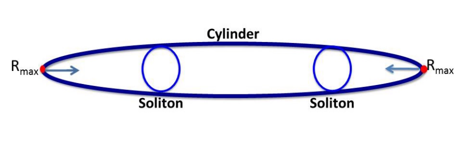
- Translating or shrining solitons with cylindrical ends:





Other Ancient and eternal solutions

- However, there exist other special ancient or eternal solutions which are not solitons.
- These, often may be visualized as obtained from the gluing as $t \to -\infty$ of two or more solitons.



Typical behavior as $t \to -\infty$ of an ancient solution

- Classifying when possible all such solutions, often leads to the better understanding of the singularities.

Geometric conditions of ancient or eternal solutions

Goal: Characterize all ancient or eternal solutions to a geometric flow under natural geometric conditions:

- Being a soliton (self-similar solution)
- Satisfying an appropriate curvature bound as $t \to -\infty$:
  i. Type I: global curvature bound after typical scaling.
  ii. Type II: solutions which are not type I.
- Satisfying a non-collapsing condition.

Liouville's theorem for the heat equation on manifolds

- Let M^n be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}\,(M^n) \geq 0$.
- Yau (1975): Any positive harmonic function u on M^n, (i.e. satisfying $\Delta u = 0$ on M^n) must be constant.
- This is the analogue of Liouville's Theorem for harmonic functions on $\mathbb{R}^n$.
- Question: Does the analogue of Yau's theorem hold for positive solutions of the heat equation

$$u_t = \Delta u \qquad \text{on } M^n?$$

- Answer: No. Example $u(x,t) = e^{x_1+t}$, $x = (x_1, \cdots, x_n)$ on $M^n := \mathbb{R}^n$.

A Liouville type theorem for the heat equation

- Souplet - Zhang (2006): Let M^n be a complete non-compact Riemannian manifold of dimension $n \geq 2$ with Ricci $(M^n) \geq 0$.

  1. If u be a positive ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that
  $$u(p,t) = e^{o(d(p)+\sqrt{|t|})} \qquad \text{as } d(p) \to \infty$$
  then u is a constant.
  2. If u be an ancient solution to the heat equation on $M^n \times (-\infty, T)$ such that
  $$u(p,t) = o(d(p) + \sqrt{|t|}) \qquad \text{as } d(p) \to \infty$$
  then u is a constant.

- Proof: By using a local gradient estimate of Li-Yau type on large appropriately scaled parabolic cylinders.

The Semi-linear heat equation

- Consider next the semilinear heat equation

$$(\star_{SL}) \quad u_t = \Delta u + u^p \quad \text{on } \mathbb{R}^n \times (0, T)$$

in the subcritical range of exponents $1 < p < \frac{n+2}{n-2}$.

- It provides a prototype for the blow up analysis of geometric flows.
- In particular in neckpinches of solutions to the Ricci flow and Mean Curvature flow.
- Also in the characterization of rescaled limits as $t \to -\infty$ of ancient solutions.

The rescaled semi-linear heat equation

- Self-similar scaling at a singularity at (a, T):

$$w(y,\tau) = (T-t)^{\frac{1}{p-1}}\, u(x,t), \;\; y = \frac{x-a}{\sqrt{T-t}}, \;\; \tau = -\log(T-t).$$

- Giga - Kohn (1985): $\|w(\tau)\|_{L^\infty(\mathbb{R}^n)} \leq C, \; \tau > -\log T.$
- The rescaled solution satisfies the equation

$$(\star) \qquad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.$$

- Objective: To analyze the blow up behavior of u one needs to understand the long time behavior of w as $\tau \to +\infty$.
- This is closely related to the classification of bounded eternal solutions of $(\star)$.

Eternal solutions of the semi-linear heat equation

- Problem: Provide the classification of bounded positive eternal solutions w of equation

$$(\star) \qquad w_\tau = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + w^p.$$

- Eternal means that $w(\cdot, \tau)$ is defined for $\tau \in (-\infty, +\infty)$.
- The only steady states of $(\star)$ are the constants:

$$w = 0 \quad \text{or} \quad w = \kappa, \quad \text{with} \;\; \kappa := (p-1)^{-\frac{1}{(p-1)}}.$$

- Theorem (Giga - Kohn '87) $\lim_{\tau \to \pm\infty} w(\cdot, \tau) = $ steady state.
- Space independent eternal solutions : $\phi(\tau) = \kappa(1 + e^\tau)^{-\frac{1}{(p-1)}}$.

Classification of Eternal solutions

- Theorem (Giga - Kohn '87 and Merle - Zaag '98)
  If w is bounded positive eternal solution of $(\star)$ defined on $\mathbb{R}^n \times (-\infty, +\infty)$, then

$$w = 0 \ \text{ or } \ w = \kappa \ \text{ or } \ w = \phi(\tau - \tau_0).$$

- Main difficulty (Merle - Zaag): Classify the orbits $w(\cdot, \tau)$ that connect the two steady states:

$$\lim_{\tau \to -\infty} w(\cdot, \tau) = \kappa \quad \text{and} \quad \lim_{\tau \to +\infty} w(\cdot, \tau) = 0.$$

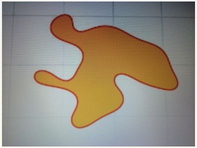
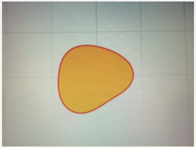
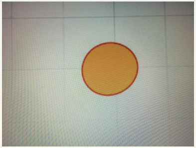
- Recently (2016) C. Collot, F. Merle, P. Raphael revisited the classification of eternal solutions to critical $(\star)$ in dimensions $n \geq 7$ in connection with type II blow up.
- Other Liouville type results related to equation $(\star_{SL})$ by:
  P. Polacik, P. Quittner, T. Bartsch, P. Souplet, E. Yanagida.

The Curve shortening flow

- Let Γ_t be a family of closed curves which is an embedded solution to the Curve shortening flow, i.e. the embedding $F : \Gamma_t \to \mathbb{R}^2$ satisfies

$$\frac{\partial F}{\partial t} = -\kappa\, \nu$$

with κ the curvature of the curve and ν the outer normal.

- M. Gage (1984); M. Grayson (1987); Gage-Hamilton (1996):
If Γ_t is closed and embedded, then it becomes strictly convex and shrinks to a round point.

Ancient Convex solutions to the CSF

- Problem: Classify the ancient closed convex embedded solutions to the Curve shortening flow.
- Evolution of curvature κ:

$$\kappa_t = \kappa_{ss} + \kappa^3 \qquad \text{or} \qquad \kappa_t = \kappa^2 \kappa_{\theta\theta} + \kappa^3$$

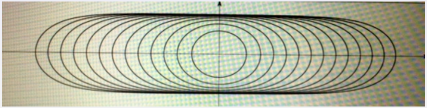
- Examples of ancient solutions:

  i. Type I solution: the contracting circles

  ii. Type II solution: the Angenent ovals (paper clips). Given by $\kappa^2(\theta, t) = \lambda \left(\frac{1}{1-e^{2\lambda t}} - \sin^2(\theta + \gamma) \right)$ and they are not solitons.

The Classification of Ancient Convex solutions to the CSF

- The Angenent ovals as $t \to -\infty$ may be visualized as two Grim reapers moving away from each other.



- Theorem (D., Hamilton, Sesum - 2010)
  The only ancient convex solutions to the CSF are the contracting spheres or the Angenent ovals.

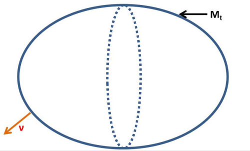
- Proof: Various monotonicity formulas + circular extinction behavior with sharp rates of convergence.

Non-Convex ancient solutions

- Question: Do they exist non convex compact embedded solutions to the curve shortening flow ?
- Angenent (2011): Presents a YouTube video of an ancient solution to the CSF built out from one Yin-Yang spiral and one Grim Reaper.

Ancient solutions to the Mean curvature flow

- Let M_t, $t \in (-\infty, T)$ be a smooth ancient solution of the Mean curvature flow



$$(\text{MCF}) \qquad \frac{\partial F}{\partial t} = -H\,\nu$$

$H(p,t)$ is the Mean curvature and ν a choice of unit normal.

- Problem: Understand ancient solutions M_t of the Mean curvature flow.
- Examples: Self-similar solutions such as self-shrinkers and translating solitons.

Self-similar solutions of MCF

- Look for homothetic (self-similar) solutions to the MCF in the form $M_t = \lambda(t) M_{t_1}$.

- Shrinking solutions (self-shrinkers): $M_t = \sqrt{-t}\, M_{-1}$, for $t \in (-\infty, 0)$ and $H = \langle x, \nu \rangle$.

- Examples: spheres, cylinders.

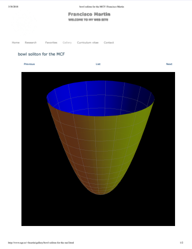
- Expanding solutions (self-expanders): $M_t = \sqrt{t}\, M_1$, for $t \in (0, \infty)$ and $H = -\langle x, \nu \rangle$.

- Translating solutions: move by translations in a direction of vector $\mathbf{v}$, that is, $H = \langle \mathbf{v}, \nu \rangle$ and $F(\cdot, t) = F(\cdot, 0) + \mathbf{v} t$.

- Example: the Bowl soliton.

Bowl soliton

- The Bowl solution is the unique convex rotationally symmetric translating solution to the MCF.



- It opens up like a paraboloid and has the maximum of mean curvature at the tip.
- It is graphical and in terms of the height function $U(x)$ it satisfies the equation

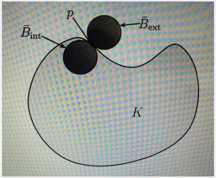
$$\frac{U_{xx}}{1+U_x^2} + \frac{(n-1)U_x}{x} = 1.$$

Non-compact ancient solutions to MCF

- Problem: Classify the non-compact ancient solutions to MCF.
- There are many solutions if you do not assume any extra natural geometric assumptions.
- Theorem (Brendle, Choi 2018): Let M_t, $t \in (-\infty, 0)$ be a noncompact, strictly convex, noncollapsed and uniformly two convex ($\lambda_1 + \lambda_2 \geq \beta H$, for $\beta > 0$) ancient solution to the MCF in $\mathbb{R}^{n+1}$. Then it is the Bowl soliton.
- Corollary: Let M_0 be a closed, 2-convex hypersurface. Evolve it by the MCF. The only possible blow up limits are: spheres, cylinders and the bowl soliton.
- Proof: They establish the rotational symmetry and then classify the radial non-compact ancient solutions.

- Weimin Sheng and Xu-Jia Wang: Introduced an α-noncollapsed condition.

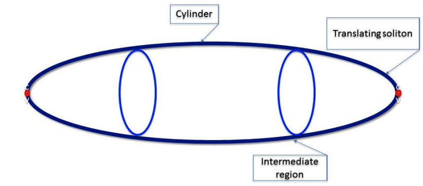
$B = B_{\frac{\alpha}{H(p)}}$



- α-noncollapsed condition is preserved by the mean curvature flow, and hence singularity models are also noncollapsed.
- Haslhofer & Kleiner (2013):
  Ancient compact + α-noncollapsed MCF solution $\Rightarrow$ convex.
- convex compact + self-similar MCF solution $\Rightarrow$ sphere.
- Ancient ovals: Any compact and α-noncollapsed ancient solution to MCF which is not self-similar.

Ancient MCF ovals

- Problem: Provide the classification of all Ancient ovals.
- B. White (2003); R. Haslhofer and O. Hershkovits (2013): Existence of ancient ovals with $O(k) \times O(l)$ symmetry. We call them White ancient ovals.
- Angenent (2012): establihes the formal matched asymptotics of all Ancient ovals as $t \to -\infty$.



- They are small perturbations of ellipsoids.

Ancient ovals

Properties of the White's ancient ovals:

- α-noncollapsed.
- $\liminf_{t\to-\infty} \inf_{M_t} \frac{\lambda_1 + \dots \lambda_{n-j+1}}{H} > 0, \quad j < n-1.$
- $\limsup_{t\to-\infty} \frac{\mathrm{diam}(M_t)}{\sqrt{|t|}} = \infty.$
- $\limsup_{t\to-\infty} \sqrt{|t|}\, \sup_{M_t} |A| = \infty.$

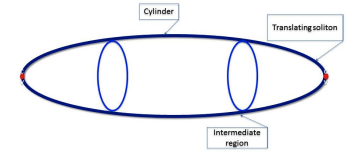
Characterization of the sphere:

(Huisken-Sinestrari, Haslhofer-Hershkowitz) α-noncollapsed solution such that at least one of the following holds.

- $\liminf_{t\to-\infty} \inf_{M_t} \frac{\lambda_1}{H} > 0$
- $\limsup_{t\to-\infty} \frac{\mathrm{diam}(M_t)}{\sqrt{|t|}} < \infty$
- $\limsup_{t\to-\infty} \sqrt{|t|}\, \sup_{M_t} |A| < \infty.$

Unique asymptotics of Ancient MCF ovals

- S. Angenent, D., and N. Sesum (2015): All ancient ovals with $O(1) \times O(n)$ symmetry have unique asymptotics as $t \to -\infty$, and satisfy Angenent's precise matched asymptotics:



- Geometric properties $t \to -\infty$: type II ancient solutions

$$\mathrm{diam}(t) \approx \sqrt{8|t|\log|t|} \quad \text{and} \quad H_{\max}(t) \approx \frac{\sqrt{\log|t|}}{\sqrt{2|t|}}.$$

Uniqueness of Ancient MCF ovals

- Conjecture 1: The Ancient ovals which are $O(n)$ invariant are uniquely determined by their asymptotics at $t \to -\infty$.

  Hence: they are unique (up to dilations and translations).

- Conjecture 2: All uniformly 2-convex Ancient ovals are rotationally symmetric.

- Theorem (Angenent, D., -Sesum (2018)):
  Assume that M_t, $t \in (-\infty, 0)$, is a uniformly two-convex Ancient oval. Then, up to ambient isometries, translations and parabolic rescaling, M_t is either a family of shrinking spheres or it is the White's Ancient oval.

Ancient compact solutions to the 2-dim Ricci flow

- Consider an ancient solution of the Ricci flow

$$\text{(RF)} \qquad \frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}$$

  on a compact manifold M^2 which exists for all time $-\infty < t < T$ and becomes singular at time T.
- In dim 2, we have $R_{ij} = \frac{1}{2} R\, g_{ij}$, where R is the scalar curvature.
- Hamilton (1988), Chow (1991): After re-normalization, the metric becomes spherical at $t = T$.
- Problem: Provide the classification of all ancient compact solutions.

Ancient compact solutions to the 2-dim Ricci flow

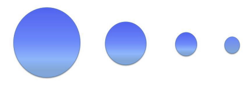
- Choose a parametrization $g_{S^2} = d\psi^2 + \cos^2\psi\, d\theta^2$ of the limiting spherical metric.
- We parametrize our solution as $g(\cdot,t) = u(\cdot,t)\, g_{S^2}$.
- Then the (RF) becomes equivalent to the fast-diffusion equation:

$$u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (-\infty, T).$$

- Provide the classification of all ancient solutions.

Examples of compact solutions on S^2

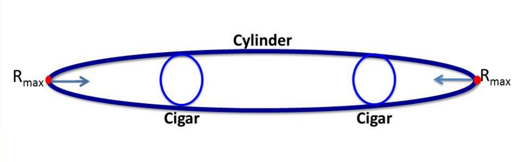
- Type I solution: the contracting spheres.



- Type II solution: the King-Rosenau solution of the form:

$$u(\psi, t) = [a(t) + b(t)\sin^2\psi]^{-1}, \quad t < T.$$

As $t \to -\infty$ the King-Rosenau solution looks like two cigar solitons glued together.



The classification result

Theorem: (D., Hamilton, Sesum - 2012)

The only ancient solutions to the Ricci flow on S^2 are the contracting spheres and the King-Rosenau solutions.

Proof: combines geometry and analysis.

i. a monotonicity formula and uniform a priori $C^{1,\alpha}$ estimates that allow us to pass to the limit as $t \to -\infty$.

ii. geometric arguments that allow us to classify the backward limit as $t \to -\infty$.

iii. maximum principle arguments that allow us to characterize the King-Rosenau solutions among type II solutions.

iv. an isoperimetric inequality that allows us to characterize the contracting spheres among type I solutions.

The 3 dimensional Ricci flow - Open problems

- 3-dim Ricci flow: The analogue of the 2-dim King-Rosenau solutions have been shown to exist by G. Perelman. They are type II and k-noncollapsed.
- Other collapsed compact solutions in closed form have been found by V.A. Fateev in a paper dated back to 1996.
- Conjecture: The only k-noncollapsed ancient and compact solutions to the 3-dim Ricci flow are the contracting spheres and the Perelman solutions.

Ancient solutions to the Yamabe flow

- We will conclude by discussing ancient solutions $g = g_{ij}$ of the Yamabe flow on S^n, $n \geq 3$.
- The Yamabe flow may be viewed as the higher dimensional analogue of the 2-dim Ricci flow.
- It is the evolution of metric $g(\cdot, t)$ conformally equivalent to the standard metric on S^n by

$$\frac{\partial g}{\partial t} = -R\, g \qquad \text{on } -\infty < t < T$$

  where R denotes the scalar curvature of g.
- Question: Is it possible to provide the classification of all such ancient solutions ?

Ancient solutions to the Yamabe flow on S^n

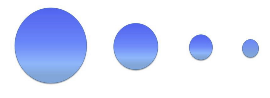
- Let $g = v^{\frac{4}{n-2}} g_{S^n}$ be an ancient solution to the Yamabe flow, which is conformal to the standard metric on S^n.
- The function v evolves by the fast diffusion equation

$$(v^{\frac{n+2}{n-2}})_t = \Delta_{S^n} v - c_n v.$$

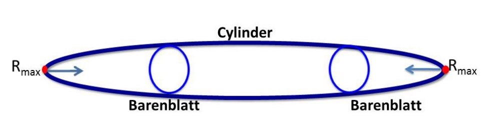
- Problem:
  Provide the classification of ancient solutions $g = v^{\frac{4}{n-2}} g_{S^n}$ to the Yamabe flow, conformal to the standard metric on S^n.

Examples of compact Type I solutions on S^n

- The contracting spheres: given by $v_S(p,t) = c_n\,(T-t)^{\frac{n-2}{4}}$.



- King (1993): there exist non-self similar ancient compact solutions in closed form.



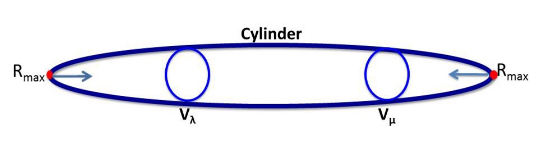
Behavior of King solutions as $t \to -\infty$

- As $t \to -\infty$ the King solutions resemble two Barenblatt self-similar solutions joined with a cylinder.

- Question 1:
  Are the contracting spheres and the King solutions the only examples of type I ancient solutions ?
- Question 2:
  Are there any type II ancient solutions ?

New Type I solutions to the Yamabe flow

- D., del Pino, J. King and N. Sesum (2016)
  There exist infinite many other type I ancient solutions.
- As $t \to -\infty$ they look as two self-similar solutions v_λ, v_μ connected by a cylinder and moving with speeds $\lambda > 0, \mu > 0$.

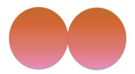


- Our solutions are not given in closed form but we show very sharp asymptotics.
- In similar spirit to the work by Hamel and Nadirashvili (1999) where they construct ancient solutions for the KPP equation

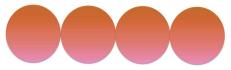
$$u_t = u_{xx} + f(u), \qquad x \in R.$$

Ancient towers of moving bubbles - type II solutions

- Question: Are there any type II ancient solutions to (YF) ?
- D., del Pino and Sesum (2013):
  We construct a class of ancient solutions of the Yamabe flow on S^n which (after re-normalization) converge as $t \to -\infty$ to a tower of n-spheres. They are rotationally symmetric.

$t \to -\infty$  $t > -\infty$ 

- The curvature operator in these solutions changes sign and they are of type II.
- Our construction also holds for any number of bubbles.

$t \to -\infty$ 

- Our construction may be viewed as a parabolic analogue of the elliptic gluing technique.
- Elliptic gluing: pioneering works by Kapouleas '90 -'95 and by Mazzeo, Pacard, Pollack, Ulhenbeck among many others.
- Brendle & Kapouleas (2014): construct new ancient compact solutions to the 4-dim Ricci flow by parabolic gluing.
- Future research direction: apply parabolic gluing on other geometric flows.

Conclusion

- We discussed ancient solutions to geometric parabolic PDE.
- Typical examples are either solitons or other special solutions obtained from the gluing as $t \to -\infty$ of solitons.
- The classification of ancient solutions often contributes to the better understanding of the formation of singularities.
- Most of the existing classification results heavily rely on knowing the exact form of these ancient solutions.
- Future research direction: develop new techniques that allow us to characterize and construct other types of ancient or eternal solutions.

THANK YOU !!!