Motion by curvature of networks in the plane/1

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- *Annibale Magni & Matteo Novaga, 2010 – 2014*
- *Matteo Novaga, Alessandra Pluda & Felix Schulze, 2014 –*
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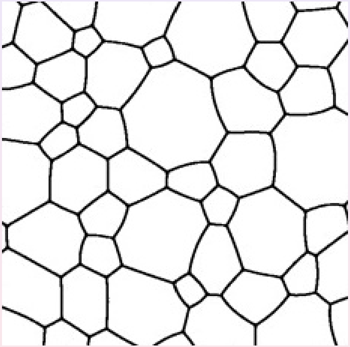

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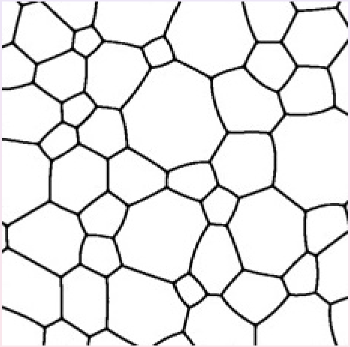

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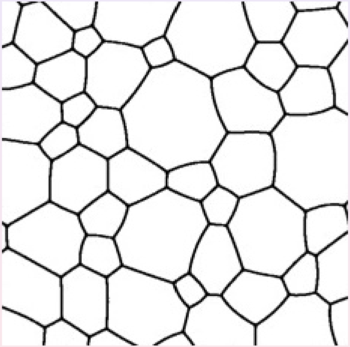

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Even if it is still possible to use several ideas from the “parametric” classical approach to mean curvature flow (differential geometry/maximum principle), some extra variational methods are needed due to the presence of multi–points, that makes the network a singular set (possibly, the simplest).

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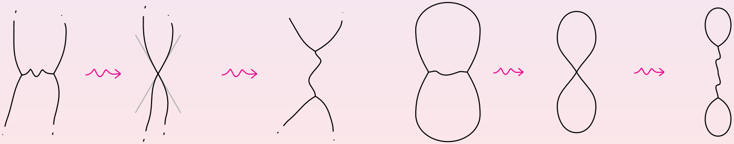

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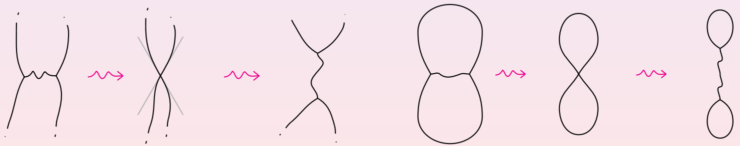
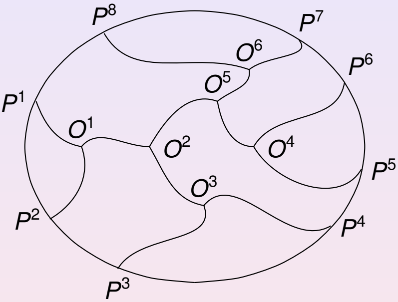


Actually, despite the (apparently) simple problem/behavior/statements, to show in a mathematically satisfactory way these observations, a lot of “technology” from analysis and geometry is needed.

Regular networks

Let Ω be an open, regular and convex subset of $\mathbb{R}^2$.



Definition

A regular network $\mathbb{S} = \bigcup_{i=1}^{n} \sigma^i([0,1])$ in Ω is a connected set described by a finite family of curves $\sigma^i : [0,1] \to \overline{\Omega}$ (sufficiently regular) such that:

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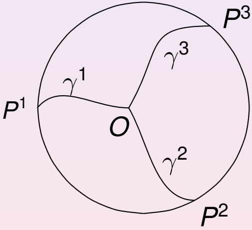
1. the curves cannot intersect each other or self–intersect in their “interior”, but they can meet only at their end–points;
2. if a curve of the network “touches” the boundary of Ω at a fixed point $P \in \partial\Omega$, no other end–point of a curve can coincide with that point;
3. all the junctions points $O^1, O^2, \ldots, O^m \in \Omega$ have order three, considering $\mathbb{S}$ as a planar graph, and at each of them the three concurring curves $\{\sigma^{pi}\}_{i=1,2,3}$ meet in such a way that the external unit tangent vectors τ^{pi} satisfy $\sum \tau^{pi} = 0$ (the curves form three angles of 120 degrees at O^p).

Examples: The triod and the spoon

A triod $\mathbb{T}$ is a network composed only by three regular, embedded curves $\gamma^i : [0, 1] \to \overline{\Omega}$.



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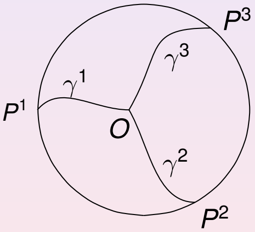
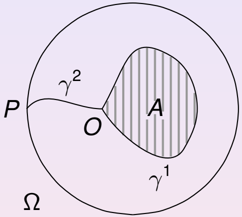
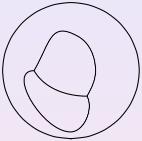
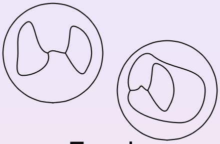
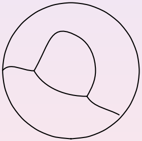
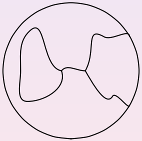
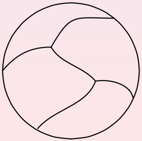




A spoon $\Gamma = \gamma^1([0, 1]) \cup \gamma^2([0, 1])$ is the union of two regular, embedded curves $\gamma^1, \gamma^2 : [0, 1] \to \overline{\Omega}$.

Examples: Networks with two triple junctions

	0 closed curves	1 closed curve	2 closed curves
0 end–points on $\partial\Omega$	 Theta		 Eyeglasses
2 end–points on $\partial\Omega$	 Lens	 Island	
4 end–points on $\partial\Omega$	 Tree		

Motion by curvature

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We say that a regular network moves by curvature if each of its time–dependent curves $\gamma^i : [0, 1] \times [0, T) \to \mathbb{R}^2$ *satisfies*

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To be more precise, a family of regular networks $\mathbb{S}_t$ *is a motion by curvature in a time interval* $[0,T)$ *if the functions* $\gamma^i : [0,1] \times [0,T) \to \overline{\Omega}$ *are at least* C^2 *in space and* C^1 *in time and satisfy the following system:*

$$\begin{cases} \gamma^i_x(x,t) \neq 0 & \\ \sum \tau^i(O,t) = 0 & \text{at every 3–point} \\ \gamma^i_t = k^i\nu^i + \lambda^i\tau^i & \text{for some continuous functions } \lambda^i \end{cases} \qquad (\star)$$

with fixed end–points on $\partial\Omega$*.*

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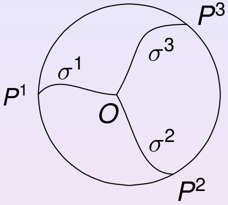
A curvature flow γ^i *for the initial, regular* C^2 *network* $\mathbb{S}_0 = \bigcup_{i=1}^n \sigma^i([0,1])$ *which satisfies*

$$\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}$$

for every $t > 0$ *will be called a* special curvature flow *of* $\mathbb{S}_0$*.*

Short time existence (Triod)



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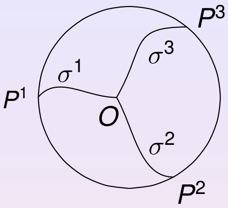

Theorem (Bronsard–Reitich, 1992 & CM–Novaga–Tortorelli, 2004)

For any initial, regular $C^{2+\alpha}$ *triod* $\mathbb{T}_0 = \bigcup_{i=1}^3 \sigma^i([0,1])$*, with* $\alpha \in (0,1)$*, which is* 2–compatible*, there exists a unique special flow in the class* $C^{2+\alpha,1+\alpha/2}([0,1] \times [0,T))$*, in a maximal time interval* $[0,T)$*.*

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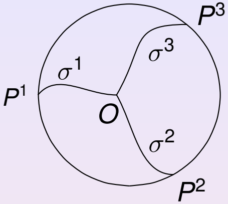

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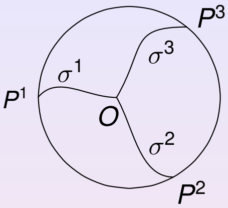

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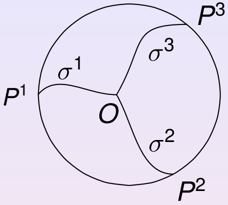

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Viceversa, if the sum of the curvatures is zero at the 3–point there is a reparametrization making the triod 2–compatible (geometric 2–compatibility).

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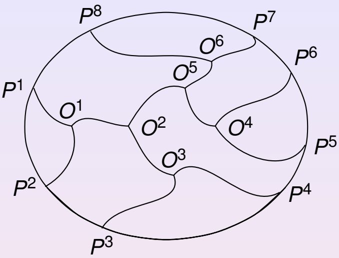

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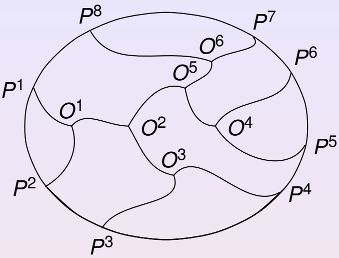

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Theorem

For any initial smooth, regular network $\mathbb{S}_0$ *there exists a unique smooth special flow in a maximal time interval* $[0,T)$*.*

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$\mathbb{T}_0 \in C^{2+\alpha}$, 2–compatible
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A network is *geometrically 2–compatible* if

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Such flow $\mathbb{S}_t = \bigcup_{i=1}^n \gamma^i([0,1],t)$ is a smooth flow for every time $t > 0$.

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- The relevance of this theorem is that the initial network is not required to satisfy any additional condition (2–compatibility), but only to have angles of 120 degrees between the concurring curves at every 3–point, that is, to be regular. Clearly, the curvature function is no more necessarily continuous at $t = 0$ at the triple junctions.

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- The relevance of this theorem is that the initial network is not required to satisfy any additional condition (2–compatibility), but only to have angles of 120 degrees between the concurring curves at every 3–point, that is, to be regular. Clearly, the curvature function is no more necessarily continuous at $t = 0$ at the triple junctions.
- The geometric uniqueness of the solution found in this theorem is an open problem.

Short time existence and uniqueness

How much one could weaken the hypotheses to still obtain existence and uniqueness of the flow?

Theorem

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- The geometric uniqueness of the solution found in this theorem is an open problem.
- General problem with uniqueness: it depends on the class of curves where we look for solutions. No proof of uniqueness in the "natural" class of C^2 in space, C^1 in time curves. Lack of maximum principle.

Short time existence for non–regular networks

All these results and the method work exclusively for *regular* networks, moreover, in order to be able to "restart" the flow after some "collapse" (change of topological structure of the network) we really need an existence theorem also for *non–regular* networks.

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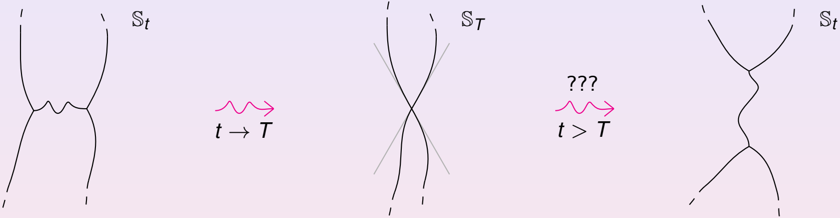

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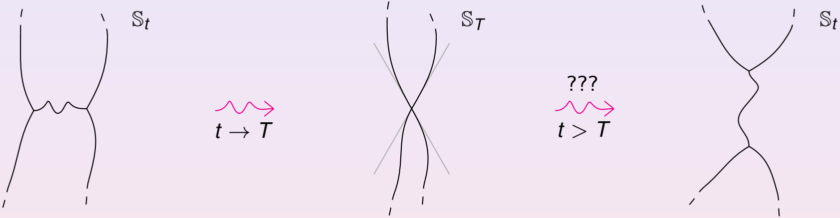

Theorem (Ilmanen–Neves–Schulze, 2014)

For any initial network of non–intersecting curves there exists a (possibly non–unique) Brakke flow by curvature in a positive maximal time interval such that for every positive time the evolving network is smooth and regular.

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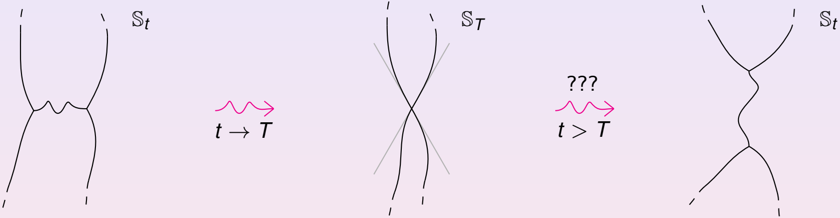

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No hope for uniqueness. Conjecturally, the flow is unique for "generic" initial networks.