Motion by curvature of networks in the plane/2

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Long time behavior

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Theorem

If $T < +\infty$ is the maximal time interval of smooth existence of the curvature flow of a network $\mathbb{S}_t$, then at least one of the following holds:

1. *the length of at least one curve of $\mathbb{S}_t$ goes to zero when $t \to T$,*
2. *the curvature is not bounded as $t \to T$.*

Gagliardo–Nirenberg estimates

Theorem (Niremberg, ”On elliptic partial...”, Ann. SNS 13, 1959 Section 3, pp. 257–263)

Let γ be a C^∞, regular curve in $\mathbb{R}^2$ with finite length L. If u is a C^∞ function defined on γ and $m \geq 1$, $p \in [2, +\infty]$, we have the estimates

$$\|\partial_s^n u\|_{L^p} \leq C_{n,m,p} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m,p}}{L^{m\sigma}} \|u\|_{L^2}$$

for every $n \in \{0, \dots, m-1\}$ where

$$\sigma = \frac{n + 1/2 - 1/p}{m}$$

and the constants $C_{n,m,p}$ and $B_{n,m,p}$ are independent of γ.
In particular, if $p = +\infty$,

$$\|\partial_s^n u\|_{L^\infty} \leq C_{n,m} \|\partial_s^m u\|_{L^2}^{\sigma} \|u\|_{L^2}^{1-\sigma} + \frac{B_{n,m}}{L^{m\sigma}} \|u\|_{L^2} \qquad \text{with} \qquad \sigma = \frac{n + 1/2}{m}$$

Evolution of a triod

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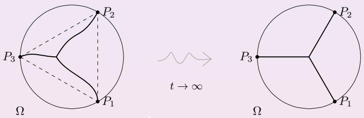

If all the angles of the triangle with vertices the three end–points on the boundary are less than 120 degrees and the initial triod is contained in the triangle, then the triod converges in infinite time to the Steiner configuration connecting the three points.

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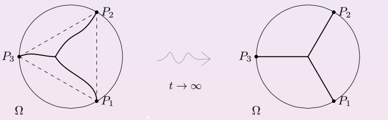

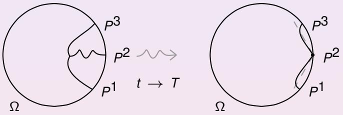

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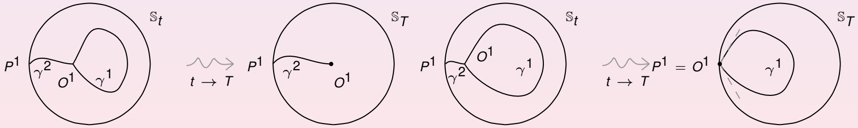
If the fixed end–points on the boundary form a triangle with an angle of more than 120 degrees, then the length of a curve goes to zero in finite time.

Evolution of a spoon

Theorem

The maximal time of existence of a smooth evolution of a spoon is finite and one of the following situations occurs:

1. *the closed loop shrinks to a point in a finite time and the maximum of the curvature goes to $+\infty$, as $t \to T$;*
2. *the "open" curve vanishes and there is a 2–point formation on the boundary of the domain of evolution, but the curvature remains bounded.*

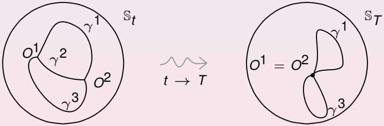


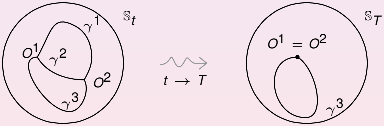
Evolution of a "theta" (double cell)

Theorem

The maximal time of existence of a smooth evolution of a "theta" is finite and one of the following situations occurs:

1. *the length of a curve that connects the two 3–points goes to zero as $t \to T$ and the curvature remains bounded;*
2. *the length of the curves composing one of the loops go to zero as $t \to T$ and the maximum of the curvature goes to $+\infty$.*





In any case the network cannot completely vanish shrinking to a point as $t \to T$ (not easy to show).

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But NO examples in which the lengths of all the curves of the network remain uniformly bounded away from zero during the evolution and the curvature is unbounded, as $t \to T$.

Conjecture

If no length of the curves of the network goes to zero as $t \to T$, then T is not a singular time (maximal time of smooth existence).

We are actually able to show the following:

Theorem

If no length goes to zero and the "Multiplicity–One Conjecture" below is valid, then the curvature is bounded. Hence, T is not a singular time and the flow is smooth.

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Every possible limit of rescaled networks of the flow is a network with multiplicity one.

To show this theorem and in general to understand the nature of the singularities of the flow, we will employ a blow–up technique where the validity of such conjecture will play a key role.