§1 Introduction, preliminaries

Make connection.

+ Brakke's characterization of MCF via inequality '78
+ De Giorgi's minimizing movements ~ 90s
+ Thresholding scheme of Merriman, Bence & Osher '92

Informal motivation: Gradient flows

$$\frac{dx}{dt} = -\operatorname{grad} E(x)$$

dissipate energy

$$\frac{d}{dt} E(X) = -|\operatorname{grad} E(X)|^2 = -\left|\frac{dx}{dt}\right|^2.$$

Can be characterized by

$$E(X(T)) + \frac{1}{2}\int_0^T |\operatorname{grad} E(x)|^2 dt + \frac{1}{2}\int_0^T \left|\frac{dx}{dt}\right|^2 dt \le E(X(0)) \quad (*)$$

$$\text{l.h.s} = E(X(T)) + \frac{1}{2}\int_0^T \left|\frac{dx}{dt} + \operatorname{grad} E(x)\right|^2 dt - \underbrace{\int_0^T \underbrace{\operatorname{grad} E(x)\cdot\frac{dx}{dt}}_{=\frac{d}{dt}E(X)} dt}_{=E(X(T))-E(X(0))}$$

$$= E(X(0)) + \frac{1}{2}\int_0^T \left|\frac{dx}{dt} + \operatorname{grad} E(x)\right|^2 dt$$

LHS terms appear lower semi-continuous
— potentially interesting for existence proofs.

Gradient flows allow for natural time discretization:
Time step size h, time steps $0, 1, 2, \ldots$
times $0, h, 2h,$

$$X^n \text{ minimizes } \frac{1}{2h}|X - X^{n-1}|^2 + E(X).$$

(Euler-Lagrange eqn

$$\frac{1}{h}(x^n - x^{n-1}) + \operatorname{grad} E(x^n) = 0$$

⊂ implicit Euler scheme

This can be formulated even in a metric setting

x^n minimizes $\frac{1}{2h} d^2(x, x^{n-1}) + E(x)$ ← metric/distance fct

Take x^{n-1} as competitor

$$E(x^n) + \frac{1}{2h} d^2(x^n, x^{n-1}) \le E(x^{n-1})$$

Get natural a priori estimate

$$E(x^N) + \sum_{n=1}^{N} \frac{1}{2h} \underbrace{d^2(x^n, x^{n-1})}_{\substack{\text{Euclidean case} \\ = |x^n - x^{n-1}|^2 \\ \approx h^2 \left|\frac{dx}{dt}\right|^2}} \le E(x^0)$$

$$\approx \frac{1}{2} \int_0^{Nh} \left|\frac{dx}{dt}\right|^2 dt$$

Non-optimal by factor of 2.

Misses the $\frac{1}{2}\int_0^T |\operatorname{grad} E(x)|^2 dt$ term in (*)

<u>Lemma 1</u> (De Giorgi, Ambrosio-Gigli-Savaré '04)

(X, d) compact metric space, E cont. fct on (X, d).

$x \in X$.

i) "variational interpolation": for $t > 0$ let $u(t)$ be the minimiser of

$$E(u) + \frac{1}{2t} d^2(u, x),$$

which exists by continuity and compactness. Then missing term

$$E(u(t)) + \frac{1}{2t} d^2(u(t), x) + \int_0^t \frac{1}{2s^2} \underbrace{d^2(u(s), x)}_{\text{is increasing, thus measurable}} ds \le E(x).$$

ii) "metric slope". For $u \in X$ define

$$|\partial E|(u) = \limsup_{v:\, d(u,v) \to 0} \frac{(E(u) - E(v))_+}{d(u,v)} \in [0, +\infty]$$

Then

$$|\partial E(u(t))| \le \frac{1}{t} d(u(t), x).$$

<u>Proof of Lemma 1</u>

① Setting $e(t) := E(u(t)) + \frac{1}{2t} d^2(u(t), x)$ we have for all $0 < s < t$

$$\frac{1}{2ts} d^2(u(s), x) \le \frac{e(s) - e(t)}{t - s} \le \frac{1}{2ts} d^2(u(t), x).$$

Hence $(0, \infty) \ni t \mapsto d(u(t), x)$ is increasing (and thus cont. besides a countable set), and $(0, \infty) \ni t \mapsto e(t)$ is lipschitz, and

$$\frac{de}{dt}(s) + \frac{1}{2s^2} d^2(u(s), x) = 0 \quad \text{where} \ldots \text{is continuous}$$

Integrating over $s \in (z, t)$ we obtain the desired

$$\underset{\substack{\| \\ E(u(t)) + \frac{1}{2}\frac{1}{t} d^2(u(t), x)}}{e(t)} + \frac{1}{2} \int_z^t \frac{1}{s^2} d^2(u(s), x)\, ds = \underset{\substack{\wedge \\ E(x)}}{e(z)}$$

② We have

$$\frac{(E(u(t)) - E(v))_+}{d(u(t), v)} \le \frac{1}{t} d(u(t), x) \left(1 + \frac{1}{2} d(u(t), v)\right)$$

and thus as desired

$$|\partial E(u(t))| \le \frac{1}{t} d(u(t), x)$$

Argument for ①: We first note for all $s, t > 0$

$$e(t) = E(u(t)) + \frac{1}{2t} d^2(u(t), x)$$
$$\le E(u(s)) + \frac{1}{2t} d^2(u(s), x)$$
$$= e(s) + \underbrace{\left(\frac{1}{2t} - \frac{1}{2s}\right)}_{= \frac{s-t}{2st}} d^2(u(s), x),$$

Exchanging the roles of s, t

$$e(s) \le e(t) + \frac{t-s}{2st} d^2(u(t), x)$$

Restricting to $s < t$ and dividing by $t-s$, as desired

$$\frac{1}{2st} d^2(u(s), x) \le \frac{e(s) - e(t)}{t-s} \le \frac{1}{2st} d^2(u(t), x) .$$

Argument for ②:

$$E(u(t)) - E(v) \le \left(e(t) - \frac{1}{2t} d^2(u(t), x)\right) - \left(e(t) - \frac{1}{2t} d^2(v, x)\right)$$
$$= \frac{1}{2t} \underbrace{\left(d^2(v, x) - d^2(u(t), x)\right)}_{= \underbrace{(d(v, x)}_{\le d(u(t), x) + d(u(t), v)} + d(u(t), x)) \underbrace{(d(v, x) - d(u(t), x))}_{\le d(u(t), v)}}$$
$$\le \frac{1}{t}\left(d(u(t), x) + \frac{1}{2} d(u(t), v)\right) d(u(t), v).$$

Take $(\cdot)_+$, divide by $d(u(t), v)$.

The thresholding scheme: $h>0$ time step size;
$\chi^n \in \{0,1\}$ characteristic fct of set Ω^n, $\partial\Omega^n$ evolves by MCF at time nh

Convolution step

$$u^n := G_h * \chi^{n-1}, \quad G_h := \text{heat kernel at time } h$$
$= $ centered Gaussian of variance h

$$G_h(z) := \frac{1}{\sqrt{h}^d} G_1\left(\frac{z}{\sqrt{h}}\right), \quad G_1(z) := \frac{1}{\sqrt{2\pi}^d}\exp\left(-\frac{|z|^2}{2}\right).$$

thresholding step

$$\chi^n = I(u^n > \tfrac{1}{2}) = \begin{cases} 1 & \text{on } u^n > \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

Preserves order $\chi^{n-1} \le \tilde{\chi}^{n-1} \Rightarrow \chi^n \le \tilde{\chi}^n$

From now on $\mathbb{R}^d \rightsquigarrow [0,1)^d$ torus.

We will use a lot the semi-group property $G_h = G_{h/2} * G_{h/2}$ and symmetry of G_h

<u>Lemma 2</u> (Esedoglu & O. '15)

χ^n minimizes
$$E_h(u) + \frac{1}{2h} d_h^2(u, \chi^{n-1})$$
among all $u: [0,1)^d \to [0,1]$ where

$$E_h(u) := \frac{1}{\sqrt{h}} \int (1-u)\, G_h * u,$$

$$\frac{1}{2h} d_h^2(u,u') := \frac{1}{\sqrt{h}} \int (u-u')\, G_h * (u-u') = \frac{1}{\sqrt{h}} \int \left(G_{h/2} * (u-u')\right)^2.$$

Furthermore (X, d_h) with $X := \{u: [0,1)^d \to [0,1] \text{ measurable}\}$ is a compact metric space.

convergence to MCF in the sense of viscosity (Evans '92, Barles & Georgelin '95, Ishii, Pires, Souganidis '99...)

<u>Proposition 1</u> (Alberti & Bellettini '98)

$$E_h \xrightarrow{\Gamma} E_0 := \begin{cases} c_0 \int |\nabla\chi| & \text{for } \chi \in \{0,1\} \\ +\infty & \text{else} \end{cases} \quad c_0 := \frac{1}{\sqrt{2\pi}}$$

under $u \xrightarrow{w} \chi$ in $L^1([0,1)^d)$.

Proof of Lemma 2

$$\frac{1}{2h} d_h^2(u,\chi) + E_h(u)$$

$$= \frac{1}{\sqrt{h}} \int (u-\chi)\, G_h * (u-\chi) + \frac{1}{\sqrt{h}} \int (1-u)\, G_h * u$$

$$\hat{=} \langle u-\chi, u-\chi \rangle + \langle 1-u, u \rangle$$

$$= - \langle u, 2\chi - 1 \rangle + \langle \chi, \chi \rangle$$

$$= -\frac{1}{\sqrt{h}} \int u \underbrace{G_h * (2\chi - 1)}_{= 2G_h * \chi - 1} + \underbrace{\frac{1}{\sqrt{h}} \int \chi\, G_h * \chi}_{\text{independent on } u \text{ – irrelevant}}$$

$$\begin{cases} \geq - \frac{1}{\sqrt{h}} \int (2G_h * \chi - 1)_+ & \text{always} \\ = - \frac{1}{\sqrt{h}} \int (2G_h * \chi - 1)_+ & \text{if } u = I(G_h * \chi > \frac{1}{2}) \end{cases}$$

For the compactness statement, we only have to show for $(u_n)_{n\uparrow\infty}, u \subset X$:

$$u_n \xrightarrow{w} u \text{ in } L^2([0,1)^d) \iff d_h(u_n, u) \to 0$$

"$\Rightarrow$": $u_n \xrightarrow{w} u$ in L^2 $\Rightarrow$ $\left.\begin{array}{l} G_{h/2} * u_n \to G_{h/2} * u \text{ pointwise} \\ G_{h/2} * u_n \in [0,1] \end{array}\right\}$

$$\Rightarrow G_{h/2} * u_n \to G_{h/2} * u \text{ in } L^2 \Rightarrow d_h(u_n, u) \to 0$$

"$\Leftarrow$": $0 \leftarrow d_h^2(u_n, u) = \int |G_{h/2} * (u_n - u)|^2 = \sum_{k \in (2\pi\mathbb{Z})^d} |\mathcal{F}(G_{h/2} * (u_n - u))(k)|^2$

$$= \underbrace{|\mathcal{F}G_{h/2}(k)|^2}_{= \mathcal{F}G_1(\frac{\sqrt{h}}{2}k) = \exp(-\frac{h}{2}|k|^2) > 0} |\mathcal{F}(u_n - u)(k)|^2$$

$$\Rightarrow \left.\begin{array}{l} \mathcal{F}u_n(k) \to \mathcal{F}u(k) \text{ for all } k \in (2\pi\mathbb{Z})^d \\ u_n \in [0,1] \end{array}\right\}$$

$$\Rightarrow u_n \xrightarrow{w} u \quad L^2$$

Besides consistency, Γ-convergence follows from

<u>Lemma 3</u> For any $h>0$, $\chi \in \{0,1\}$

i) $E_h(\chi) \le E_0(\chi)$,

ii) $E_{N^2h}(\chi) \le E_h(\chi)$ for all $N \in \mathbb{N}$, (EO '15)

iii) $\int |\nabla (G_h * \chi)| \lesssim E_h(\chi)$
up to $C(d)$.

<u>Lemma 4</u> For any $h>0$, $u, u' \in [0,1]$

i) $\int |G_h * u - u| \le 2\sqrt{h}\, E_h(u)$

ii) $\int (u-u')^2 \le \frac{1}{2\sqrt{h}} d_h^2(u,u') + 2\sqrt{h}\,(E_h(u)+E_h(u'))$

iii) $|E_h(u) - E_h(u')|^2 \le \frac{1}{h\, 2\sqrt{h}} d_h^2(u,u')$,
in particular, E_h is (Lipschitz) continuous w.r.t. d_h

<u>Lemma 5</u>

Let χ^0 be s.t. $E_0(\chi^0) < \infty$. Let $(\chi^n)_{n\in\mathbb{N}}$ be a solution of the thresholding scheme with initial data χ^0 and time step size $h>0$. Consider the piecewise constant interpolation

$\chi^h(t) = \chi^n$ for $t \in [nh, (n+1)h)$.

Then, as $h \downarrow 0$, χ^h is compact wrt $L^1_{loc}([0,\infty)\times[0,1)^d)$.

<u>Proof of Lemma 3</u>

symmetry of $G_h *$

i) $E_h(\chi) = \frac{1}{\sqrt{h}} \int (1-\chi)\, G_h * \chi = \frac{1}{2\sqrt{h}} \int (1-\chi)\, G_h * \chi + \chi\, G_h * (1-\chi)$

$= \frac{1}{2\sqrt{h}} \int G_h(z) \int \underbrace{(1-\chi)(x)\,\chi(x-z) + \chi(x)\,(1-\chi)(x-z)}_{= |\chi(x-z) - \chi(x)| \text{ since } \chi \in \{0,1\}} dx\, dz$

$\underbrace{\qquad}_{\leq \int |\nu \cdot z|\, |\nabla \chi|}$

$\leq \int \underbrace{\frac{1}{2\sqrt{h}} \int G_h(z)\, |\nu \cdot z|\, dz}\, |\nabla \chi|$

$= \frac{1}{2\sqrt{h}} \int G_h(z)\, |z_1|\, dz = \frac{1}{2} \int G_1(z)\, |z_1|\, dz = \frac{1}{2} \int G_1^{1-d}(z_1)\, |z_1|\, dz_1$

$= \int_0^\infty \underbrace{G_1^{1-d}}(z_1)\, z_1\, dz_1 = G_1^{1-d}(0) = \frac{1}{\sqrt{2\pi}} = c_0$

$= \underbrace{\frac{1}{\sqrt{2\pi}} \exp(-\frac{z_1^2}{2})}$

$= -\frac{d}{dz_1} G_1^{1-d}(z_1)$

ii) $E_h(\chi) = \frac{1}{2\sqrt{h}} \int G_h(z) \int |\chi(x-z) - \chi(x)|\, dx\, dz$

$= \frac{1}{2\sqrt{h}} \int G_1(z) \int |\chi(x - \sqrt{h} z) - \chi(x)|\, dx\, dz,$

$E_{N^2 h}(\chi) = \frac{1}{2\sqrt{h}} \int G_1(z)\, \frac{1}{N} \int \underbrace{|\chi(x - N\sqrt{h} z) - \chi(x)|}\, dx$

$\leq \underbrace{\sum_{n=1}^{N} |\chi(x - n\sqrt{h} z) - \chi(x - (n-1)\sqrt{h} z)|}$

$\leq \int |\chi(x - \sqrt{h} z) - \chi(x)|\, dx.$

iii) We first argue that

$\sqrt{4h}\, |\nabla G_h| \leq G_{4h},$

which by scaling reduces to

$|\nabla G_1| \leq G_4$

and is seen as follows

$$\underbrace{|\nabla G_1(t)|}_{= -t\, G_1(t)} \le \underbrace{|t|}_{\le \frac{1}{2^d}\exp\left(\frac{|t|^2}{4}\right)} \underbrace{G_1(t)}_{= \frac{1}{\sqrt{2\pi}^d}\exp\left(-\frac{|t|^2}{2}\right)} \le \underbrace{\frac{1}{\sqrt{2\pi 4}^d}\exp\left(-\frac{|t|^2}{4}\right)}_{= G_4(t)},$$

From this we obtain via

$$\nabla(G_h * \chi)(x) = \int \nabla G_h(t)\,(\chi(x-t)-\chi(x))\,dt,$$

$$|\nabla(G_h * \chi)(x)| \le \int |\nabla G_h(t)|\,|\chi(x-t)-\chi(x)|\,dt$$

$$\le \frac{1}{\sqrt{h}}\int G_{4h}(t)\,|\chi(x-t)-\chi(x)|\,dt$$

$$\int |\nabla(G_h * \chi)| \le \frac{1}{\sqrt{4h}}\int G_{4h}(t)\int |\chi(x-t)-\chi(x)|\,dx\,dt$$

$$= E_{4h}(\chi) \overset{ii)}{\le} E_h(\chi).$$

Proof of Lemma 4

i) $\int |G_h * u - u| \le \int G_h(z) \int \underbrace{|u(x-z) - u(x)|}_{\le (1-u)(x)u(x-z) + (1-u)(x-z)u(x)} dx\,dz$

$\le 2 \int (1-u)(x)\, u(x-z)\,dx$

$\le 2\sqrt{h}\, E_h(u).$

ii) $\int (u-u')^2$

$= \int (u-u') G_h * (u-u') - \int \underbrace{(u-u')}_{\in[-1,1]} (G_h * u - u) + \int (u-u')(G_h * u' - u')$

$\le \underbrace{\int (u-u') G_h * (u-u')}_{= \sqrt{h} \frac{1}{2h} d_h^2(u,u')} + \underbrace{\int |G_h * u - u|}_{\overset{i)}{\le} 2\sqrt{h} E_h(u)} + \int |G_h * u' - u'|$

iii) $\int (1-u) G_h * u - \int (1-u') G_h * u'$

$\hat{=} \langle 1-u, u\rangle - \langle 1-u', u'\rangle$

$= -\langle u-u', u-u'\rangle + \langle 1-2u', u-u'\rangle$

$= -\underbrace{\int \left(G_{h/2} * (u-u')\right)^2}_{\ge 0} + \int \underbrace{G_{h/2} * \underbrace{(1-2u')}_{\in[-1,1]}}_{\in[-1,1]} G_{h/2} * (u-u')$

$\le \int |G_{h/2} * (u-u')| \le \left(\int |G_{h/2} * (u-u')|^2 \right)^{1/2}$

turns into

$\sqrt{h}\left(E_h(u) - E_h(u')\right) \le \left(\sqrt{h} \frac{1}{2h} d_h^2(u,u') \right)^{1/2}.$

Proof of Lemma 5

By the minimizing movements interpretation (Lemma 2) and the elementary a priori estimate (Lemma 3 i)) we have

$$\sup_N \left(E_h(\chi^N) + \sum_{n=1}^{N} \frac{1}{2h} d_h^2(\chi^n, \chi^{n-1}) \right) \le E_h(\chi^0) \le E(\chi^0) =: E_0 < \infty,$$

By the Fréchet-Kolmogorov criterion, compactness follows from:

① $\sup_t \int |\chi^h(t, x+y) - \chi^h(t,x)| \, dx \lesssim E_0 (|y| + \sqrt{h})$ for all $y \in \mathbb{R}^d$,

i.e. spatial Lipschitz modulus of continuity in $L^\infty_t(L^1_x)$ down to scales $\sqrt{h}$

② $\int_0^T \int |\chi^h(t+\tau, x) - \chi^h(t,x)| \, dx\,dt \lesssim E_0 \left(\tau + T(\underbrace{\tau \vee \sqrt{h}}_{\max}) \right)$

for all $\tau \in [0,T]$,

i.e. temporal Lipschitz modulus of continuity in $L^1_t(L^1_x)$ down to scales $\sqrt{h}$

Argument for ①:

$$\text{lhs} \le 2 \sup_t \underbrace{\int |G_{h/2} * \chi^h - \chi^h| \, dx}_{\overset{\text{Lemma 4 i)}}{\le} 2\sqrt{h} \underbrace{E_h(\chi^h)}_{\le E_0}} + |y| \sup_t \underbrace{\int |\nabla (G_{h/2} * \chi^h)|}_{\overset{\text{Lemma 3 iii)}}{\lesssim} \underbrace{E_h(\chi^h)}_{\le E_0}}$$

For ② we need the intermediate

③ $\int_0^T \int (\chi^h(t+\tau, x) - \chi^h(t,x))^2 \, dx\,dt \lesssim E_0 \left((\tau + h) \frac{\tau}{\sqrt{h}} + T\sqrt{h} \right)$

Argument for ③:

$$\int (\chi^h(t+\tau,x)-\chi^h(t,x))^2\,dx$$

Lemma 4 ii)

$$\le \frac{1}{2\sqrt{h}}\underbrace{d_h^2(\chi^h(t+\tau,\cdot),\chi^h(t,\cdot))}+2\sqrt{h}\,\underbrace{\left(E_h(\chi^h(t+\tau,\cdot))+E_h(\chi^h(t,\cdot))\right)}_{\le 2E_0}$$

by Δ-inequ.

$$\le \Big(\sum_{\substack{n:\,nh>t\\ nh\le t+\tau}} d_h(\chi^n,\chi^{n-1})\Big)^2$$

$$\le \Big(\frac{\tau}{h}+1\Big)\sum_{\substack{n:\,nh>t\\ nh\le t+\tau}} d_h^2(\chi^n,\chi^{n-1})$$

and thus

$$\int_0^T\int (\chi^h(t+\tau,x)-\chi^h(t,x))^2\,dx\,dt$$

$$\le \frac{1}{2\sqrt{h}}\Big(\frac{\tau}{h}+1\Big)\underbrace{\int_0^\infty \sum_{\substack{n:\,nh>t\\ nh\le t+\tau}} d_h^2(\chi^n,\chi^{n-1})\,dt}+T\,4\sqrt{h}\,E_0$$

$$= \tau\underbrace{\sum_{n=1}^\infty d_h^2(\chi^n,\chi^{n-1})}_{\le 2hE_0}.$$

Argument for ②: Writing $\tau = \underset{\in\{0,1,\dots\}}{N}\sqrt{h} + [0,\sqrt{h})$, we see by the Δ-inequality in $L^1_{t,x}$ that it is enough to show

$$\int_0^{2T}\int |\chi^h(t+\tau,x)-\chi^h(t,x)|\,dx\,dt \lesssim E_0(\tau+T\sqrt{h})$$

for $\tau\in[0,\sqrt{h}]$.

This follows since

$$\text{l.h.s} = \int_0^{2T}\int (\underbrace{\chi^h}_{\in\{0,1\}}(t+\tau,x)-\chi^h(t,x))^2\,dx\,dt$$

$$\overset{③}{\lesssim} E_0\Big(\underbrace{(\tau+h)\frac{\tau}{\sqrt{h}}}_{\le 1}+\overset{\text{absorb}}{T}\sqrt{h}\Big)$$