2018-ICTP Summer School on Extrinsic Curvature Flows

The thresholding scheme for mean curvature flow as minimizing movement scheme

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The thresholding scheme

Merriman & Bence & Osher '92:

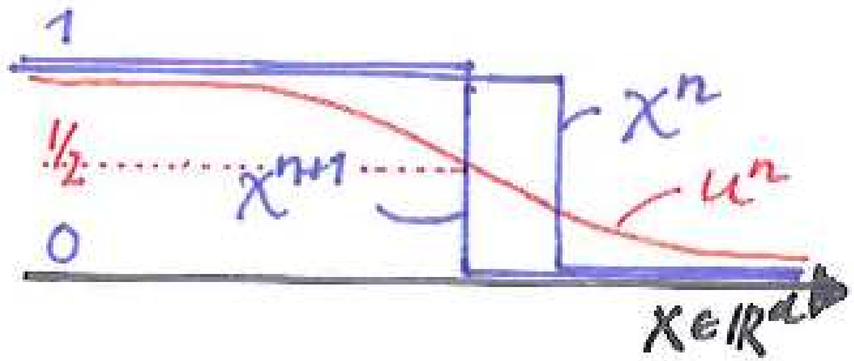
Computational scheme for flow by mean curvature (MCF)

Here just time discretization; time-step size h; $\chi \in \{0, 1\}$

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$

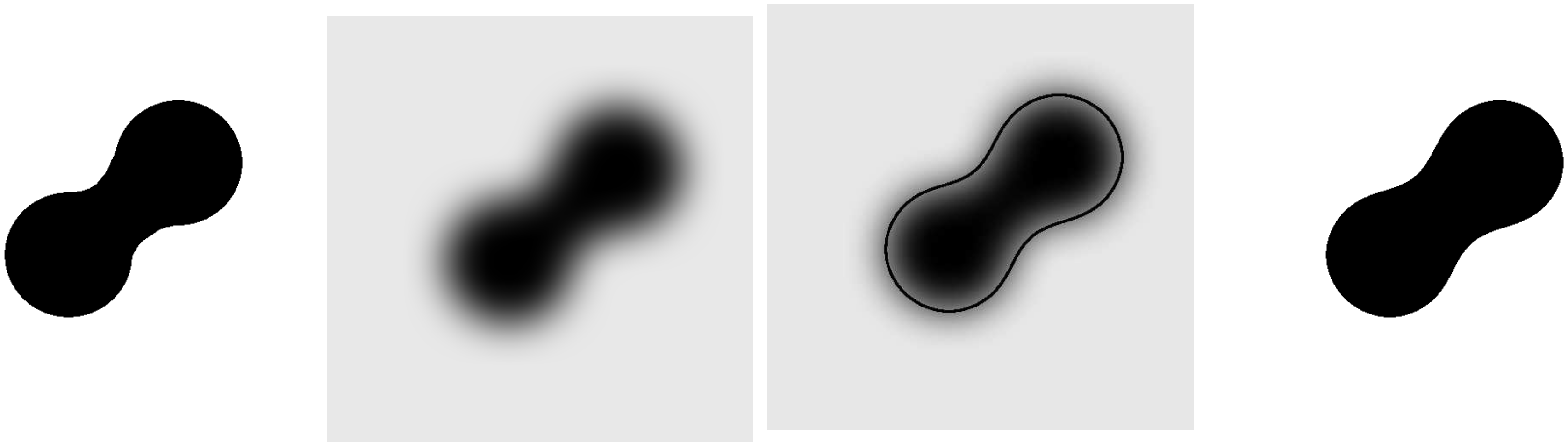
G_h heat kernel at time h

$=$ Gaussian of variance h



Easy to implement

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$



χ^{n-1} $\quad u^n$ $\quad \{u^n = \frac{1}{2}\}$ $\quad \chi^n$

Low complexity: Fast Fourier Transform for convolution

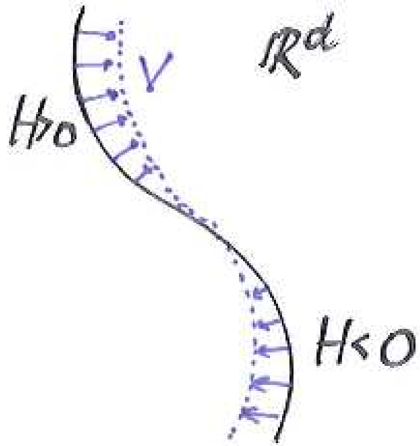
Convergence in the two-phase case

$$\chi^{n-1} \overset{\text{convolution}}{\rightsquigarrow} u^n := G_h * \chi^{n-1} \overset{\text{thresholding}}{\rightsquigarrow} \chi^n := I(u^n \geq \tfrac{1}{2})$$

Thresholding satisfies comparison principle:

$$\chi^{n-1} \leq \tilde{\chi}^{n-1} \Longrightarrow u^n \leq \tilde{u}^n \Longrightarrow \chi^n \leq \tilde{\chi}^n$$

Evans '93: Barles & Georgelin '95



convergence to MCF

in sense of viscosity solution

Straightforward extension to multi-phase version

N phases, eg $\chi = \{\chi_i\}_{i=1,\cdots,N}$ with $\sum_{i=1}^N \chi_i = 1$

$\chi^{n-1} \rightsquigarrow u^n, u_i^n := G_h * \chi_i^{n-1} \rightsquigarrow \chi^n, \chi_i^n := I(u_i^n \geq u_j^n \, \forall j)$

Application to grain growth:

eg Elsey & Esedoglu & Smereka '11

Long-time existence of multi-phase MCF:

Kim & Tonegawa via Brakke's notion of MCF '15

Lemma 1 (de Giorgi, Ambrosio & Gigli & Savaré '04). Let (X, d) be a compact metric space, E a continuous function on it, $\chi \in X$.

i) "Variational interpolation". For $t > 0$ let $u(t)$ be a minimizer of

$E(u) + \frac{1}{2t} d^2(u, \chi)$,

which exists by continuity and compactness. Then

$$E(u(t)) + \frac{1}{2t} d^2(u(t), \chi) + \int_0^t \frac{1}{2s^2} d^2(u(s), \chi) ds \leq E(\chi).$$

ii) "Metric slope". For $u \in X$ define

$|\partial E|(u) := \limsup_{v:d(u,v)\downarrow 0} \frac{(E(u)-E(v))_+}{d(u,v)} \in [0, \infty]$.

Then

$|\partial E|(u(t)) \leq \frac{1}{t} d(u(t), \chi)$.

Usage: Two interpolations for minimizing movements scheme

$\chi(nh+t) = \chi^n$ for $t \in [0, h)$ where χ^n minimizes $E(u)+\frac{1}{2h} d^2(u, \chi^{n-1})$.

$u((n-1)h+t)$ minimizes $E(u) + \frac{1}{2t} d^2(u, \chi^{n-1})$ for $t \in [0, h)$.

$$E(\chi(t+h)) + \int_t^{t+h} \frac{1}{2} |\partial E(\chi(t+h))|^2 + \frac{1}{2} |\partial E(u(s))|^2 ds \leq E(\chi(t)).$$

Thresholding: $\chi^n = I(G_h * \chi^{n-1} > \frac{1}{2})$, $G_h(z) := \frac{1}{(\sqrt{2\pi h})^d} \exp(-\frac{|z|^2}{h})$.

Lemma 2 (Esedoglu & O. '15). χ^n minimizes

$$E_h(u) + \frac{1}{2h} d_h^2(u, \chi^{n-1})$$

among all $u \in X := \{u\colon [0,1)^d \to [0,1] \text{ measurable}\}$, where

$$E_h(u) := \frac{1}{\sqrt{h}} \int (1-u) G_h * u,$$

$$d_h^2(u,u') := \frac{1}{\sqrt{h}} \int (u-u') G_h * (u-u') = \frac{1}{\sqrt{h}} \int (G_{\frac{h}{2}} * (u-u'))^2.$$

Furthermore, (X, d_h) is a compact metric space

Proposition 1 (Alberti & Bellettini '98). E_h Γ-converges to

$$E_0(\chi) := \left\{ \begin{array}{cc} c_0 \int |\nabla \chi| & \text{for } \chi \in \{0,1\} \\ +\infty & \text{else} \end{array} \right\}, \quad c_0 := \frac{1}{\sqrt{2\pi}}$$

under the underlying notion of convergence $u^h \stackrel{w}{\rightharpoonup} \chi$ in $L^1([0,1)^d)$.

Lemma 3. For any $\chi \in \{0,1\}$

i) $E_h(\chi) \leq E_0(\chi)$,

ii) $E_{N^2h}(\chi) \leq E_h(\chi)$ for all $N \in \mathbb{N}$ (Esedoglu&O. '15),

iii) $\int |\nabla G_h * \chi| \lesssim E_h(\chi)$.

Lemma 4. For any $u, u' \in [0,1]$

i) $\int |G_h * u - u| \leq 2\sqrt{h} E_h(u)$,

ii) $\int (u-u')^2 \leq \frac{1}{2\sqrt{h}} d_h^2(u,u') + 2\sqrt{h}(E_h(u) + E_h(u'))$,

iii) $\Big(E_h(u) - E_h(u')\Big)^2 \leq \frac{1}{h 2\sqrt{h}} d_h^2(u,u')$,

in particular, E_h is Lipschitz wrt d_h.

Lemma 5. Let χ^0 be st $E_0(\chi^0) < \infty$. Let $(\chi^n)_{n\in\mathbb{N}}$ be a solution of the thresholding scheme with initial data χ^0 and time step size h. Consider the piecewise constant interpolation

$$\chi^h(t) = \chi^n \quad \text{for } t \in [nh, (n+1)h).$$

Then, as $h \downarrow 0$, χ^h is compact in $L^1_{loc}([0,\infty) \times [0,1)^d)$.

Theorem 3 (Laux & O. '16). Let χ^0 with $E_0(\chi^0) < \infty$ be given. Let χ^h denote the piecewise constant interpolation of the solution $(\chi^n)_{n\in\mathbb{N}}$ of the thresholding scheme with time step size $h > 0$ and initial data χ^0. Suppose that for a subsequence $h \downarrow 0$ and some χ we have for any $T < \infty$

$$\chi^h \to \chi \text{ in } L^1([0,T] \times [0,1)^d), \quad \int_0^T E_h(\chi^h) \to \int_0^T E_0(\chi).$$

Then there exists a mean-curvature $H \in L^2(|\nabla\chi| dt)$, ie

$$\int (\nabla \cdot \xi - \nu \cdot D\xi\nu - \xi \cdot \nu H)|\nabla\chi| = 0$$

for all $\xi \in C^\infty([0,1)^d)^d$ and ae in time,

and we have $2V = H$ in the sense of Brakke's inequality

$$\int \zeta|\nabla\chi(T)| + \frac{1}{2}\int_0^T \int (\zeta H^2 + \nabla\zeta \cdot \nu H)|\nabla\chi| \le \int \zeta|\nabla\chi^0|$$

for all nonnegative $\zeta \in C^\infty([0,1)^d)$ and ae T.

Lemma 6. For any nonnegative $\zeta \in C^\infty([0,1)^d)$ we have de Giorgi's minimizing movements structure

$$\chi^n \quad \text{minimizes} \quad \tilde{E}_h(u,\chi^{n-1}) + \frac{1}{2h}\tilde{d}_h^2(u,\chi^{n-1}) \quad \text{among all } u \in X,$$

where

$$\frac{1}{2h}\tilde{d}_h^2(u,u') := \frac{1}{\sqrt{h}}\int \zeta (G_{\frac{h}{2}} * (u-u'))^2,$$

$$\tilde{E}_h(u,\chi) := \frac{1}{\sqrt{h}}\int \zeta(1-u)G_h * u + (u-\chi)[\zeta, G_h *](1-\chi)$$
$$+ (u-\chi)[\zeta, G_{\frac{h}{2}} *]G_{\frac{h}{2}} * (u-\chi).$$

Furthermore, $(X, \tilde{d}_h)$ is a compact metric space provided $\zeta > 0$.

$$E(\chi^n) + \int_{(n-1)h}^{nh} \frac{1}{2}|\partial E(\chi^n)|^2 + \frac{1}{2}|\partial E(u(t))|^2 dt \leq E(\chi^{n-1}).$$

$$\begin{aligned}&\tilde{E}_h(\chi^n, \chi^{n-1}) \\ &\quad + \int_{(n-1)h}^{nh} \frac{1}{2}|\partial \tilde{E}_h(\cdot, \chi^{n-1})|^2(\chi^n) + \frac{1}{2}|\partial \tilde{E}_h(\cdot, \chi^{n-1})|^2(\tilde{u}^h(t))dt \\ &\quad \leq \tilde{E}_h(\chi^{n-1}, \chi^{n-1}).\end{aligned}$$

$$\begin{aligned}&\tilde{E}_h(\chi^h(T), \chi^h(T)) \\ &\quad + \int_0^T \frac{1}{2}|\partial \tilde{E}_h(\cdot, \chi^h(t))|^2(\chi^h(t+h)) + \frac{1}{2}|\partial \tilde{E}_h(\cdot, \chi^h(t))|^2(\tilde{u}^h(t))dt \\ &\quad + \int_0^T \frac{1}{h}\Big(\tilde{E}_h(\chi^h(t+h), \chi^h(t)) - \tilde{E}_h(\chi^h(t+h), \chi^h(t+h))\Big)dt \leq \tilde{E}_h(\chi^0, \chi^0).\end{aligned}$$

$$\int \zeta|\nabla\chi(T)| + \frac{1}{2}\int_0^T \int \zeta H^2|\nabla\chi|dt + \frac{1}{2}\int_0^T \int \nabla\zeta \cdot \nu H|\nabla\chi|dt \leq \int \zeta|\nabla\chi^0|.$$

$|\partial E(u)| := \limsup_{v:d(u,v)\downarrow 0} \frac{(E(u)-E(v))_+}{d(u,v)}$

$\frac{1}{2}|\partial E(u)|^2 \geq \limsup_{v:d(u,v)\downarrow 0}\{(E(u)-E(v))_+ - \frac{1}{2}d^2(u,v)\}$

First variation $\delta E(u,\xi)$ of a function E in a configuration $u \in X$ in direction of a vector field $\xi \in C^\infty([0,1)^d)^d$: $\delta E(u,\xi) := \frac{d}{ds}_{|s=0} E(u_s)$ where $\partial_s u_s + \xi \cdot \nabla u_s = 0$ with $u_{|s=0} = u$.

$$\frac{1}{2}|\partial \tilde{E}_h(\cdot,\chi)|^2(u) \geq \sup_\xi \left\{ \delta \tilde{E}_h(\cdot,\chi)(u,\xi) - \sqrt{h}\int \zeta(G_{\frac{h}{2}} * (\xi\cdot\nabla)u)^2 \right\}$$

Lemma 7 (Localization and first variation commute).
For any $u, \chi \in X$ and $\xi \in C^\infty([0,1)^d)^d$

$$|\delta \tilde{E}_h(\cdot,\chi)(u,\xi) - \delta E_h(u,\zeta\xi)| \lesssim_{\zeta,\xi} h^{\frac{1}{4}} \frac{d_h(u,\chi)}{h}.$$

$\frac{d_h(u,\chi)}{h}$ is controlled by standard a priori estimate

Lemma 8. For any $u, \chi \in X$

$$\left|\frac{1}{h}\Big(\tilde{E}_h(u,\chi) - \tilde{E}_h(u,u)\Big) + \frac{1}{2}\delta \frac{1}{2h} d_h^2(\cdot,\chi)(u,\nabla\zeta)\right|$$
$$\lesssim_{\zeta,\xi} h^{\frac{1}{4}} \frac{d_h(u,\chi)}{h} + h \frac{d_h^2(u,\chi)}{h^2} + h^{\frac{1}{2}}(E_h(u) + E_h(\chi))^{\frac{1}{2}}.$$

Euler-Lagrange equation for un-localized minimizing movements interpretation of thresholding: For any $\xi \in C^\infty([0,1)^d)^d$

$$\delta \frac{1}{2h} d_h^2(\cdot,\chi^{n-1})(\chi^n,\xi) = -\delta E_h(\chi^n,\xi).$$

Lemma 9 (à la Luckhaus-Modica, Reshetniak).
For any $\{u^h\}_{h\downarrow 0} \subset X$ and $\chi \in \{0,1\}$ with

$$u^h \overset{L^1}{\to} \chi, \quad E_h(u^h) \to E_0(\chi)$$

we have for all $\zeta \in C^\infty([0,1)^d), \xi \in C^\infty([0,1)^d)^d$

$$\tilde{E}_h(u^h,u^h) = \frac{1}{\sqrt{h}} \int \zeta(1-u^h) G_h * u^h \to c_0 \int \zeta |\nabla\chi|,$$

$$\delta E_h(u^h,\xi) \to \delta E_0(\chi,\xi) = c_0 \int (\nabla\cdot\xi - \nu\cdot D\xi\nu)|\nabla\chi|.$$

Recall $\frac{1}{2}|\partial\tilde{E}_h(\cdot,\chi)|^2(u) \geq \sup_\xi \left\{\delta\tilde{E}_h(\cdot,\chi)(u,\xi) - \sqrt{h}\int \zeta(G_{\frac{h}{2}} * (\xi\cdot\nabla)u)^2\right\}$

Lemma 10 (Limit in infinitesimal metric / metric tensor).
For any $\{u^h\}_{h\downarrow 0} \subset X$ and $\chi \in \{0,1\}$ with

$$u^h \stackrel{L^1}{\to} \chi, \quad E_h(u^h) \to E_0(\chi)$$

we have for all $\zeta \in C^\infty([0,1)^d), \xi \in C^\infty([0,1)^d)^d$

$$\sqrt{h}\int \zeta(G_{\frac{h}{2}} * (\xi\cdot\nabla)u^h)^2 \to c_0 \int \zeta(\xi\cdot\nu)^2|\nabla\chi|.$$