Motion by curvature of networks in the plane/4

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Analysis of singularities – Collapse with bounded curvature

As a first remark, regions cannot collapse in this situation, otherwise the curvature goes to $+\infty$. Moreover, only “isolated” curves can vanish, by the *Multiplicity–One Conjecture*, that is, it is not possible that more than two triple junctions collide together at a single point. Indeed, in such case there must be a region collapsing.

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Theorem

If **M1** *is true, every* interior *vertex of such limit network either is a regular triple junction or it is a* 4*–point where the four concurring curves have opposite unit tangents in pairs and form angles of* $120/60$ *degrees among them.*

This is exactly what we saw in the simulation when a single curve vanishes.

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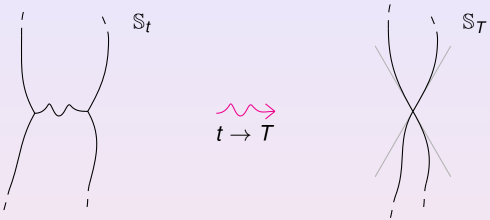

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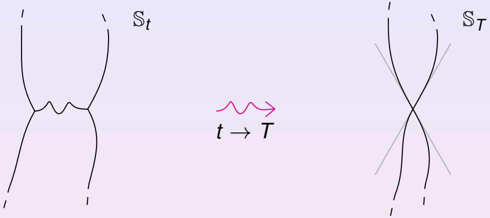

This analysis can be extended to the curves containing the fixed boundary points (*boundary curves*). If any of them vanishes, we get two curves forming an angle of 120 degrees (remember the example of the *spoon* network). In such case, the flow stops (we really “have to decide” whether/how to continue the flow).

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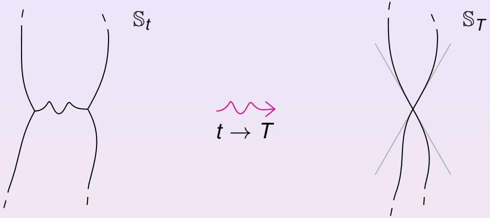

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As the limit network $\mathbb{S}_T$ contains at least one 4–junction, in order to restart the flow we need the short time existence theorem of Ilmanen, Neves and Schulze.

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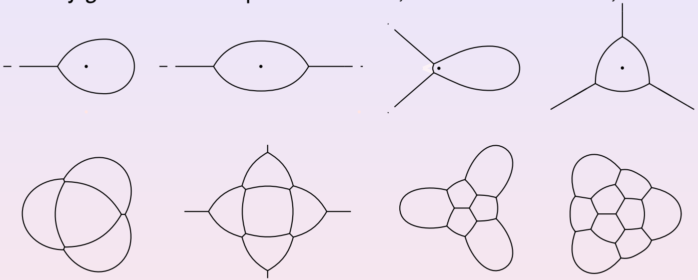
Also this corollary can be localized.

Analysis of singularities – Collapse of a region/Unbounded curvature

In the situation when the curvature is unbounded, that is, at least one region is collapsing, even if **M1** is true, using again the blow–up procedure we can obtain a very *general* blow–up limit shrinker, like the ones below, for instance:



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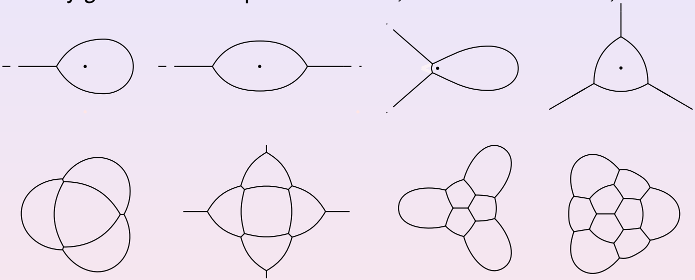

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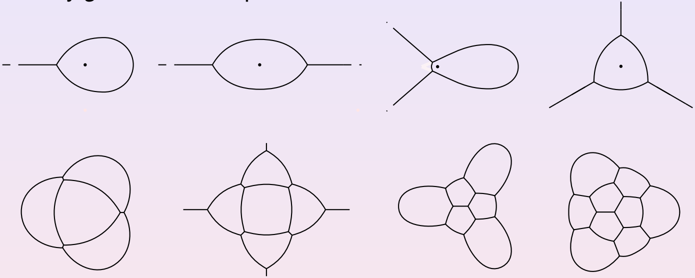

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In this case the blow–up limit is a shrinker *with regions*. These regions are the “memory” of the collapsing regions in the flow $\mathbb{S}_t$, as $t \to T$, and the unbounded halflines of the shrinker give the *limit tangents* of the *non–vanishing* curves of $\mathbb{S}_t$ arriving at the group of collapsing regions.

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Assuming such uniqueness, one can prove that (possibly after reparametrization) the *vanishing* part of $\mathbb{S}_t$ collapse to a point and the *non–vanishing* part converge in C^1 to a limit network $\mathbb{S}_T$. The point of collapse is then a multi–point of $\mathbb{S}_T$ and the curves of $\mathbb{S}_T$ concurring at such point are of class C^∞ far from the multi–point.
Moreover, the curvature of each of such curves goes like $k = o(1/d)$, where d is the arclenght distance to the multi–point along the curve.

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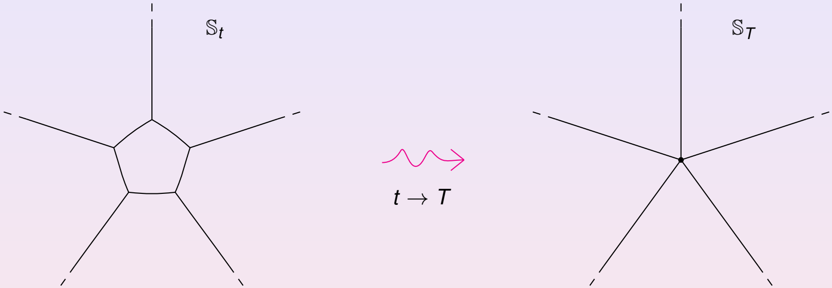

Example of a (homothetic) collapse of a (symmetric) pentagonal region of $\mathbb{S}_t$ (*five–ray star*).