§ 2 Main results, Almgren-Taylor-Wang, Brakke

Informal: MCF ($2V = H$) ← normal velocity, ← mean curvature
is gradient flow of the (the) for curvature

$$\int |\nabla \chi| \text{ "}=\text{ } \mathcal{H}^{d-1}(\partial\Omega)\text{"}$$

with first variation

$$\delta\left(\int|\nabla\chi|\right)(\xi) = \int \left(\underbrace{\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu}_{\text{tangential divergence}}\right)|\nabla\chi| \text{ "}=\text{ } \int_{\partial\Omega} H\,\nu\cdot\xi \, d\mathcal{H}^{d-1}$$

w.r.t. to the metric tensor given by "$4\times$"
the $L^2(|\nabla\chi|)$ inner product. This
formal Riemannian structure is degenerate in the
sense that induced metric is degenerate.
(Michor & Mumford '06): In order to formulate
a minimizing movement discretization,
need a proxy for induced metric.

Almgren-Taylor Wang '93:

χ^n minimizes

$$\mathcal{H}^{d-1}(\partial\Omega) + \frac{1}{2h}\int_{\Omega\Delta\Omega^{n-1}} 4\,\mathrm{dist}(\cdot, \Omega^{n-1})$$

<u>Theorem 1</u> (Luckhaus-Sturzenhecker '95)

Let $h \downarrow 0$ and χ be such that

$$\chi^h \longrightarrow \chi \text{ in } L^1, \quad \int_0^T\!\!\int|\nabla\chi^h| \longrightarrow \int_0^T\!\!\int|\nabla\chi|$$

(fixed finite time horizon)

χ^h: piecewise constant interpolation of ATW scheme for given χ^0 with $\int|\nabla\chi^0| < \infty$, cf Lemma 5

convergence assumption, for free $\int_0^T\!\!\int|\nabla\chi| \le \liminf_{h\downarrow 0}\int_0^T\!\!\int|\nabla\chi^h|$ rules out ghost interfaces

Then there exists $V \in L^2(|\nabla \chi| dt)$ that is the normal velocity in the sense of

$$\int_0^T \left(\int \zeta V |\nabla \chi| + \int \partial_t \zeta \, \chi \right) + \int \zeta(t=0) \chi^0 = 0$$

for all $\zeta \in C_0^\infty([0,T) \times [0,1)^d)$,

and we have $2V = H$ in the sense of

$$\int_0^T \int (\nabla \cdot \xi - \nu \cdot D\xi \, \nu - 2 \xi \cdot \nu V) |\nabla \chi| = 0$$

for all $\xi \in C_0^\infty((0,T) \times [0,1)^d)^d$.

<u>Theorem 2</u> (Laux & O. '16 + multiphase)

Let $h \downarrow 0$ and χ be such that

$\chi^h \longrightarrow \chi$ in L^1 $\qquad \int_0^T E_h(\chi^h) \longrightarrow \int_0^T E_0(\chi)$

(χ^h: piecewise constant interpolation of thresholding scheme with initial data χ^0 s.t. $E_0(\chi^0) < \infty$)

Then χ is as in Theorem 1.

Does not imply $2 \frac{d}{dt} \int |\nabla \chi| \leq -\int H^2 |\nabla \chi|$ dissipation inequality.

Brakke's notion of solution amounts to a localization of the dissipation inequality; based on the "kinetic identity"

$$\frac{d}{dt} \int \zeta |\nabla \chi| = \int (-\zeta H V + V \nu \cdot \nabla \zeta + \partial_t \zeta) |\nabla \chi|,$$

namely

$$2\frac{d}{dt}\int\zeta|\nabla\chi| \le \int(-\zeta H^2 + H\nu\cdot\nabla\zeta + 2\partial_t\zeta)|\nabla\chi|$$

for all $\zeta \ge 0$.

<u>Theorem 3</u> (Laux & O. '17 + multiphase)
Let $h \downarrow 0$ and χ be as in Theorem 2.
Then there exists $H \in L^2(|\nabla\chi|dt)$ that is the mean curvature in the sense of

$$\int_0^T\int(\nabla\cdot\xi - \nu\cdot\nabla\xi\,\nu - \xi\cdot\nu H)|\nabla\chi| = 0$$

for all $\xi \in C_0^\infty((0,T)\times[0,1)^d)^d$

and we have $2V = H$ in the sense of Brakke:

$$-2\int\zeta(t=0)|\nabla\chi^0| \le \int_0^T\int(-\zeta H^2 + \nabla\zeta\cdot\nu H + 2\partial_t\zeta)|\nabla\chi|$$

for all $\zeta \in C_0^\infty([0,T)\times[0,1)^d)$, $\zeta \ge 0$.

Main idea: Derive Brakke's inequality in form of

$$\int\zeta|\nabla\chi(t=T)| + \frac{1}{2}\int_0^T\int(\zeta H^2 + \nabla\zeta\cdot\nu H)|\nabla\chi| \le \int\zeta|\nabla\chi^0|$$

for fixed $\zeta \in C^\infty([0,1)^d)$, $\zeta \ge 0$, from Lemma 1

$$E(u(h)) + \int_0^h \frac{1}{2}|\partial E|^2(u(h)) + \frac{1}{2}|\partial E|^2(u(t))\,dt \le E(\chi)$$

We clearly need a localized version of the energy E_h

§3 Structure of argument

Lemma 6 (localization of Lemma 2)

For any $\zeta \in C^\infty([0,1]^d)$, $\zeta \geq 0$ we have that χ^n minimizes

$$\tilde{E}_h(u, \chi^{n-1}) + \frac{1}{2h}\tilde{d}_h^2(u, \chi^{n-1}) \text{ among all } u:[0,1]^d \to [0,1]$$

where

$$\tilde{E}_h(u,\chi) := \frac{1}{\sqrt{h}}\int \Big(\zeta(1-u)G_h * u$$
$$+ (u-\chi)[\zeta, G_h *](1-\chi)$$

commutator $\zeta G_h * - G_h * \zeta$

$$+ (u-\chi)[\zeta, G_{h/2} *]\, G_{h/2} * (u-\chi)\Big),$$

$$\frac{1}{2h}\tilde{d}_h^2(u,\chi) := \frac{1}{\sqrt{h}}\int \zeta \left(G_{h/2} * (u-\chi)\right)^2.$$

Furthermore, $(\mathcal{X}, \tilde{d}_h)$ is a compact metric space provided $\zeta > 0$.

From Lemma 1 we obtain

$$\tilde{E}_h(\chi^h(T), \chi^h(T)) + \int_0^T \frac{1}{2}|\partial \tilde{E}_h(\cdot, \chi^h(t))|^2(\chi^h(t+h))$$
$$+ \frac{1}{2}|\partial \tilde{E}_h(\cdot, \chi^h(t))|^2(\tilde{u}^h(t))\Big)\,dt$$

$$+ \int_0^T \frac{1}{h}\Big(\tilde{E}_h(\chi^h(t+h), \chi^h(t)) - \tilde{E}_h(\chi^h(t+h), \chi^h(t+h))\Big)dt$$

$$\leq \tilde{E}_h(\chi^0, \chi^0)$$

Goal: This turns into

$$\int \zeta |\nabla \chi(T)| + \frac{1}{2}\int_0^T\int \zeta H^2 |\nabla \chi|$$
$$+ \frac{1}{2}\int_0^T \int \nabla\zeta \cdot \nu H |\nabla \chi|$$
$$\leq \int \zeta |\nabla \chi^0|.$$

Proof of Lemma 6

$$\tilde{E}_h(u, x^{n-1}) + \frac{1}{2h} \tilde{d}_h^2(u, x^{n-1})$$

$$= \frac{1}{\sqrt{h}} \int \Big\{ \nabla(1-u) G_h * u + (u-x) \underbrace{[\nabla, G_h*](1-x)}_{= \nabla G_h*(1-x) - G_h*\nabla(1-x)}$$

$$+ \nabla\big(G_{h/2}*(u-x)\big)^2 + (u-x) \underbrace{[\nabla, G_{h/2}*] G_{h/2}*(u-x)}_{= \nabla G_h*(u-x) - G_{h/2}*\nabla G_{h/2}*(u-x)} \Big\}$$

Symmetry of $G_{h/2}*$, G_h*

$$= \frac{1}{\sqrt{h}} \int \nabla \Big\{ (1-u) G_h*u + (u-x) \underbrace{G_h*(1-x)}_{=1-G_h*x} - (1-x) G_h*(u-x) + (u-x) G_h*(u-x) \Big\}$$

$$-(1-x)G_h*(u-x) + (u-x)G_h*(u-x) = -(1-u) G_h*(u-x)$$

$$\ldots = (1-u) G_h*x$$

$$= -u(2G_h*x - 1) + G_h*x - x(1-G_h*x)$$

$$+ \frac{1}{\sqrt{h}} \int \underbrace{\Big\{ \nabla\big(G_{h/2}*(u-x)\big)^2 - G_{h/2}*(u-x)\, \nabla G_{h/2}*(u-x) \Big\}}_{=0}$$

$$= -\frac{1}{\sqrt{h}} \int \nabla\, u\,(2G_h*x - 1) + \frac{1}{\sqrt{h}} \int \nabla\big(G_h*x - x(1-G_h*x)\big)$$

- $\nabla \geq 0$; $u(2G_h*x-1)$ for $u \in [0,1]$ maximal for $u = I(G_h*x > \frac{1}{2})$, see Lemma 2
- second term: irrelevant, since independent of u

<u>Lemma 7</u> (localization commutes with 1st variation). For any $u, \chi \in [0,1]$

$$\Big|\underbrace{\delta\tilde{E}_h(\cdot,\chi)(u,\xi)}_{\text{first variation}} - \underbrace{\delta E_h(u,\zeta\xi)}_{\text{first variation}}\Big| \lesssim_{\zeta,\xi} h^{1/4}\underbrace{\frac{d_h(u,\chi)}{h}}_{\text{ok by energy's estimate}}$$

<u>Proof of Lemma 7</u>

(1) $\delta\tilde{E}_h(\cdot,\chi)(u,\xi) - \delta E_h(u,\zeta\xi)$

$$= \frac{2}{\sqrt{h}}\int\big(G_{h/2}*(u-\chi)\big)[\zeta, G_{h/2}*](\xi\cdot\nabla)u$$

Argument for (1): If

$$i\partial_s u_s + \xi\cdot\nabla u_s = 0, \quad u_{s=0} = u$$

then

$$\delta\tilde{E}_h(\cdot,\chi)(u,\xi) := \frac{d}{ds}\Big|_{s=0}\tilde{E}_h(u_s,\chi)$$

By definition of $\tilde{E}_h$:

$\delta\tilde{E}_h(\cdot,\chi)(u,\xi)$

$$= \frac{1}{\sqrt{h}}\int\Big\{\zeta(\xi\cdot\nabla)u\, G_h*u - \zeta(1-u)G_h*(\xi\cdot\nabla)u$$

$$-(\xi\cdot\nabla)u\,[\zeta, G_h*](1-\chi)$$

$$-(\xi\cdot\nabla)u\,[\zeta, G_{h/2}*]G_{h/2}*(u-\chi) - (u-\chi)[\zeta, G_{h/2}*]G_{h/2}*(\xi\cdot\nabla)u\Big\}$$

and thus of E_h:

$\delta E_h(u,\zeta\xi)$

$$= \frac{1}{\sqrt{h}}\int\Big\{(\zeta\xi)\cdot\nabla u\, G_h*u - (1-u)G_h*((\zeta\xi)\cdot\nabla)u\Big\}$$

So that

$$\delta \tilde{E}_h(\cdot,\chi)(u,\xi) - \delta E_h(u,\xi)$$

$$= \frac{1}{\sqrt{h}} \int \{ -(1-u)[\zeta, G_h*](\xi\cdot\nabla)u$$
$$- (\xi\cdot\nabla)u\,[\zeta, G_h*](1-\chi)$$
$$- (\xi\cdot\nabla)u\,[\zeta, G_{h/2}*]G_{h/2}*(u-\chi) - (u-\chi)[\zeta, G_{h/2}*]G_{h/2}*(\xi\cdot\nabla)u \}$$

symmetry of G_h*, $G_{h/2}*$ $\Rightarrow$ antisymmetry of $[\zeta, G_h*]$, $[\zeta, G_{h/2}*]$

$$= \frac{1}{\sqrt{h}} \int \{ \underbrace{-(1-u)[\zeta, G_h*](\xi\cdot\nabla)u + (1-\chi)[\zeta, G_h*](\xi\cdot\nabla)u}_{=(u-\chi)[\zeta, G_h*](\xi\cdot\nabla)u}$$
$$+ (u-\chi)G_{h/2}*[\zeta, G_{h/2}*](\xi\cdot\nabla)u - (u-\chi)[\zeta, G_{h/2}*]G_{h/2}*(\xi\cdot\nabla)u \}$$

$$= (u-\chi)\Big(\underbrace{[\zeta, G_h*]}_{=\zeta G_h* - G_{h/2}*G_{h/2}*\zeta} + G_{h/2}*[\zeta, G_{h/2}*] - \underbrace{[\zeta, G_{h/2}*]G_{h/2}*}_{=-\zeta G_h* + G_{h/2}*\zeta G_{h/2}*}\Big)(\xi\cdot\nabla)u$$

$$= G_{h/2}*[\zeta, G_{h/2}*]$$

$$= 2\,G_{h/2}*[\zeta, G_{h/2}*]$$

$$= \frac{2}{\sqrt{h}} \int (u-\chi)\, G_{h/2}*[\zeta, G_{h/2}*](\xi\cdot\nabla)u$$

$$= \frac{2}{\sqrt{h}} \int \big(G_{h/2}*(u-\chi)\big)[\zeta, G_{h/2}*](\xi\cdot\nabla)u$$

(2) $$\left| \frac{2}{\sqrt{h}} \int G_{h/2}*(u-\chi)\,[\zeta, G_{h/2}*](\xi\cdot\nabla)u \right|$$

$$\lesssim \big(\|\nabla\zeta\|_\infty \|\xi\|_\infty + \|\zeta\|_\infty \|\nabla\cdot\xi\|_\infty\big)\, h^{1/4}\, \frac{d_h(u,\chi)}{h}$$

Argument for ②: By Cauchy-Schwarz

$$\left|\frac{2}{\sqrt{h}}\int G_{h/2}*(u-\chi)\,[\zeta, G_{h/2}*](\xi\cdot\nabla)u\right|$$

$$\leq \frac{2}{\sqrt{h}}\Big(\underbrace{\int (G_{h/2}*(u-\chi))^2}_{=\frac{1}{2\sqrt{h}}d_h^2(u,\chi)}\int ([\zeta, G_{h/2}*](\xi\cdot\nabla)u)^2\Big)^{1/2},$$

so that it remains to show

$$\sup|[\zeta, G_{h/2}*](\xi\cdot\nabla)u| \lesssim \|\nabla\zeta\|_\infty\|\xi\|_\infty + \|\zeta\|_\infty\|\nabla\cdot\xi\|_\infty$$

The latter is easy to see:

$$([\zeta, G_{h/2}*](\xi\cdot\nabla)u)(x) = \int G_{h/2}(z)(\zeta(x)-\zeta(x-z))\underbrace{(\xi\cdot\nabla)u(x-z)}_{=\xi(x-z)\cdot\nabla u(x-z)}dz$$

$$= \int\Big(\underbrace{(\zeta(x)-\zeta(x-z))}_{|\cdot|\leq|z|\|\nabla\zeta\|_\infty}\big(\nabla G_{h/2}(z)\cdot\xi(x-z) - G_{h/2}(z)\nabla\cdot\xi(x-z)\big) + G_{h/2}(z)\nabla\zeta(x-z)\xi(x-z)\Big)\underbrace{u(x-z)}_{|\cdot|\leq 1}dz$$

so that

$$|([\zeta, G_{h/2}*](\xi\cdot\nabla)u)(x)|$$

$$\leq \|\nabla\zeta\|_\infty\|\xi\|_\infty\underbrace{\int(|z||\nabla G_{h/2}(z)| + G_{h/2}(z))dz}_{=\int(|z||\nabla G_{1/2}(z)|+G_{1/2}(z))dz\lesssim 1},$$

$$+2\|\zeta\|\,\|\nabla\cdot\xi\|_\infty\underbrace{\int G_{h/2}(z)\,dz}_{=1}$$

<u>Lemma 8</u> (dependance on second vector) for $u, x \in [0,1]$

$$\left| \frac{1}{h}\left(\tilde{E}_h(u,x) - \tilde{E}_h(u,u)\right) + \frac{1}{2}\delta \frac{1}{2h} d_h^2(\cdot, x)(u, \nabla\zeta) \right|$$

$$\lesssim_{\zeta} h^{1/4} \underbrace{\frac{d_h(u,x)}{h}} + h \underbrace{\frac{d_h^2(u,x)}{h^2}} + h^{1/2}\underbrace{\left(E_h(u) + E_h(x)\right)^{1/2}}$$ of by a priori estimate

Will later use that by Euler-Lagrange equ for Optimal minimizing movements,

$$-\frac{1}{2h}\delta d_h^2(\cdot, X^{n-1})(X^n, \zeta) = \delta E_h(X^n, \zeta)$$

<u>Proof of Lemma 8</u>

① Expanding the commutator:

$$[\zeta, G_h *]V + h \nabla\zeta \cdot \nabla G_h * V = O\left(h \|\nabla^2\zeta\|_\infty \|V\|_\infty\right),$$

$$[\zeta, G_h *]V + h \nabla G_h * V \nabla\zeta - \frac{h}{2}(G_h id + h \nabla^2 G_h) * V \nabla^2\zeta = O\left(h^{3/2} \|\nabla^3\zeta\|_\infty \|V\|_\infty\right)$$

Argument for ①: Wlog only 2nd estimate:

$$([\zeta, G_h *]V)(x) = \int G_h(z)\big(\zeta(\underset{= x-z+z}{x}) - \zeta(x-z)\big) V(x-z)\, dz$$

$$= z\cdot\nabla\zeta(x-z) + \frac{1}{2} z\otimes z : \nabla^2\zeta(x-z) + O(|z|^3 \|\nabla^3\zeta\|_\infty)$$

$$= G_h(z) z \cdot (V\nabla\zeta)(x-z) + \frac{1}{2} G_h(z) z\otimes z : (V\nabla^2\zeta)(x-z)$$
$$+ O(|z|^3 G_h(z) \|\nabla^3\zeta\|_\infty \|V\|_\infty).$$

Note in addition that from $G_h(z) = \frac{1}{\sqrt{2\pi h}^d} \exp\left(-\frac{|z|^2}{h}\right)$:

$$\nabla G_h(z) = -G_h(z)\frac{z}{h} \implies G_h(z) z = -h \nabla G_h(z)$$

$$\nabla^2 G_h(z) = -G_h(z)\frac{id}{h} + G_h(z)\frac{z}{h}\otimes\frac{z}{h} \implies G_h(z) z\otimes z = h(G_h(z) id + h\nabla^2 G_h(z))$$

Get as desired

$$([\zeta, G_h *]V)(x) = \int\left(-h\nabla G_h(z)\cdot(V\nabla\zeta)(x-z) + \frac{h}{2}(G_h(z) id + h\nabla^2 G_h(z)) : (V\nabla^2\zeta)(x-z)\right) dz$$

$$+ O\Big(\underbrace{\int |z|^3 G_h(z)\, dz}_{\approx h^{3/2}} \|\nabla^3\zeta\|_\infty \|V\|_\infty\Big)$$

② $\frac{1}{h}\left(\tilde{E}_h(u,\chi)-\tilde{E}_h(u,u)\right)+\frac{1}{2}\delta\frac{1}{2h}d_h^2(\cdot,\chi)(u,\nabla\zeta)$

$$= \frac{1}{\sqrt{h}}\int \left(G_{h/2}*(u-\chi)\right)\frac{1}{2}\left(G_{h/2}\,id + h\nabla^2 G_{h/2}\right)*(1-u)\nabla^2\zeta + O\left(\|\nabla^3\zeta\|_\infty\int|u-\chi|\right)$$

$$= \frac{1}{\sqrt{h}}\int\frac{1}{4}\left(G_{h/2}*(u-\chi)\right)^2\Delta\zeta + O\left(\|\nabla^2\zeta\|_\infty\frac{1}{\sqrt{h}}\int\left|G_{h/2}*(u-\chi)\right|\right)$$

Argument for ②: By definition of $\tilde{E}_h$:

$$\frac{1}{h}\left(\tilde{E}_h(u,\chi)-\tilde{E}_h(u,u)\right)$$

$$= \frac{1}{\sqrt{h}}\int (u-\chi)\frac{1}{h}\underbrace{[\zeta, G_h*]}_{\text{skew symmetric}}(1-\chi) + (u-\chi)\frac{1}{h}[\zeta, G_{h/2}*]G_{h/2}*(u-\chi)$$

$$= \frac{1}{\sqrt{h}}\int (u-\chi)\frac{1}{h}[\zeta, G_h*](1-u) + (u-\chi)\frac{1}{h}[\zeta, G_{h/2}*]G_{h/2}*(u-\chi).$$

By definition of $\frac{1}{2h}d_h^2$ and 1st variation:

$$-\frac{1}{2}\delta\frac{1}{2h}d_h^2(\cdot,\chi)(u,\nabla\zeta)$$

$$= -\frac{1}{\sqrt{h}}\int (u-\chi)\cdot G_h * \underbrace{(\nabla\zeta\cdot\nabla)u}_{=-\nabla\zeta\cdot\nabla(1-u) = -\nabla\cdot(1-u)\nabla\zeta + (1-u)\Delta\zeta}$$

$$= \frac{1}{\sqrt{h}}\int (u-\chi)\nabla G_h*(1-u)\nabla\zeta - (u-\chi)G_h*(1-u)\Delta\zeta.$$

Hence

$$\frac{1}{h}\left(\tilde{E}_h(u,\chi)-\tilde{E}_h(u,u)\right)+\frac{1}{2}\delta\frac{1}{2h}d_h^2(\cdot,\chi)(u,\nabla\zeta)$$

$$= \frac{1}{\sqrt{h}}\int (u-\chi)\underbrace{\left(\frac{1}{h}[\zeta, G_h*](1-u)+\nabla G_h*(1-u)\nabla\zeta\right)}_{\overset{①}{=}\ \frac{1}{2}\underbrace{\left(G_h\,id + h\nabla^2 G_h\right)}_{=G_{h/2}*\left(G_{h/2}\,id + h\nabla^2 G_{h/2}\right)}*(1-u)\nabla^2\zeta + O\left(\sqrt{h}\,\|\nabla^3\zeta\|_\infty\right)}$$

$$= \frac{1}{\sqrt{h}}\int\left(G_{h/2}*(u-\chi)\right)\frac{1}{2}\left(G_{h/2}\,id + h\nabla^2 G_{h/2}\right)*(1-u)\nabla^2\zeta + O\left(\|\nabla^3\zeta\|_\infty\int|u-\chi|\right)$$

$$+\frac{1}{\sqrt{h}}\int\Big((u-x)\,\frac{1}{h}\,[\zeta,G_{h/2}*]\,G_{h/2}*(u-x)-(u-x)\,\underbrace{G_h}_{=G_{h/2}*G_{h/2}}*(1-u)\Delta\zeta\Big)$$

$$=-\frac{1}{\sqrt{h}}\int\Big(G_{h/2}*(u-x)\Big)\Big(\underbrace{\frac{1}{h}[\zeta,G_{h/2}]*(u-x)+G_{h/2}*(1-u)\Delta\zeta}_{\overset{(4)}{=}-\frac{1}{2}\nabla\zeta\cdot\nabla G_{h/2}*(u-x)+O(\|\nabla^2\zeta\|_\infty)}\Big)$$

$$=\frac{1}{\sqrt{h}}\int\underbrace{G_{h/2}*(u-x)\,\frac{1}{2}\nabla\zeta\cdot\underbrace{\nabla G_{h/2}*(u-x)}_{=\nabla(G_{h/2}*(u-x))}}_{=\nabla\zeta\cdot\nabla\frac{1}{4}(G_{h/2}*(u-x))^2}+O\Big(\|\nabla^2\zeta\|_\infty\frac{1}{\sqrt{h}}\int|G_{h/2}*(u-x)|\Big)$$

$$=-\frac{1}{\sqrt{h}}\int\frac{1}{4}(G_{h/2}*(u-x))^2\Delta\zeta$$

③ Conclusion:

$$\Big|\frac{1}{h}\big(\tilde{E}_h(u,x)-\tilde{E}_h(u,u)\big)+\frac{1}{2}\delta\frac{1}{2\sqrt{h}}d_h^2(\cdot,x)(u,\nabla\zeta)\Big|$$

$$\lesssim\|\nabla^2\zeta\|_\infty\Big(\frac{1}{\sqrt{h}}\underbrace{\int|G_{h/2}*(u-x)|}_{\le\left(\int(G_{h/2}*(u-x))^2\right)^{1/2}}+\frac{1}{\sqrt{h}}\underbrace{\int(G_{h/2}*(u-x))^2}_{=\frac{1}{2\sqrt{h}}d_h^2(u,x)}\Big)$$

$$+\|\nabla^3\zeta\|_\infty\underbrace{\int|u-x|}_{\le\left(\int(u-x)^2\right)^{1/2}}$$

$$\lesssim\Big(\frac{1}{2\sqrt{h}}d_h^2(u,x)+2\sqrt{h}\big(E_h(u)+E_h(x)\big)\Big)^{1/2}$$

by lemma 4 ii)

$$\leq \|\nabla^2 \gamma\|_\infty \left(h^{1/4} \frac{d_h(u,x)}{h} + h \cdot \frac{d_h^2(u,x)}{h^2} \right)$$

$$+ \|\nabla^3 \gamma\|_\infty \left(h^{3/4} \frac{d_h(u,x)}{h} + h^{1/4} \left(E_h(u) + E_h(x) \right)^{1/2} \right)$$