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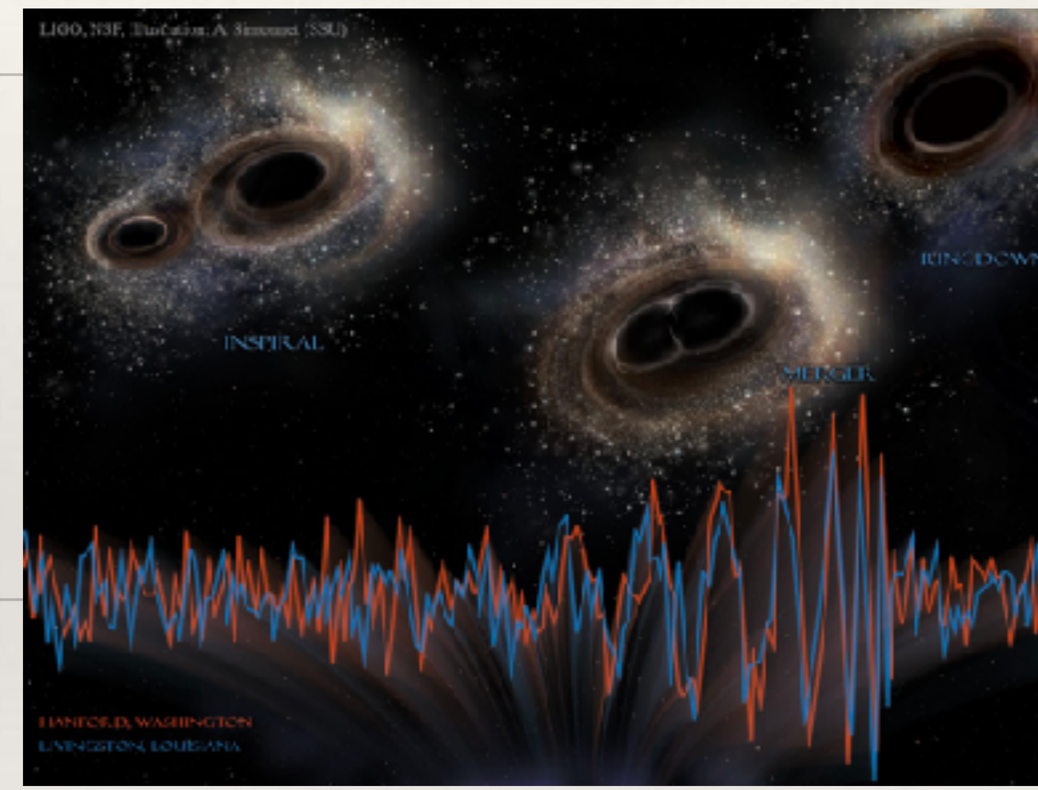
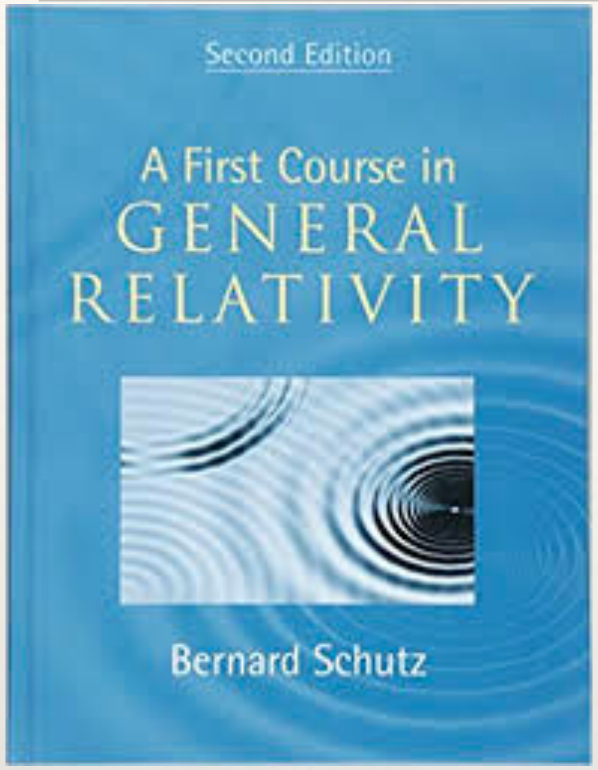
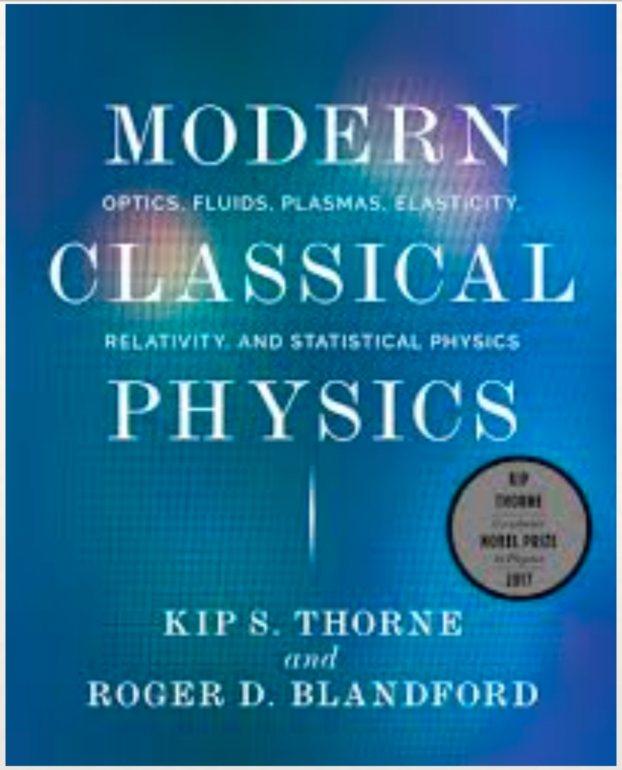
Gravitational waves Part I



ICTP, 18-22 June 2018

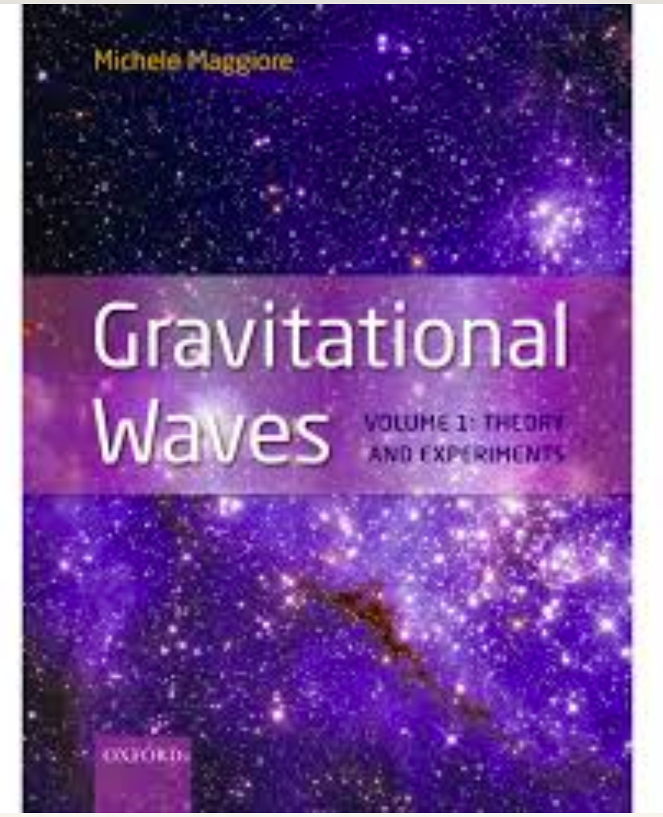
Literature

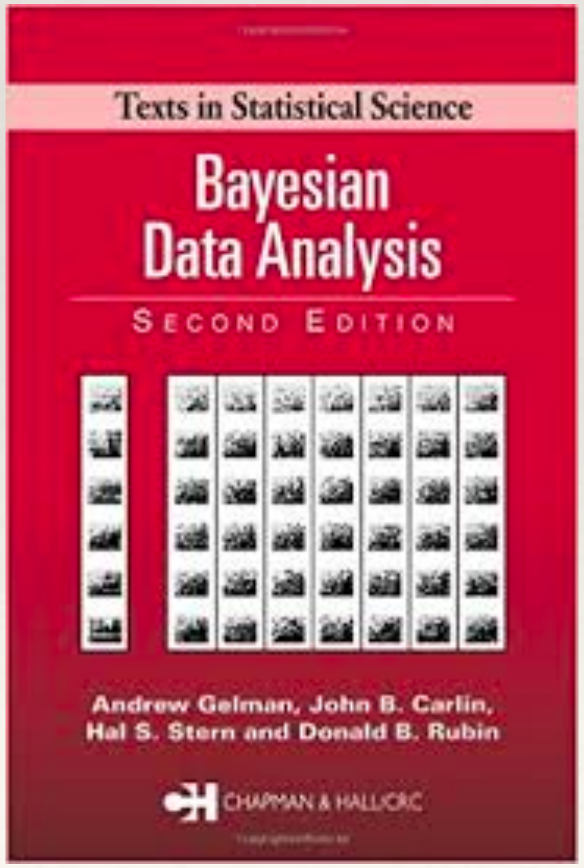


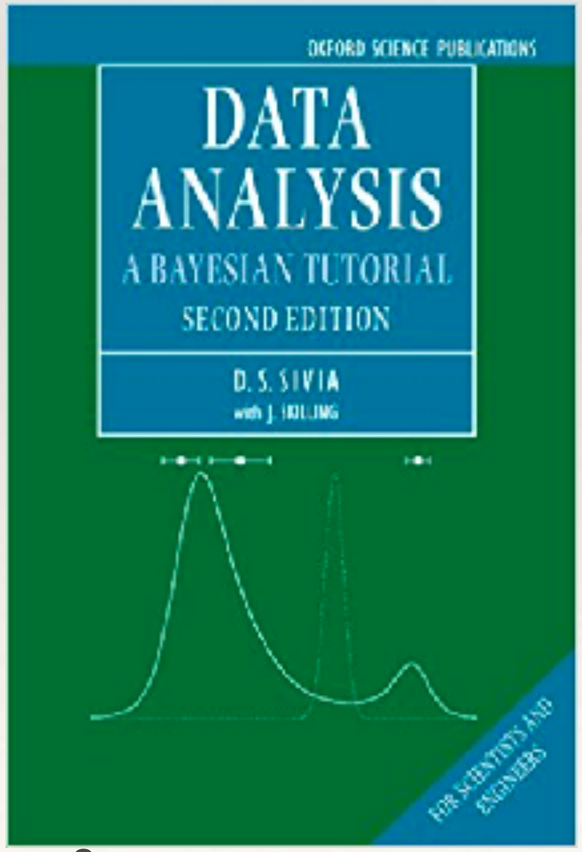


Some figures in these lectures are borrowed from theses books and these articles

LSC+Virgo, Phys. Rev. Lett. 116, 221101 (2016), LSC+Virgo, Phys. Rev. X 6, 041015 (2016), LSC+Virgo, Phys.Rev.Lett. 116 241102 (2016), LSC+Virgo, Phys.Rev. X6 041014 (2016), LSC+Virgo, Phys. Rev. Lett. 118, 221101 (2017), Berti et al., Class.Quantum Grav. 32, 243001 (2015), LSC+VIRGO, ArXiV: 1805.11579, S. Khan+, Phys.Rev. D93 (2016) 044007, LSC+Virgo arXiv: 1805.11579 , LIGO_Virgo, Astrophys.J. 848 (2017) L12, LIGO+Virgo Phys.Rev.Lett. 119 (2017) 161101, Babak+ Phys.Rev. D95 (2017) 103012, LISA consortium arXiv:1702.00786, Klein+ Phys.Rev. D93 (2016) 024003, LISA consortium, arXiv:1305.5720, Amaro-Seoane+ Class.Quant.Grav. 29 (2012) 124016 .



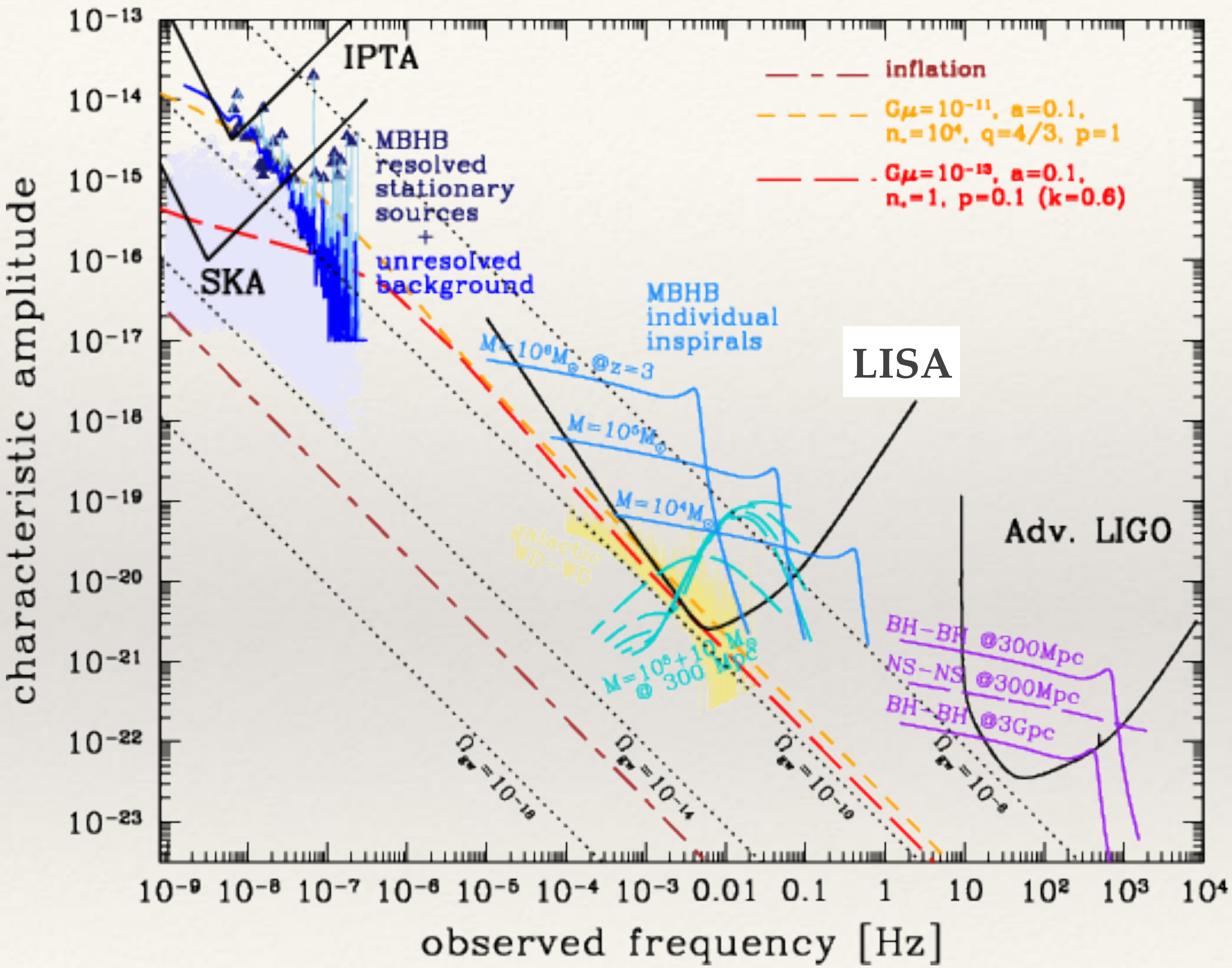




Lecture 1

- Notations and brief reminder of General Relativity
- Weak gravitational waves (GW) in vacuum
- Generation of GWs (quadrupole expression).

Gravitational Waves across frequency



[Credits A. Sesana}

Main features of GWs

- Gravitational waves (GWs) exist in every covariant theory of gravity
- In GR:
  - GWs propagate with the speed of light
  - GWs are acting as time varying tidal forces
  - GWs have two polarization state (in case of linearly polarized wave: $h_+, \ h_\times$
  - GWs weakly interact with the matter ($G = 6.674 \times 10^{-11}$) - are not scattered or absorbed, but hard to detect

Special relativity (reminder)

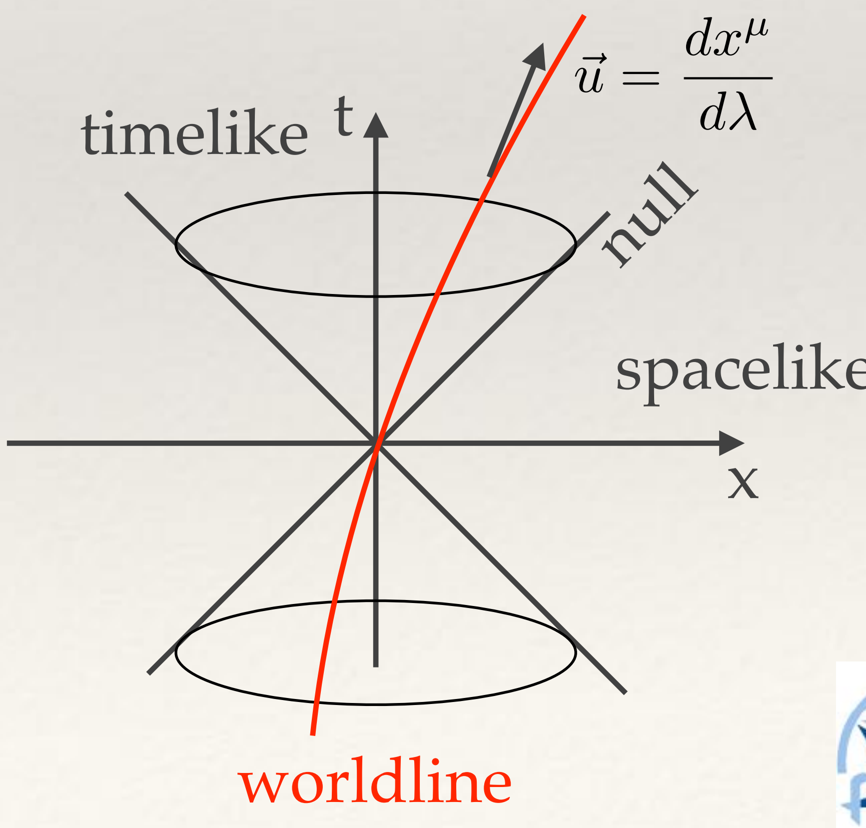
- Existence of inertial coordinate systems (not accelerated)
- Equivalence of inertial coordinate frames
- Universality of the speed of light

We use **G=c=1** (mass, length, time in seconds)

interval: $$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \equiv \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ Minkowski metric,

$$\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \equiv \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$



Differential geometry (reminder)

Non-flat geometry

Local flatness theorem: the metric can be brought by a coordinate transformation at a point to Minkowski form, and all its first derivatives can be made = zero at that point. (but not the second derivatives!)

Metric defines the distances and angles:

arbitrary metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

flat (Minkowski)

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

We can introduce Local Inertial Frame: approximately flat in a vicinity of a point (for the scale mauch less than radius of curvature of spacetime) $dx \ll \mathcal{R}$

Differential geometry (reminder)

Vector field: vector is defined continuously at each point: requires/involves comparison between vectors at two points.

$$\frac{d\vec{V}}{dx^\mu} = \frac{d(V^\nu \vec{e}_\nu)}{dx^\mu} = \frac{dV^\nu}{dx^\mu}\vec{e}_\nu + V^\nu \frac{d\vec{e}_\nu}{dx^\mu}$$

basis vector could be different at different points

$$\frac{d\vec{e}_\nu}{dx^\mu} = \Gamma^\alpha_{\mu\nu}\vec{e}_\alpha$$

Christoffel symbols (not a tensor)

Covariant derivative: $V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + V^\mu \Gamma^\alpha_{\mu\beta}$

Covariant derivative of metric=0 => $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$

Differential geometry (reminder)

Local flatness: at a given point we can make Christoffel symbols=0:
covariant derivative -> partial

Parallel transport of the vector along the curve: we transport the vector along the curve and we preserve its length and keep it parallel in nearby points.

Local inertial frame (LIF)

$$\frac{dV^\alpha}{d\lambda} = 0 = V^\alpha{}_{,\beta} u^\beta$$

parameter along the curve

tangent to the curve

covariant extension

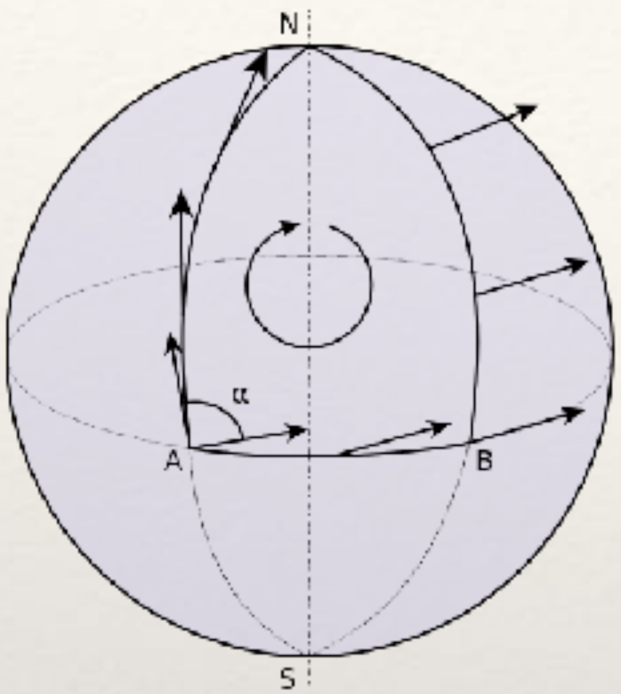
$$= u^\beta V^\alpha{}_{;\beta} = 0$$

Geodesic: the curve with the shortest distance connecting two points. Line in flat spacetime: how to extend? Line as straight as possible: tangent to a curve is parallel at nearby points - > parallel transport of a tangent vector.

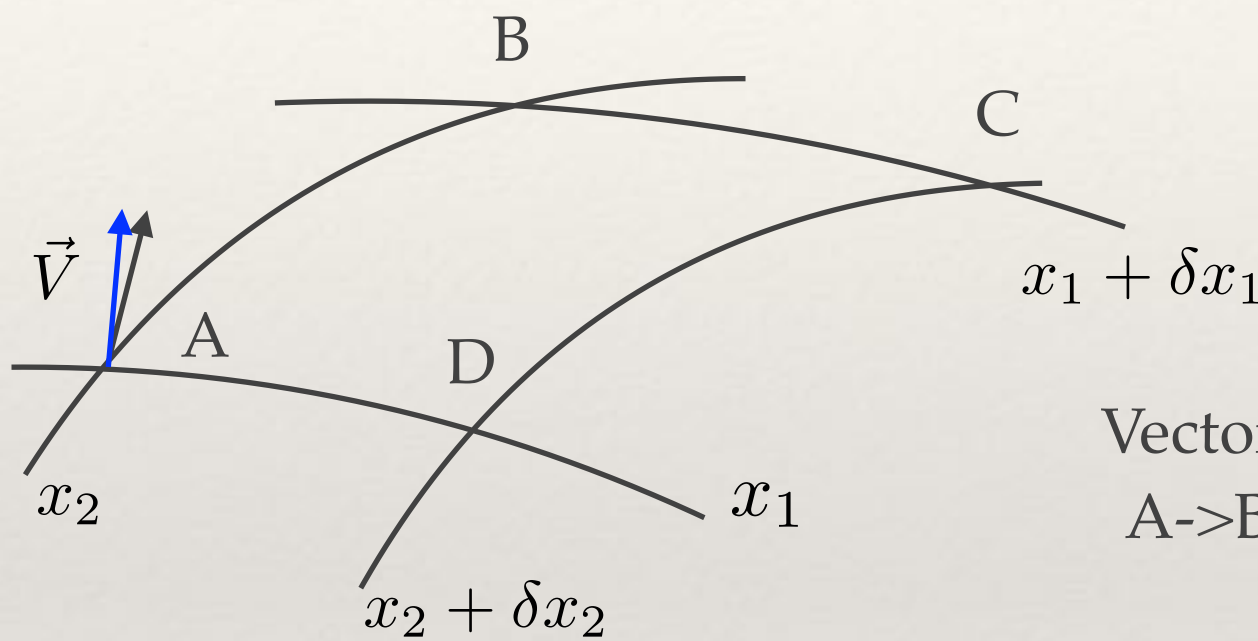
$$u^\beta u^\alpha{}_{;\beta} = 0$$

Differential geometry (reminder)

Curvature tensor



The vector parallel transported along the loop does not return to itself if space is curved



Vector propagated A->B->C->D->A

$$\delta V \propto \delta x_1 \delta x_2 R V$$

curvature

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu}$$

Differential geometry (reminder)

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu}$$

Properties of Riemann (curvature) tensor:

- Depends on the second derivatives of metric: cannot be eliminated by coord. transformation.
- Flat spacetime <-> Riemann=0
- Second covariant derivatives do not commute (~ Riemann)
- Not all components independent (symmetries + differential Bianchi identity)
- In LIF:

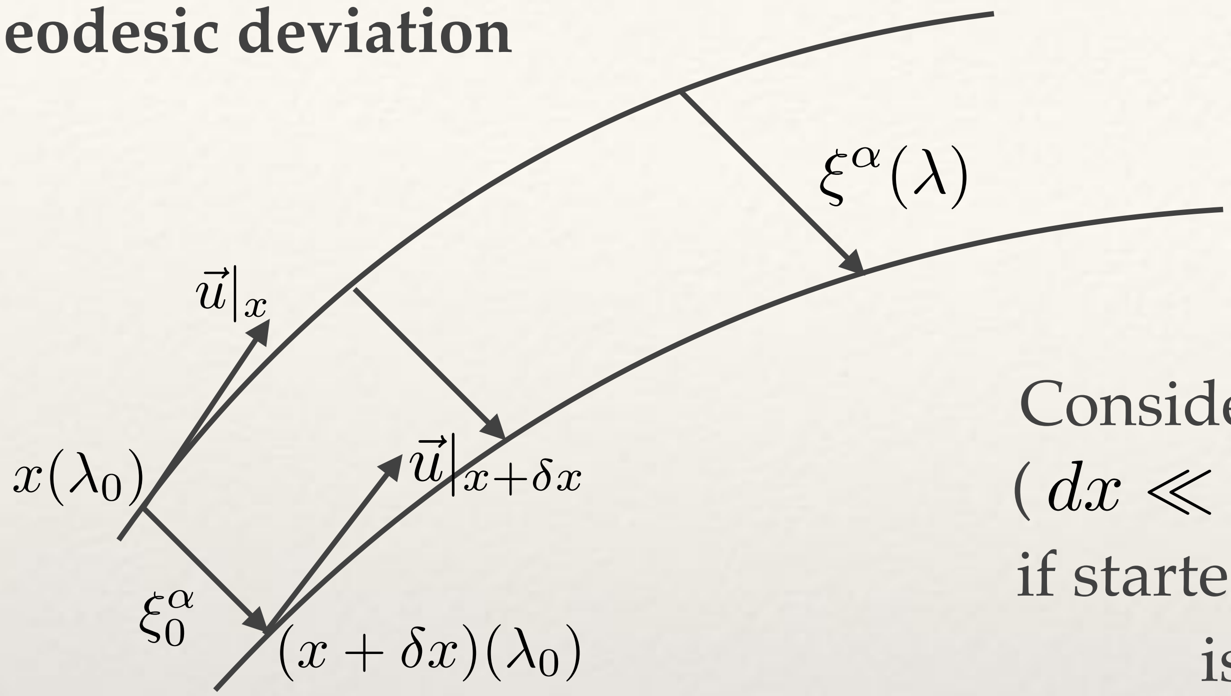
$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}\right]$$

Ricci tensor, Ricci scalar, Einstein tensor:

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} \qquad R = g^{\alpha\beta}R_{\alpha\beta} \qquad G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$$

Differential geometry (reminder)

Geodesic deviation



Consider two geodesic, due to to local flatness ($dx \ll \mathcal{R}$) they remain parallel for some time if started in the nearby points, but if spacetime is not flat they diverge/converge.

$$\nabla_u \nabla_u \xi^\alpha = R^\alpha{}_{\mu\nu\beta} u^\mu u^\nu \xi^\beta$$

$$\nabla_u \xi^\alpha = u^\beta \xi^\alpha{}_{;\beta}$$

covar. deriv along the curve

General Relativity

- No particles neutral to grav. interaction
- Weak equivalence principle: freely falling particles move on timelike geodesics.
- Einstein equivalence principle: any *local* physical experiment not involving gravity has the same result in LIF and in flat spacetime

Einstein equations: should be covariant, should have Newtonian limit

$$R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R + \Lambda g^{\alpha\beta} = \kappa T^{\alpha\beta}$$

Stress-energy tensor

Bianchi identities:

$$G^{\alpha\beta}{}_{;\beta} \equiv 0 \longrightarrow T^{\alpha\beta}{}_{;\beta} = 0$$

Covaiant conservation law

↓ (from $G^{\alpha\beta}{}_{;\beta} \equiv 0$)

4 identities: choice of coordinate frame

Gauge transformation

Consider weak gravitational field: $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

In the LIF $$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[h_{\alpha\nu,\beta\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu} - h_{\beta\nu,\alpha\mu}\right]$$

gauge transformation: we keep fixed Minkowski part but modify the field:

$$h^{new}_{\alpha\beta} \rightarrow h^{old}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$ xi is arbitrary but preserving $|h^{new}_{\mu\nu}| \ll 1$

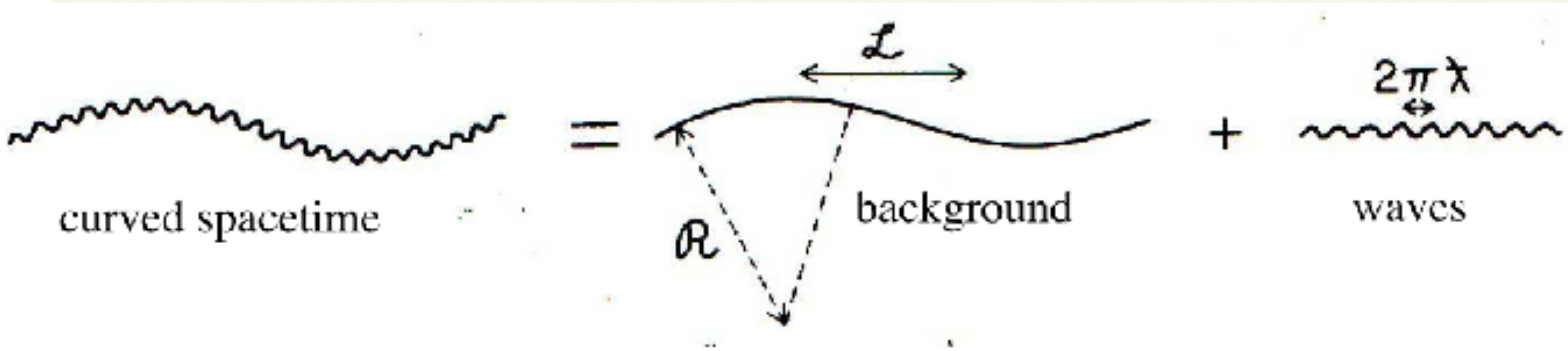
$$R_{\alpha\beta\mu\nu}(h^{new}) = R_{\alpha\beta\mu\nu}(h^{old})$$

In terms of full metric it corresponds to the coordinate transformation:

$$x^{\bar{\alpha}} = x^{\alpha} + \xi^{\alpha}(x^{\nu})$$

GW in linearized gravity

We need to seprate the background geometry and GWs



$$g_{\alpha\beta} = g^{B}_{\alpha\beta} + h_{\alpha\beta}, \qquad R_{\alpha\beta\mu\nu} = R^{B}_{\alpha\beta\mu\nu} + R^{GW}_{\alpha\beta\mu\nu}$$

$$g^{B}_{\alpha\beta} = < g_{\alpha\beta} >, \qquad R^{B}_{\alpha\beta\mu\nu} = < R_{\alpha\beta\mu\nu} >$$

Averaging over several GW wavelength

We will work with the metric, $h_{\alpha\beta}$ - GWs

We use LIF for the background (!) geometry (LIF is also called sometimes local Lorentz frame)

Valid on the scales: $l \ll \mathcal{R}$ still assume that $\lambda^{GW} \ll \mathcal{R}$

Weak GWs

In LIF (w.r.t.) background: $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2)$

Consider Einstein equations in vacuum (far from the source) $R_{\alpha\beta} = 0$

Introduce trace reverse metric $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h, \qquad h = h^{\mu}{}_{\mu}$

Choose gauge (coordinate frame): $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ <— harmonic gauge

Einstein's field equations (linear): $\Box \bar{h}^{\mu\nu} = 0$

Remaining gauge freedom: $\bar{h}^{\mu\nu}_{new\,,\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \Box\xi^{\mu} = 0, \quad \text{if} \quad \Box\xi^{\mu} = 0$

$$\bar{h}^{0\alpha} = 0, \quad h = 0$$

T(ransverse)T(raceless) gauge: For the GW travelling in z-direction

$$h_{\mu\nu} = h^{TT}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Weak GWs

Consider the plane wave solution:

$$\bar{h}^{\mu\nu} = A^{\mu\nu} e^{ik_\alpha x^\alpha}, \quad k^\alpha = \{\omega, k^i\}$$ 4-wave vector

Einstein equations: $k_\alpha k^\alpha = -\omega^2 + \vec{k}.\vec{k} = 0$ propagates with tthe speed of light.

transverse & traceless $A_{\mu\nu} k^\nu = 0 \quad A^\mu{}_\mu = 0$ Moreover: $A^{0\mu} = 0$

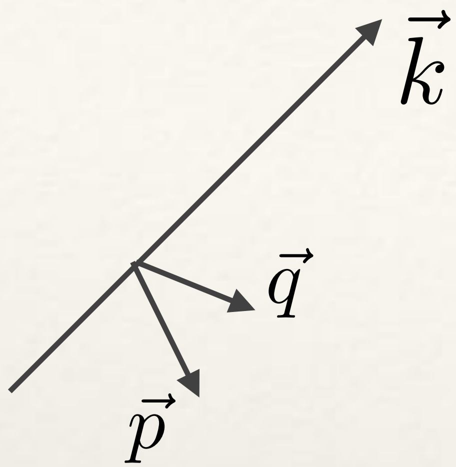
Note that in general the linearized Einstein equations have also static (non-radiative) solution (Newtonian potential). To get the radiative part we can take the "TT" part of the spatial metric by applying the projection operator:

$$h^{TT}_{jk} = Pr_{jklm} h^{lm},$$

$$Pr_{jklm} = P_{jl} P_{mk} - \frac{1}{2} P_{jk} P_{lm}, \quad P_{jk} = \delta_{jk} - n_j n_k,$$

$$n^i = k^i / |\vec{k}|$$

Weak GW, polarization



We can choose polarization basis vectors arbitrary in the plane orthogonal to the direction of GW propagation

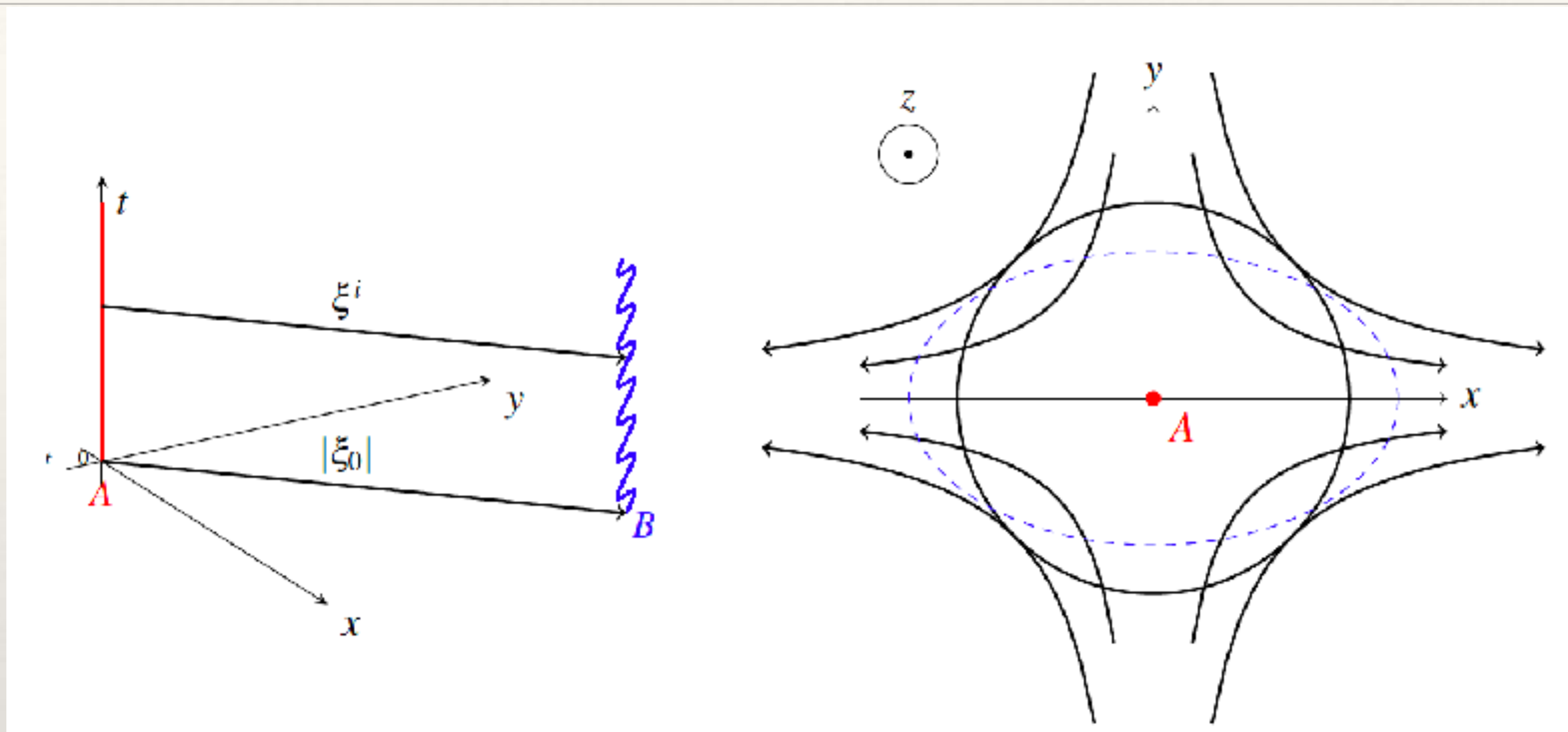
Introduce basis:
$$\epsilon_{+}^{ij} = p^i p^j - q^i q^j$$
$$\epsilon_{\times}^{ij} = p^i q^j + p^j q^i$$

$$h^{ij} = \epsilon_{+}^{ij} h_{+} + \epsilon_{\times}^{ij} h_{\times}$$

Two polarization

finally, Riemann tensor for GW: $R_{0j0k}^{GW} = -\frac{1}{2}\ddot{h}_{jk}^{TT}$

Geodesic deviation in GW



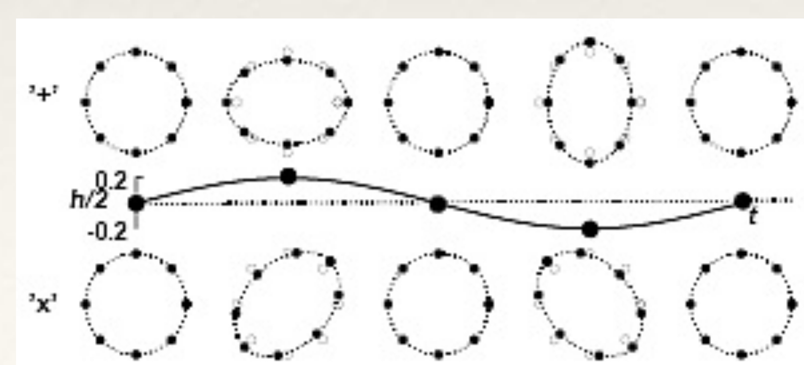
Consider two observers (A, B) moving on geodesic. We use LIF associated with the observer A, which is at rest. Consider GW propagating in z-direction, then the distance between A & B:

$$\frac{d^2\xi^i}{dt^2} = R^i{}_{00j}\xi^j = \frac{1}{2}\ddot{h}^{jk}_{TT}\xi_k$$

Introduce: $\delta\xi^j = \xi^j - \xi^j_0 \longrightarrow \delta\xi^j = \frac{1}{2}h^{jk}_{TT}\xi_k$

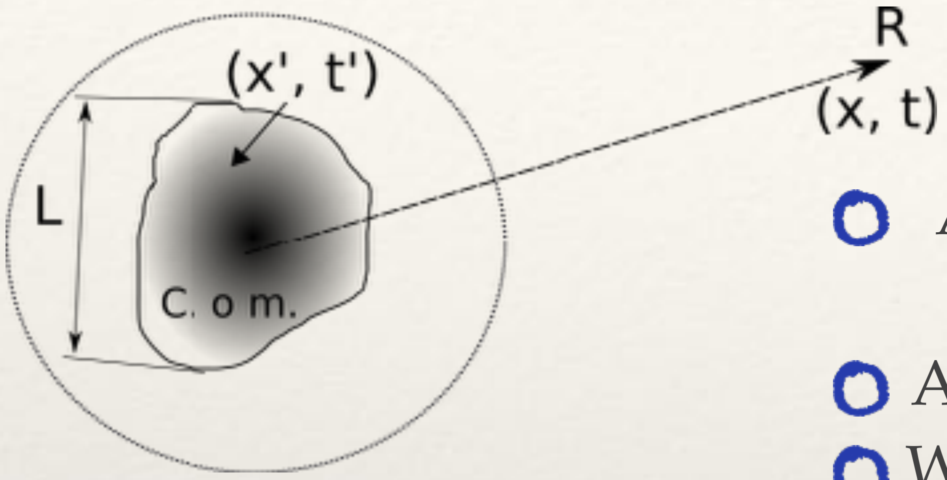
$$\delta x = \frac{1}{2}x_0 h_+ \quad \delta y = -\frac{1}{2}y_0 h_+$$

$$\delta x = \frac{1}{2}y_0 h_\times \quad \delta y = \frac{1}{2}x_0 h_\times$$



Generation of GWs

Consider ***isolated*** source with weak internal gravity.



Observer (field point)

- Assume that the observer is far away (far zone), we can use LIF of the observer
- As before we use trace-reverse form of the metric
- We use harmonic gauge (coordinates)

Einstein equations: $$\Box\bar{h}^{\mu\nu} = -16\pi(T^{\mu\nu} + t^{\mu\nu})$$

Contains all non-linear terms from the r.h.s

- We start with the linear order (neglecting the red term)

Solution in form of retarded potentials

$$\bar{h}^{\mu\nu} = 4\int \frac{T^{\mu\nu}(x', t' = t - |\vec{x}' - \vec{x}|)}{|\vec{x}' - \vec{x}|} d^3\vec{x}'$$

Generation of GWs

- Assume slow motion (v<<c)
- Take observer far away (far zone) $|\vec{x} - \vec{x}'| \approx R \gg \lambda^{GW}$
- We are interested in the radiative part of gravitational potentials => take "TT" part

$$h_{jk}^{TT} = \left[\frac{4}{R}\int T_{jk}(x', t' = t - R)d^3x'\right]^{TT}$$

- Use the conservation law $T^{\mu\nu}{}_{,\nu} = 0$

$$h_{jk}^{TT} = \left[\frac{2}{R}\frac{d^2}{dt^2}\mathcal{M}_{jk}(t - R)\right]^{TT}$$

Quadrupole formula (Landau & Lifshitz)

$$\mathcal{M}^{jk} = \int T^{00}\left(x^j x^k - \frac{1}{3}\delta^{jk} r\right) d^3x$$

mass quadrupole moment

Generation of GWs

- Besides leading order (mass quadrupole) other moments also give contribution. There are two types of moments: mass-moments and current-moments

$I_l \sim ML^l$ mass moments

$S_l \sim MvL^l$ current moments

$$h_{+,\times} \sim \frac{1}{R}\left[\frac{d^2 I_2}{dt^2} \& \frac{d^3 I_3}{dt^3} \& .... \& \frac{d^2 S_2}{dt^2} \& \frac{d^3 S_3}{dt^3} ...\right] \qquad \frac{1}{R}\frac{d^l I_l}{dt^l} \sim \frac{M}{R} v^l$$

- Einstein equations are non-linear: grav field is its own source (the red term which we have neglected). Post-Newtonian expansion: $\varepsilon = v/c \ll 1$

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} + \varepsilon^2 h^{(2)}_{\mu\nu} + ...$$

Solving Einstein equations iteratively updating the equation of motion at each step