Cosmological Structure Formation

Linear theory

The transfer function

Structure formation:
The shape of P(k)

Three possible metrics for homogeneous and isotropic 3-space

$$ds^2 = dr^2 + S_\kappa(r)^2 d\Omega^2 \ ,$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$$

$$S_\kappa(r) = \begin{cases} R\sin(r/R) & (\kappa = +1) \\ r & (\kappa = 0) \\ R\sinh(r/R) & (\kappa = -1) \end{cases}$$

Changing from r to x = S_κ(r) makes this:

$$ds^2 = \frac{dx^2}{1 - \kappa x^2/R^2} + x^2 d\Omega^2$$

Robertson-Walker metric

(If homogeneity and isotropy did not exist, it would be necessary to invent them!)

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2$$ Minkowski metric

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dx^2}{1-\kappa x^2/R_0^2} + x^2d\Omega^2\right]$$

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2d\Omega^2\right]$$

Much of Observational Cosmology dedicated to determining κ, a(t), R_0

Connection to GR

$$G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2 = 8\pi G\, T_{\mu\nu}$$

Homogeneity/isotropy:

$$T_{\mu\nu} = \text{diagonal} = (\rho,-p,-p,-p)$$

Conservation of stress-energy:

$$\nabla_\nu (T_{\mu\nu}) = 0$$

Using FRW metric:

$$d(\rho a^3) = -p\, d(a^3)$$

Since $a^3 \propto V$ this is like 1st Law of thermodynamics.

So, if $p(\rho)$ then can solve for $\rho(t)$:

Evolution depends on 'equation of state'

Equation of state

Consider: $p(t) = w\,\rho(t)$ w independent of t

Then $d(\rho V)/dt = V\,(d\rho/dt) + \rho\,(dV/dt) = -p\,(dV/dt)$

So $V\,(d\rho/dt) = -\,(\rho+p)\,(dV/dt)$

$(d\ln\rho/dt) = -\,(1+p/\rho)\,(d\ln V/dt)$

So $\rho(t) \propto a^{-3(1+w)}$

Special cases:

Non-relativistic matter: $p = 0$ so $w = 0$ so $\rho \propto a^{-3}$

Radiation: $w = 1/3$ so $\rho \propto a^{-4}$

Vacuum energy: $w = -1$ so ρ constant

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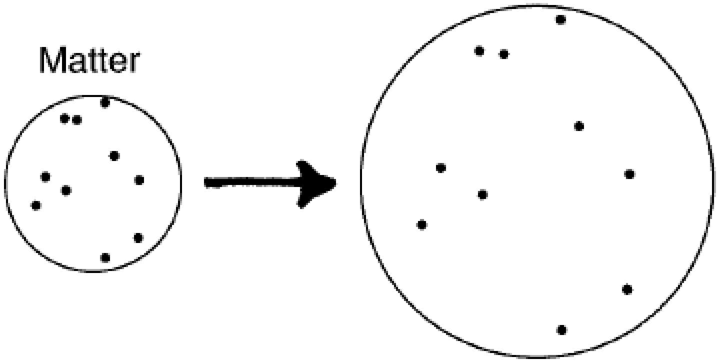
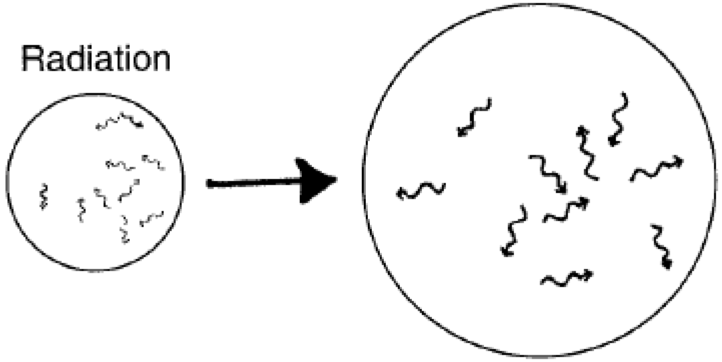


Radiation:

$w = 1/3$ so $\rho \propto a^{-4}$



Vacuum energy:

$w = -1$ so ρ constant

If Universe contains all three, then different ones dominate at different t

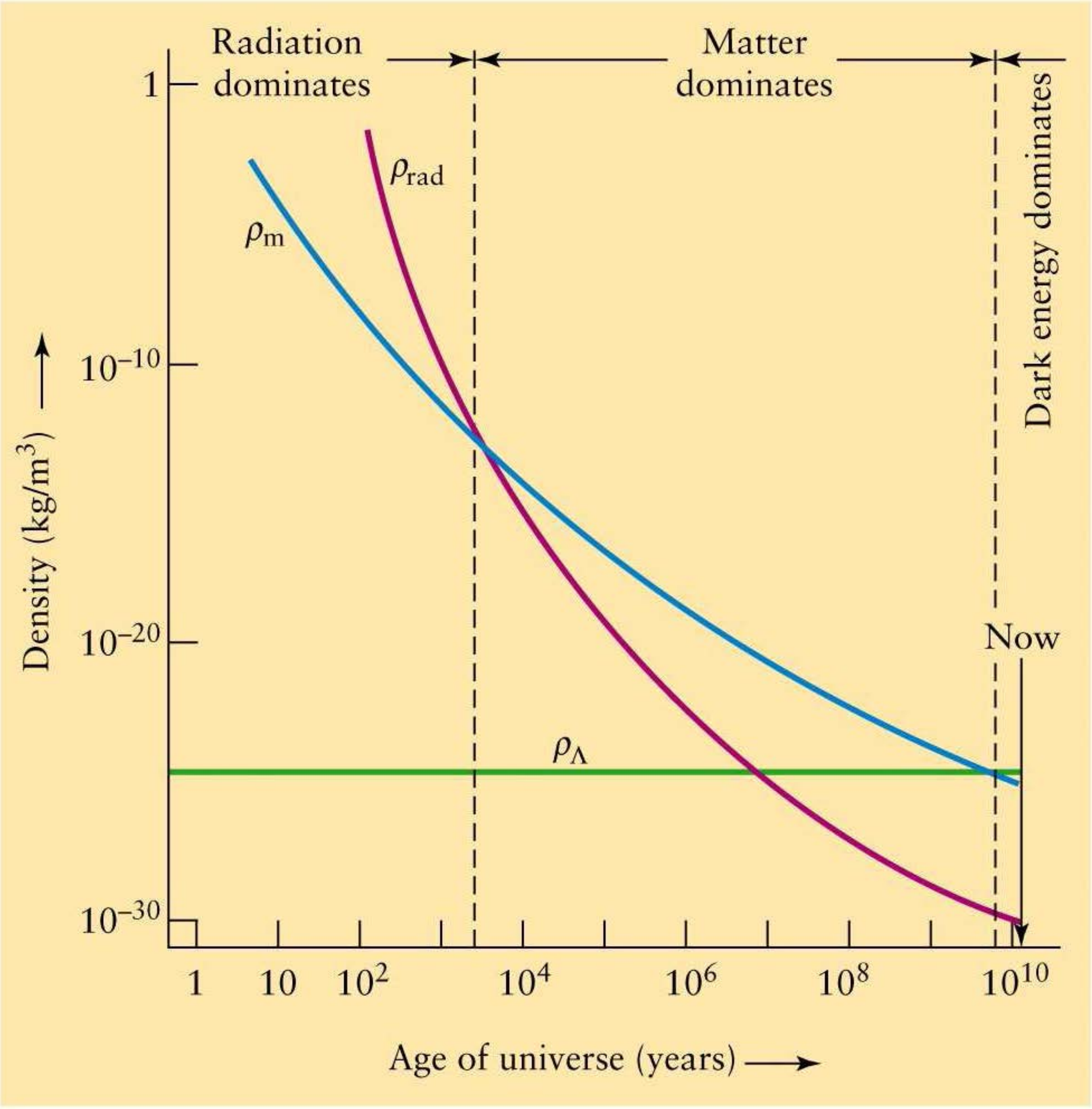
Conventional to define:

$\Omega_m = \rho_m/\rho_c$

$\Omega_r = \rho_r/\rho_c$

$\Omega_\Lambda = \rho_\Lambda/\rho_c$

$\rho_c = 3H^2/8\pi G$



Matter-radiation equality

$1 + z_{eq} = \Omega_{m0} / \Omega_{r0} = 0.3/(8.5e\text{-}5) = 3570$

Length scale: $c\, t_{eq} = c\, t_0\, (t_{eq}/t_0)$

$= c\ 13.7\ \text{Gyrs}/3570^{1.5}$

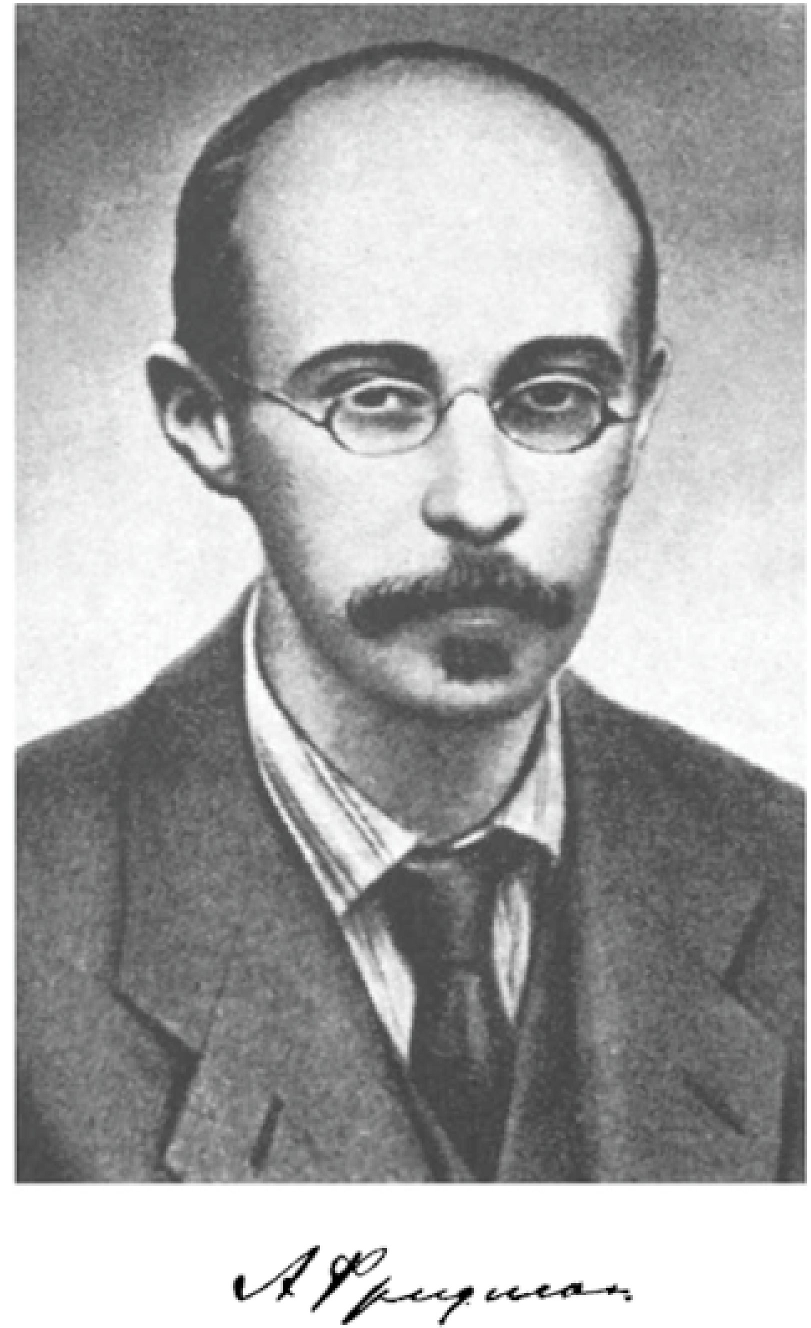
Stretched by factor of $(1 + z_{eq})$, so today is

$3 \times 10^5\ (\text{km/s}) \times 13.7\ \text{Gyrs}/3570^{0.5} = 70\ \text{Mpc}$

Friedmann equations

From 00 element of Einstein equations with RW metric (relates expansion rate to density and curvature);

And from time derivative of it (relates acceleration to density and pressure).



Friedmann equation

$$(d\ln a/dt)^2 + (\kappa c^2/R_0{}^2 a(t)^2) = (8\pi G/3)\,\rho$$

$$H^2 = (8\pi G/3)\,\rho - (\kappa c^2/R_0{}^2 a(t)^2)$$

$$1 - \Omega(t) = -\,\kappa\,[c/H(t)]^2/R_0{}^2 a(t)^2$$

Knowing Ω = knowing sign of curvature
Flat Universe (κ =0) has $\Omega(t) = 1$;
it has energy density $3H^2/(8\pi G)$.
Note that Ω is sum of all components
(matter + radiation + dark energy) .

Empty Universe: $\Omega=0$

$$1 = -\kappa\,[c/H(t)]^2/R_0^{\,2}a(t)^2$$

$$(aH)^2 = -\kappa\,(c/R_0)^2$$

$\kappa=0$ requires a = constant

$\kappa=1$ not allowed

$\kappa=-1$ requires da/dt = constant; $a = ct/R_0$

Flat Universe: $\Omega = 1$

Suppose $a \propto t^q$

Then $H = q/t$ so $\rho \propto a^{-3(1+w)} \propto H^2 \propto t^{-2}$

means $q = 2/3(1+w)$

Matter ($w=0$): $a \propto t^{2/3}$

Radiation ($w=⅓$): $a \propto t^{1/2}$

Dark Energy ($w=-1$)?? $a \propto e^{Ht}$

(because $\rho \propto a^{-3(1+w)} \propto H^2 \propto$ constant)

Λ (w=-1):

$a \propto e^{Ht}$

Empty:

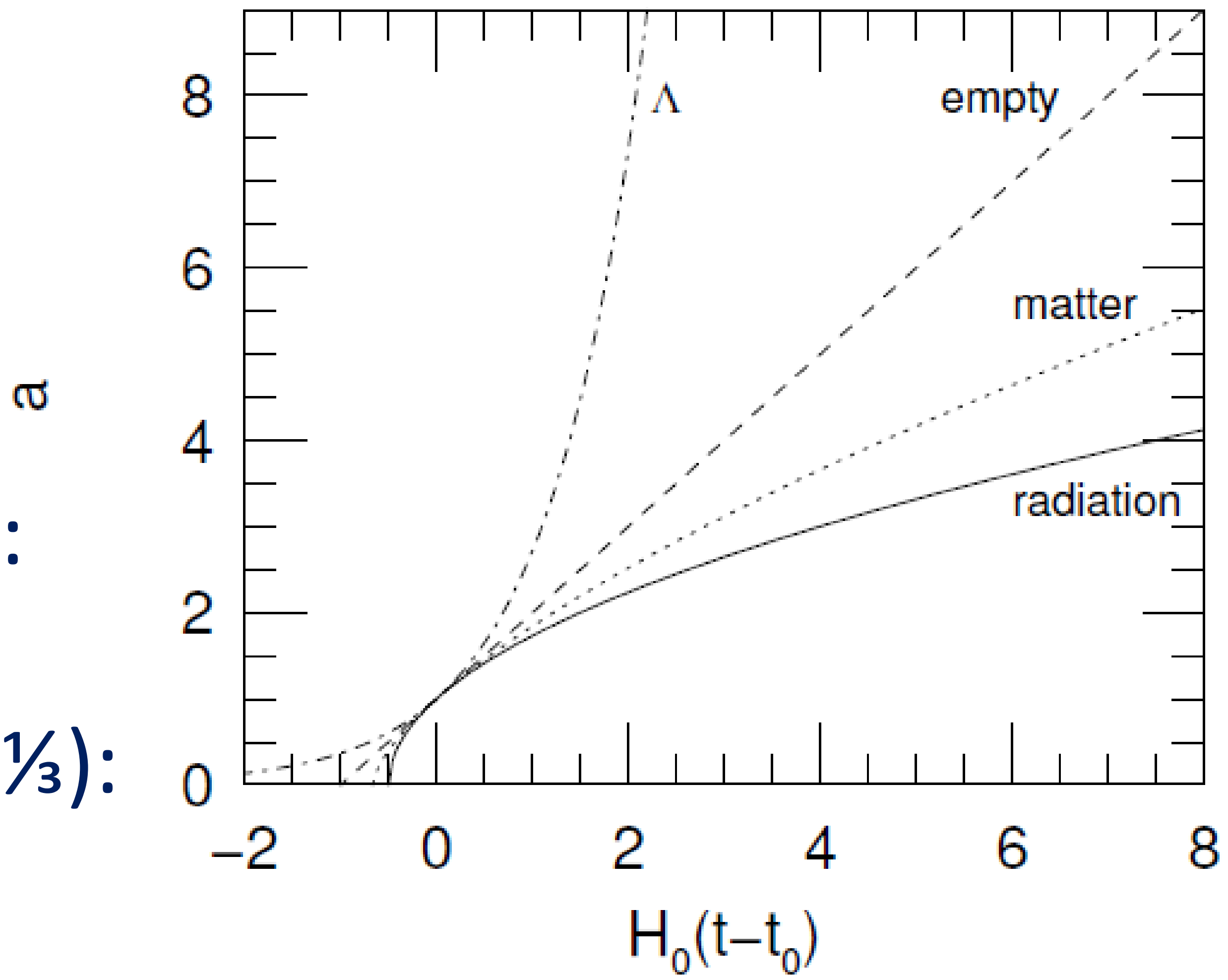
$a \propto t$

Matter (w=0):

$a \propto t^{2/3}$

Radiation(w=⅓):

$a \propto t^{1/2}$



From these, can work out $d_L(z|\Omega,\Lambda)$

Matter + curvature + Λ

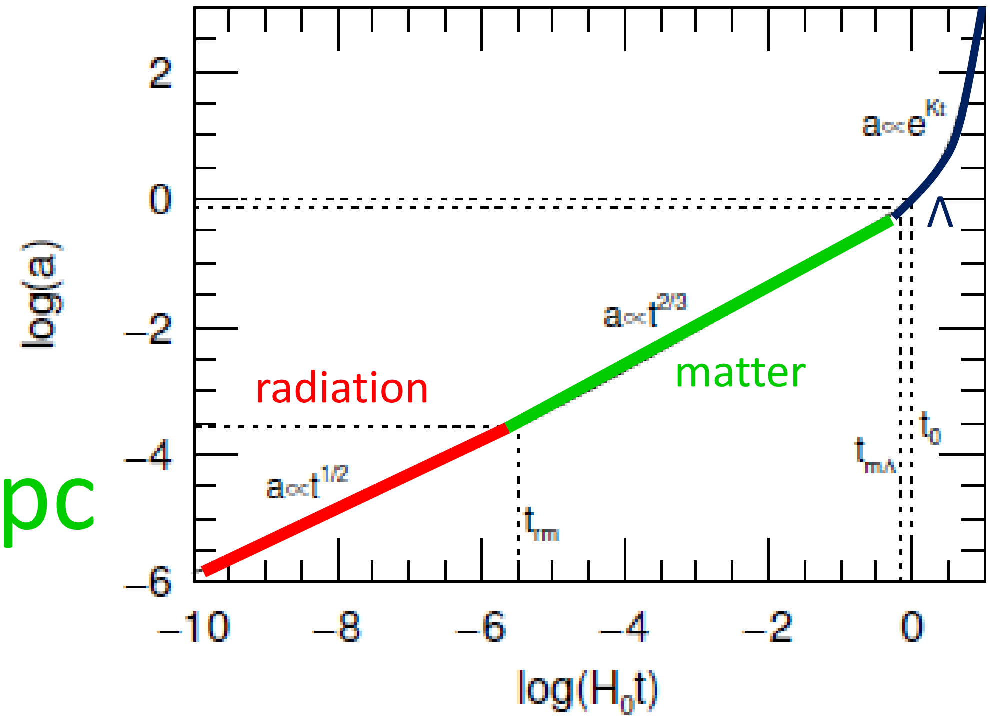
$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} + \frac{1 - \Omega_{m,0} - \Omega_{\Lambda,0}}{a^2} + \Omega_{\Lambda,0}$$

Flat

$\Omega_{\Lambda 0} = 0.7$

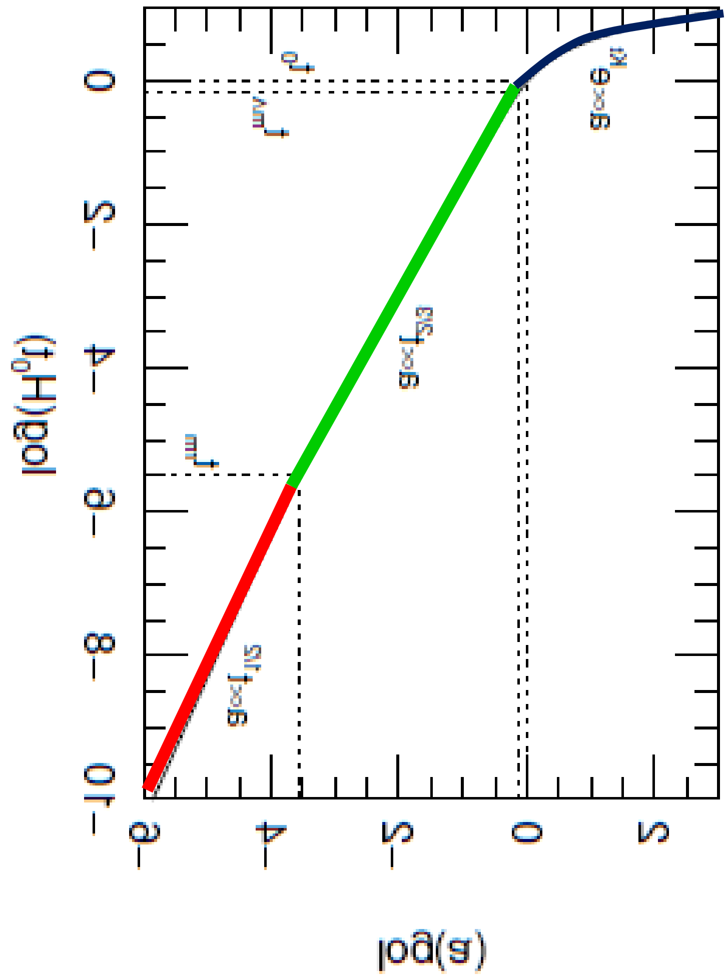
$T_0 = 2.725K$

$H_0 = 70$ km/s/Mpc

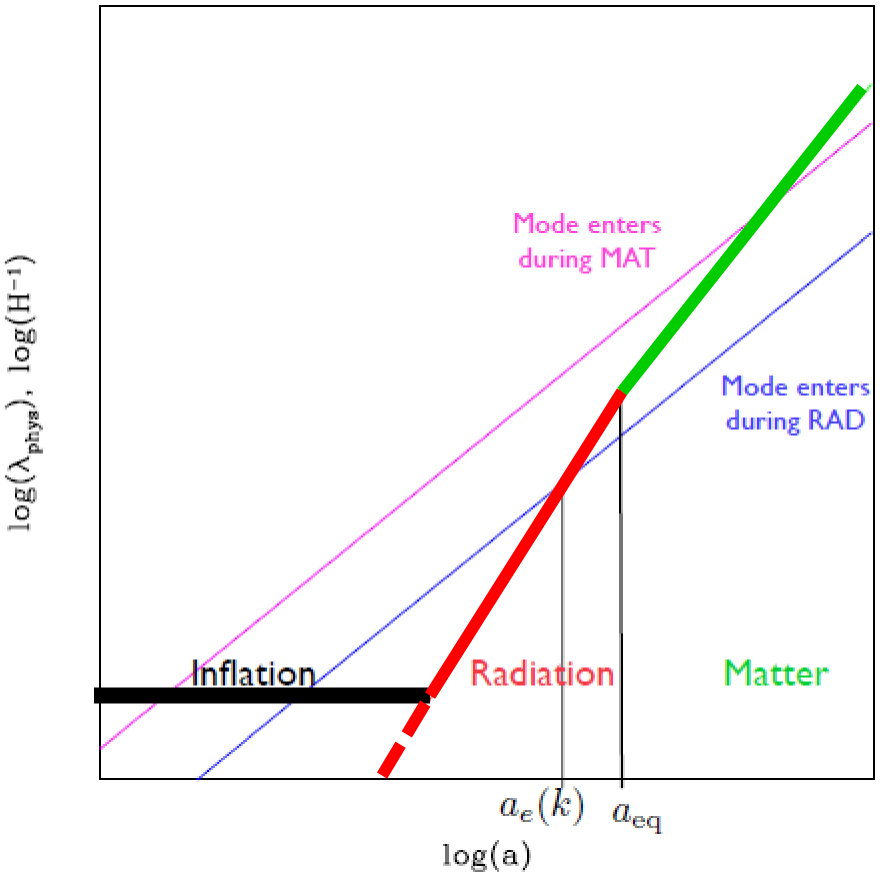


Horizon grows ~ t

In the future, a will increase rapidly



Longer wavelengths 'enter' (are smaller than) horizon later



Sub-horizon: Linear theory

- Newtonian analysis:

  $d^2R/dt^2 = - GM/R^2(t) = - (4\pi/3)\, G\rho(t)R(t)\, [1+\delta(t)]$

- M constant means $R^3 \propto \rho^{-1} [1+\delta]^{-1} \propto a^3 [1+\delta]^{-1}$
- I.e., $R \propto a [1+\delta]^{-1/3}$ so $dR/dt \propto HR - d\delta/dt\, (R/3)\, [1+\delta]^{-1}$ and when $|\delta| << 1$ then

  $(d^2R/dt^2)/R = (d^2a/dt^2)/a - (d^2\delta/dt^2)/3 - (2/3)H\, (d\delta/dt)$

  $= - (4\pi/3)\, G\rho(t)\, [1+\delta(t)]$

- Friedmann equation: $(d^2a/dt^2)/a = - (4\pi/3)\, G\rho(t)$ so

  $(d^2\delta/dt^2) + 2H\, (d\delta/dt) = 4\pi\, G\rho(t)\, \delta(t) = (3/2)\, \Omega_m H^2\, \delta(t)$

Linear theory (contd.)

- When radiation dominated ($H = 1/2t$):

$$(d^2\delta/dt^2) + 2H\,(d\delta/dt) = (d^2\delta/dt^2) + (d\delta/dt)/t = 0$$

$$\delta(t) = C_1 + C_2 \ln(t) \quad \text{(weak growth)}$$

- In distant future (H = constant):

$$(d^2\delta/dt^2) + 2H_\Lambda(d\delta/dt) = 0$$

$$\delta(t) = C_1 + C_2 \exp(-2H_\Lambda t)$$

- If flat matter dominated ($H = 2/3t$):

$$\delta(t) = D_+\, t^{2/3} + D_-\, t^{-1} \propto a(t) \quad \text{at late times}$$

- Because linear growth just multiplicative factor, it cannot explain non-Gaussianity at late times

Super-horizon 'growth'

- Start with Friedmann equation when κ=0:

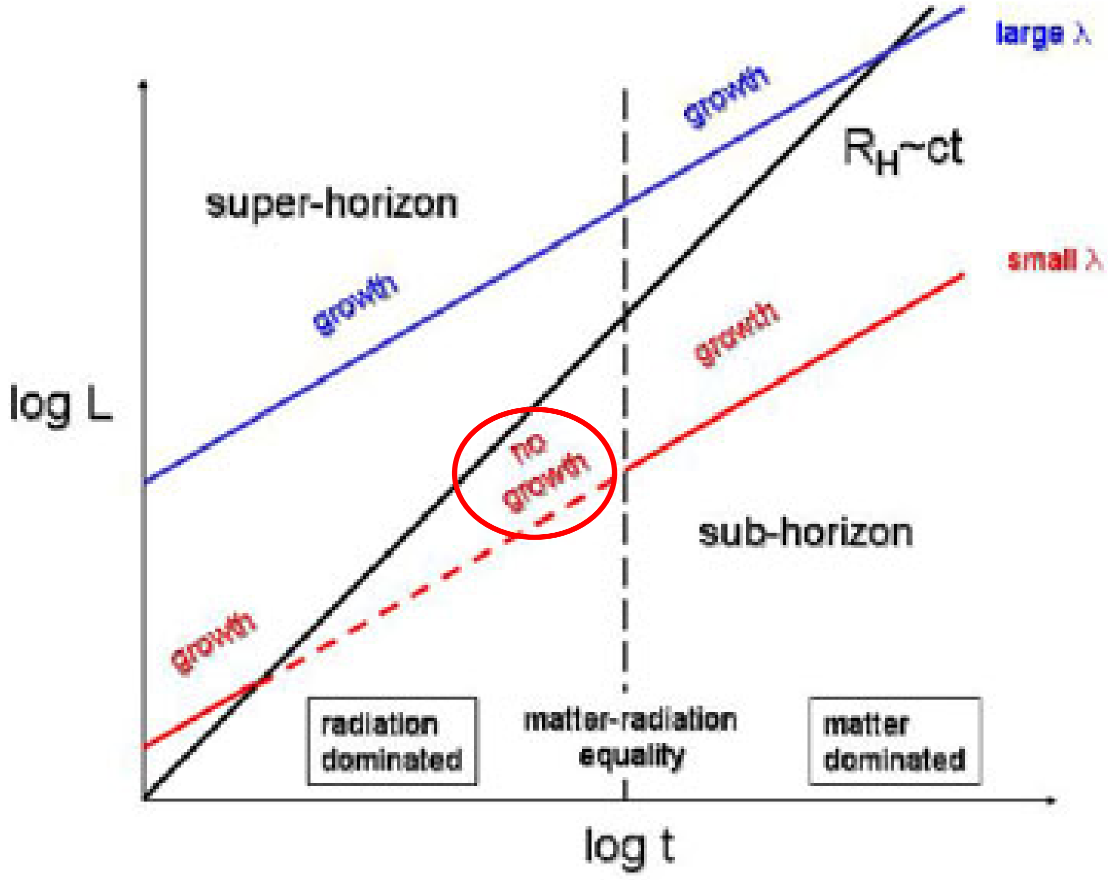
$$H^2 = (8\pi G/3)\,\rho$$

- Now consider a model with same H but slightly higher ρ (so it is a closed universe):

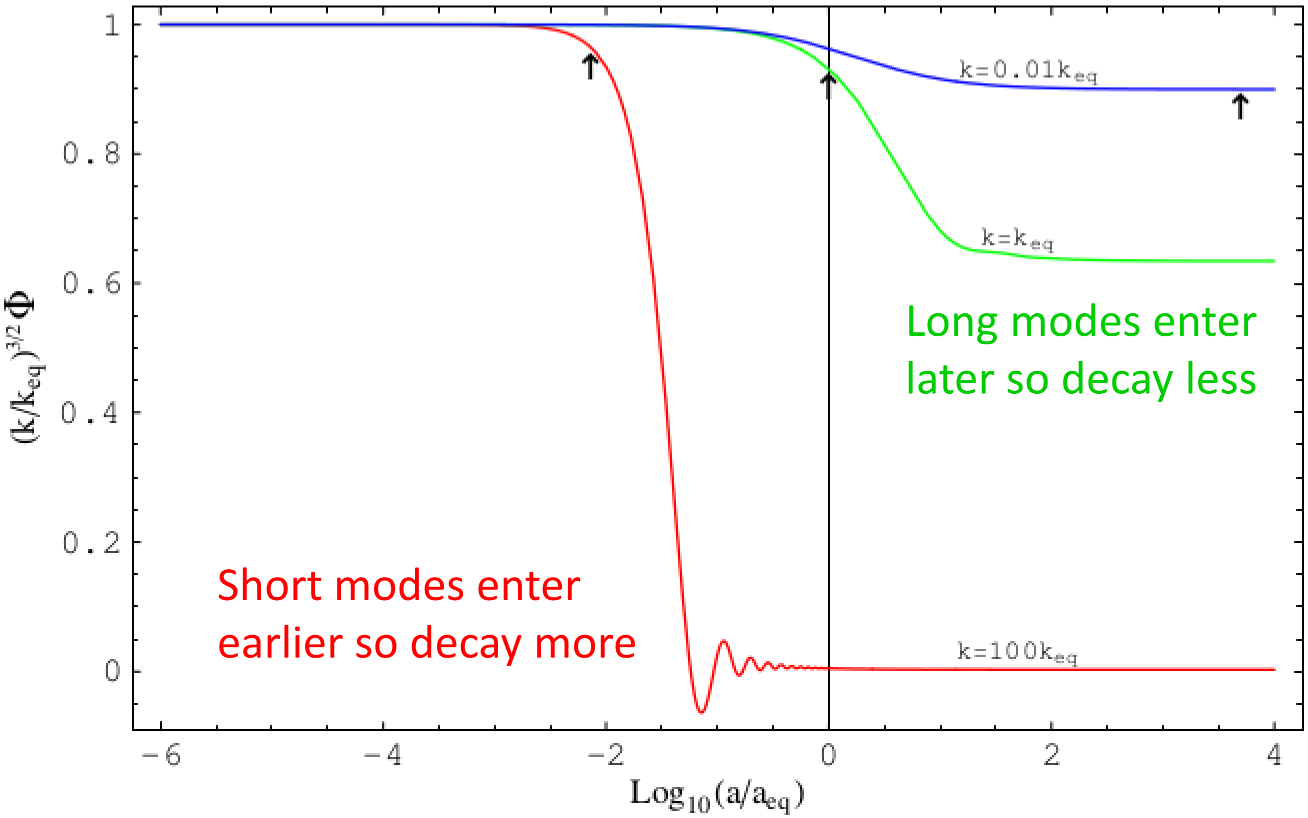
$$H^2 = 8\pi G\rho_1/3 - \kappa/a^2$$

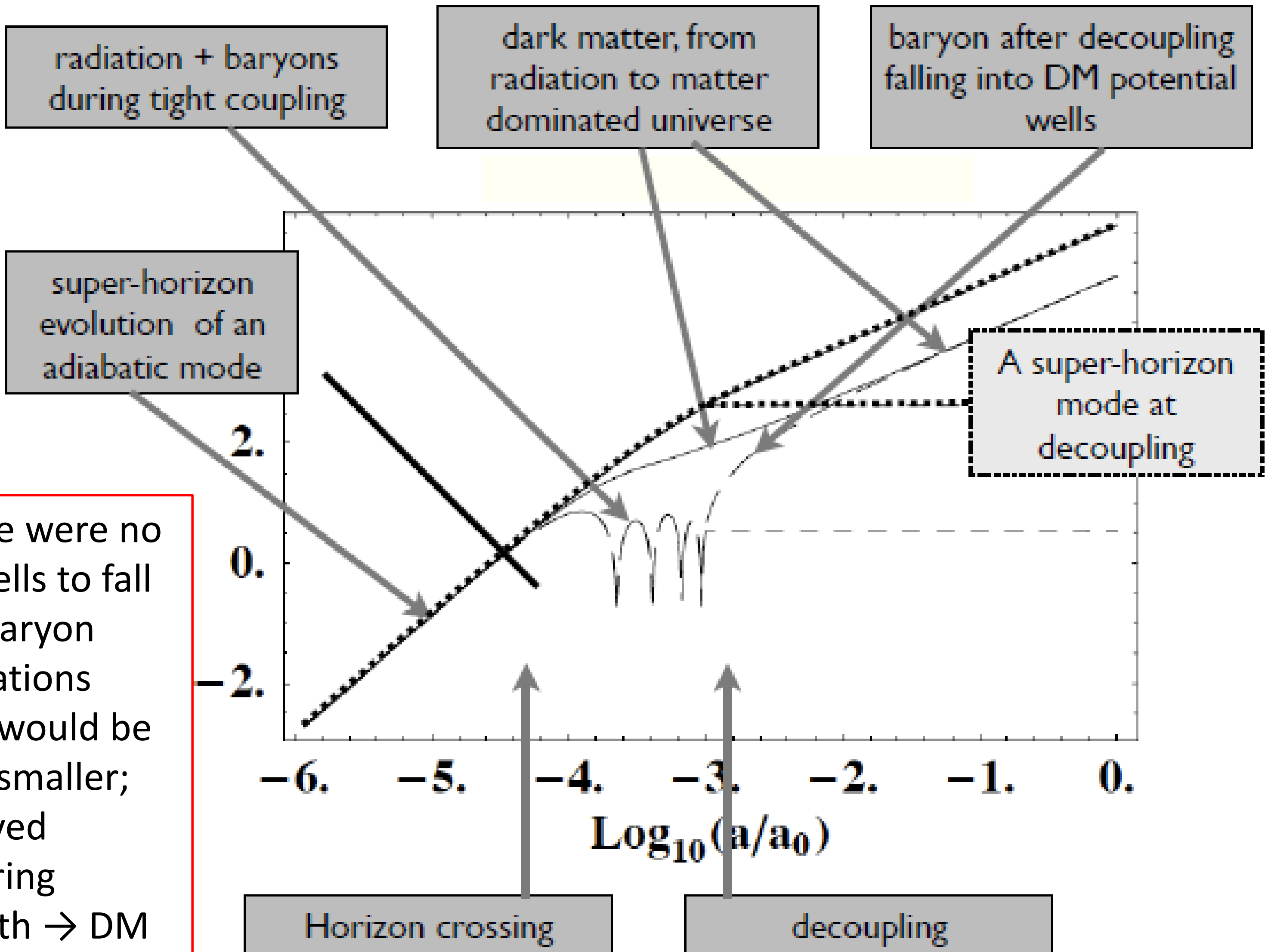
- Then $\delta = (\rho_1 - \rho)/\rho = (\kappa/a^2)/(8\pi G\rho/3)$
- For small δ we have $\delta \propto a$ (matter dominated) but $\delta \propto a^2$ (radiation dominated)

Long and short modes enter horizon at different times, so will grow differently



Potential
outside horizon = constant
inside horizon = decay during rad. dom





If there were no DM wells to fall into, baryon fluctuations today would be much smaller; observed clustering strength → DM must exist!

Putting it together

- Consider two modes, λ_1 and $\lambda_2 < \lambda_1$, which entered at $a_1/a_2 = \lambda_1/\lambda_2$ while radiation dominated
- Their amplitudes will be $(a_1/a_2)^2 = (k_2/k_1)^2$ so expect suppression of power $\propto k^{-2}$ at $k>k_{eq}$ (i.e. for the short wavelength modes which entered earlier)
- After entering horizon, dark matter grows only logarithmically until matter domination, after which it grows $\propto a$
- Baryons oscillate (i.e. don't grow) until decoupling, after which they fall into the deeper wells defined by the dark matter

Transfer function is approximately

$T_{CDM}(k) \propto 1/[1+(k/k_{eq})^2]$

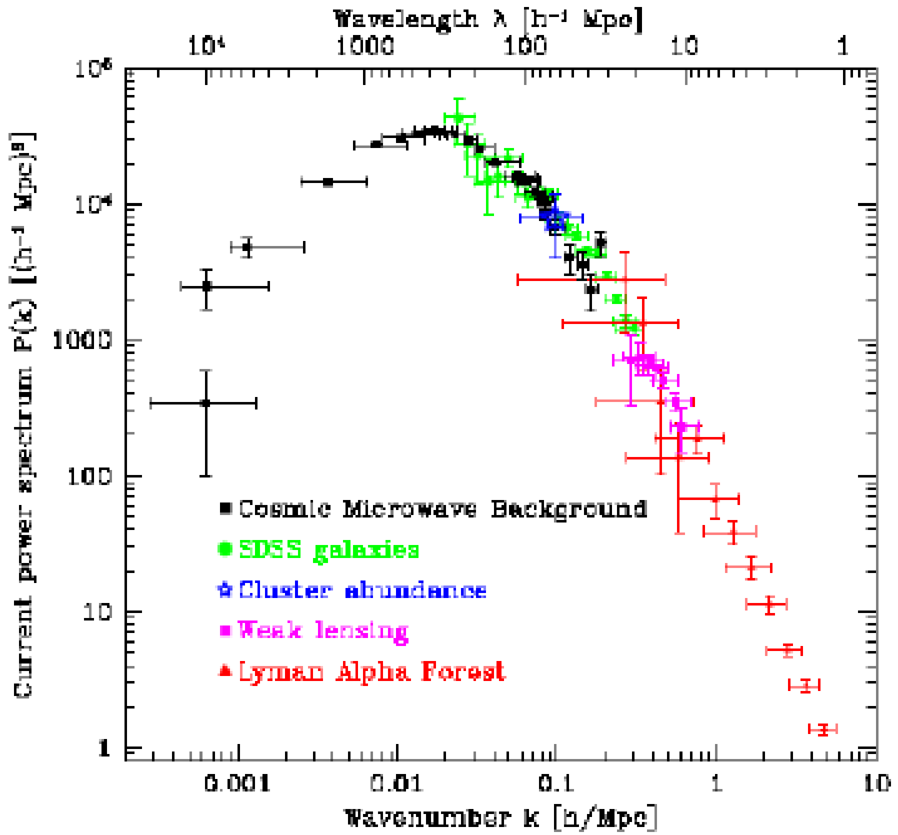
$P(k) \propto k\, T_{CDM}^2(k)$

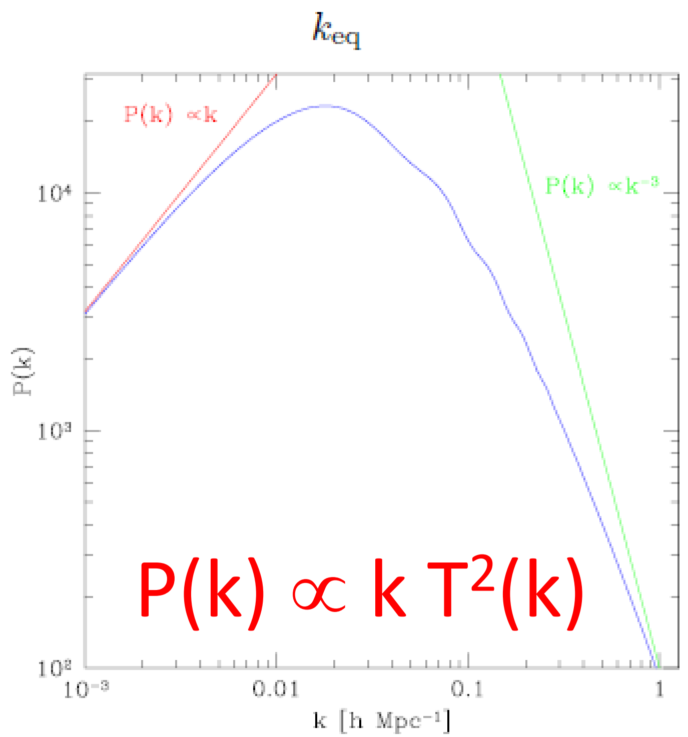
FT of $T_{CDM} = (k_{eq}^{-3}/4\pi)\, \exp(-rk_{eq})/rk_{eq}$

so might wish to think of T_{CDM} as describing 'smoothing' on scale R_{eq}

Similarly, sometimes useful to think of P(k) as 'smoothing' of 'white-noise' field to obtain field with correlations

Transfer function:
$T_{CDM}(k) \propto 1/[1+(k/k_{eq})^2]$





$P(k) \propto k\, T^2(k)$

$T_{WDM}(k) \approx T_{CDM}(k)\ [1 + (\alpha k)^2]^{-5}$

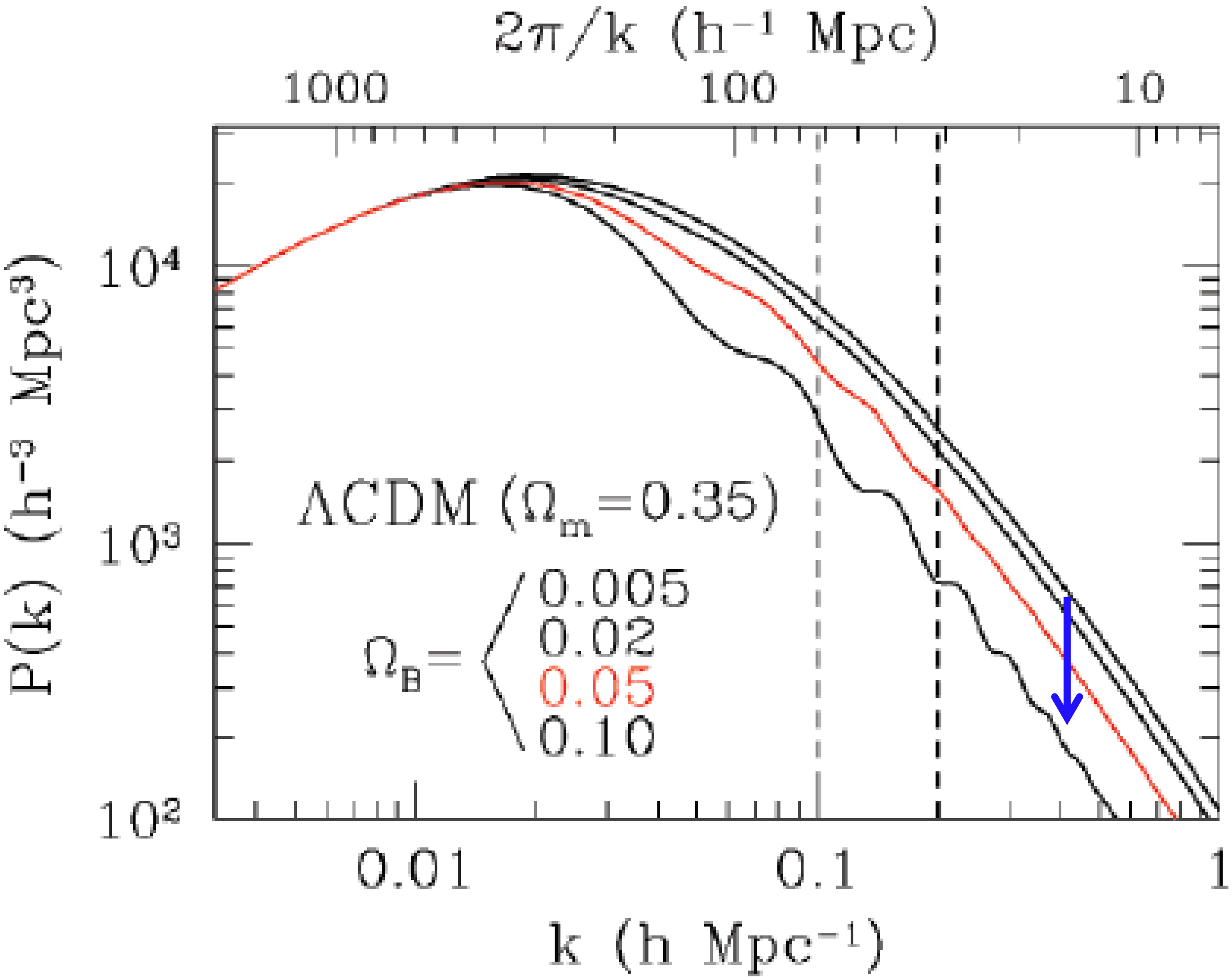
$$\alpha \equiv 0.05 \left(\frac{\Omega_m}{0.4}\right)^{0.15} \left(\frac{h}{0.65}\right)^{1.3} \left(\frac{m_{\rm dm}}{1\,{\rm keV}}\right)^{-1.15} h^{-1}{\rm Mpc}$$

Each species will have its own transfer function. E.g., baryons have $T_b(k)$, so

$$P(k) \propto k\,[\Omega_{CDM} T_{CDM}(k) + \Omega_b T_b(k)]^2$$

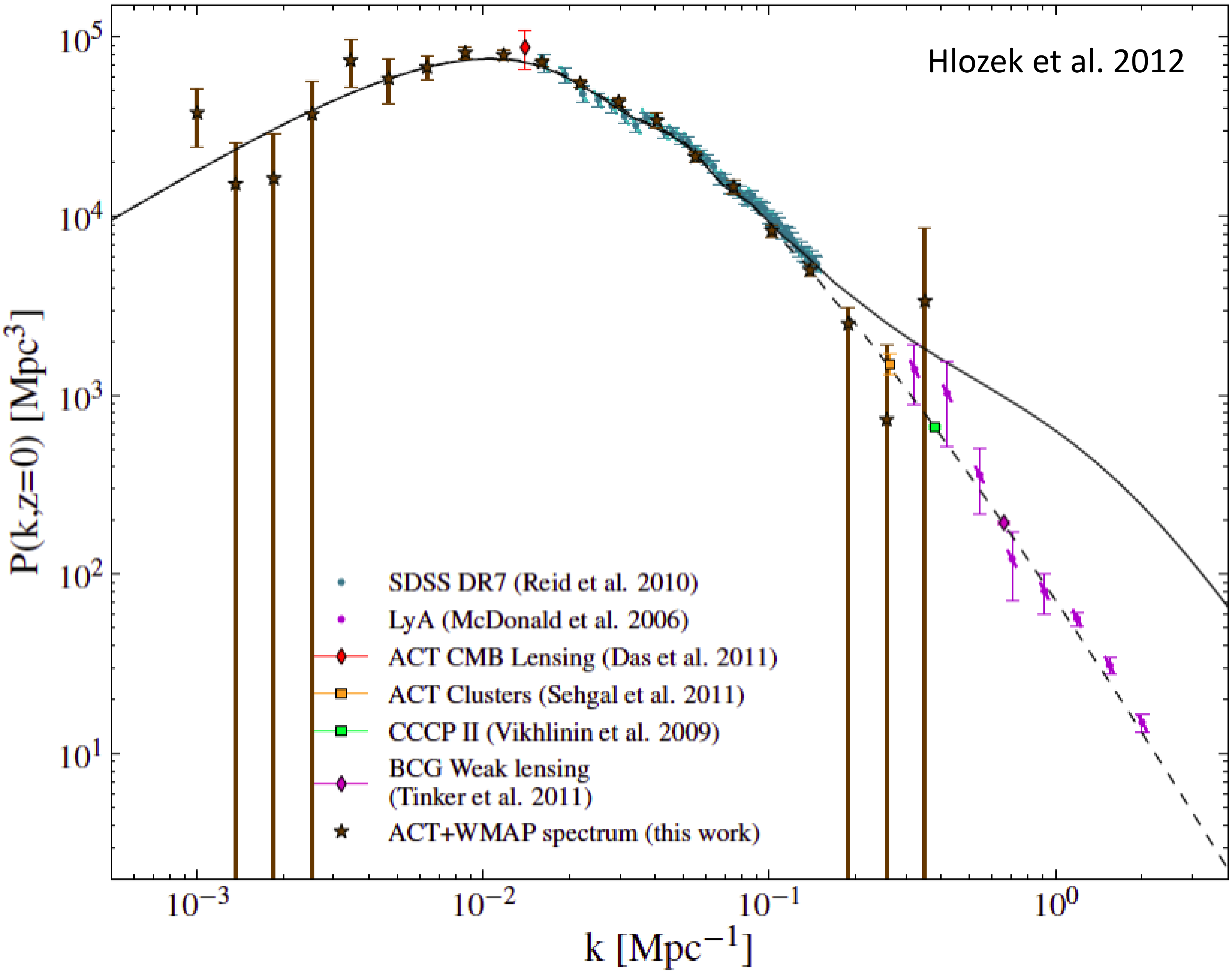
$$T_{CDM}(k) \propto 1/[1+(k/k_{eq})^2]$$

$$T_b(k) \propto j_0(k r_{BAO})$$

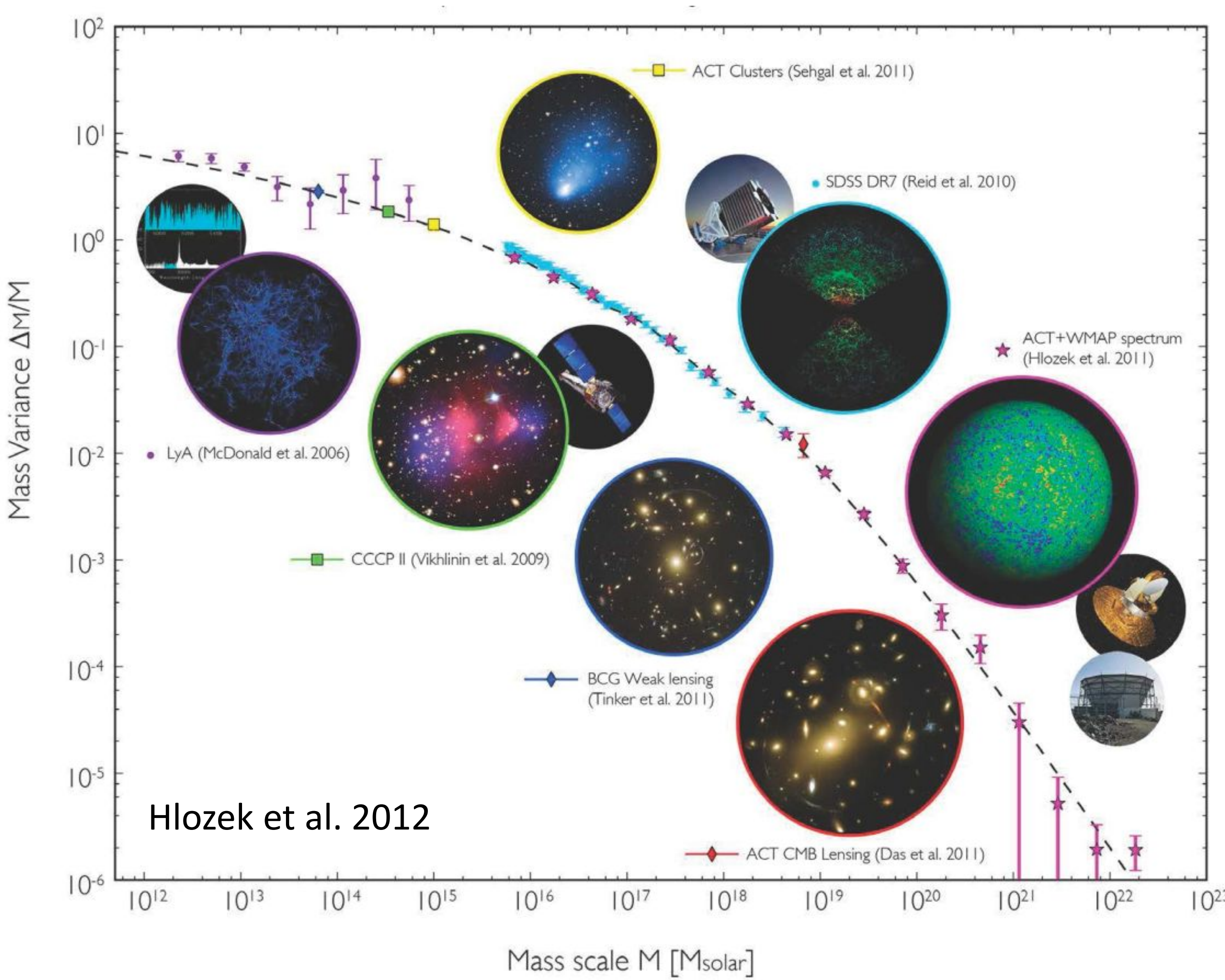


If all matter baryonic, power below 200 Mpc/h is suppressed

Need nonbaryonic gravitating dark matter to explain structure formation



Hlozek et al. 2012
P(k,z=0) [Mpc^3]
k [Mpc^-1]
10^5
10^4
10^3
10^2
10^1
10^-3
10^-2
10^-1
10^0
SDSS DR7 (Reid et al. 2010)
LyA (McDonald et al. 2006)
ACT CMB Lensing (Das et al. 2011)
ACT Clusters (Sehgal et al. 2011)
CCCP II (Vikhlinin et al. 2009)
BCG Weak lensing
(Tinker et al. 2011)
ACT+WMAP spectrum (this work)



ACT Clusters (Sehgal et al. 2011)
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ACT CMB Lensing (Das et al. 2011)
Hlozek et al. 2012
Mass Variance ΔM/M
Mass scale M [Msolar]
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{12}
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
10^{18}
10^{19}
10^{20}
10^{21}
10^{22}
10^{23}