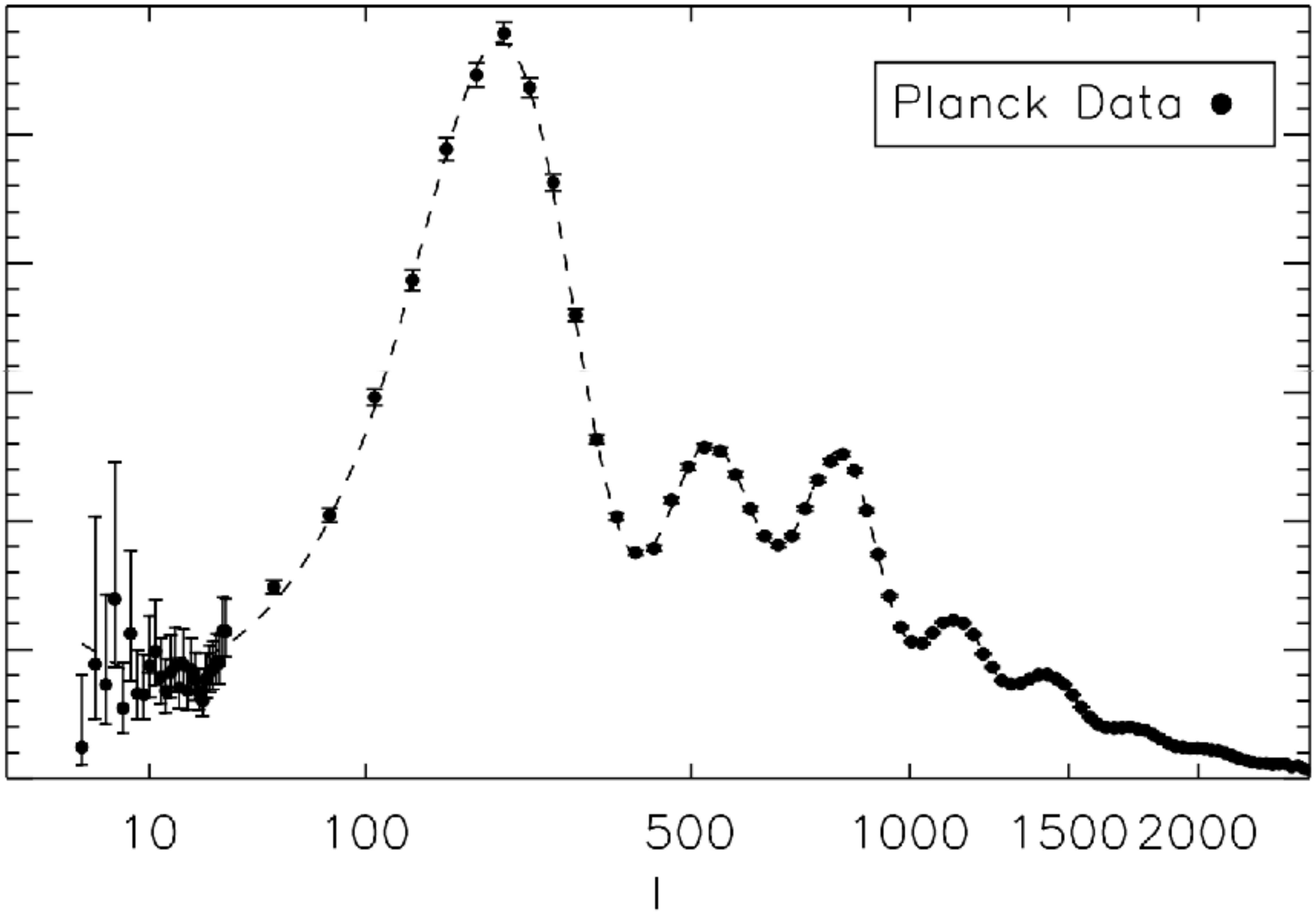
Lecture 2

- Temperature anisotropy from sound waves

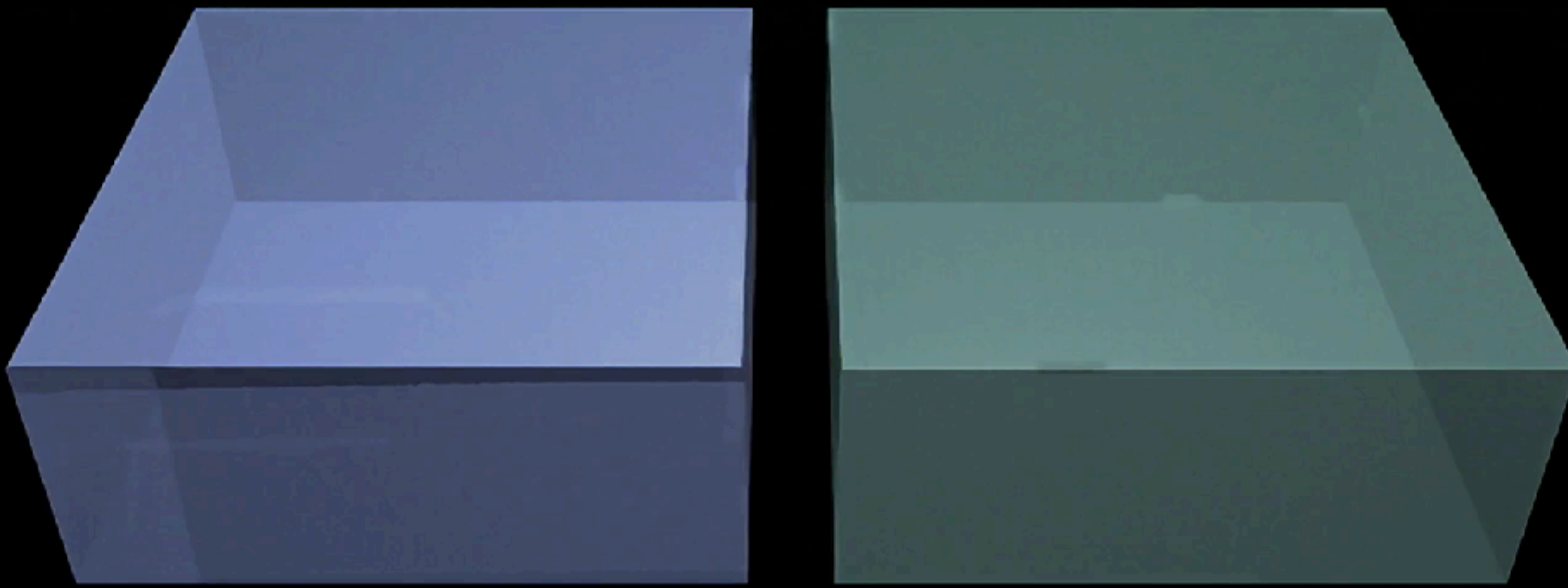
Planck Collaboration (2016)

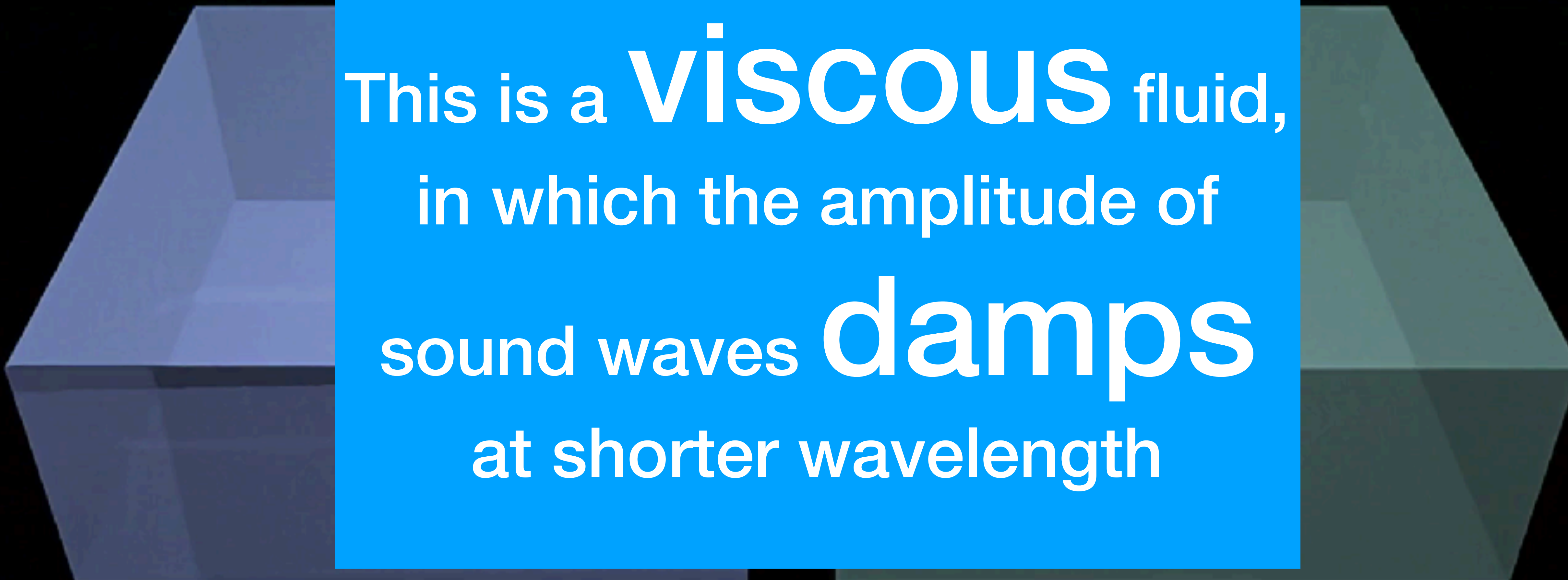
Planck 29-mo Power Spectrum



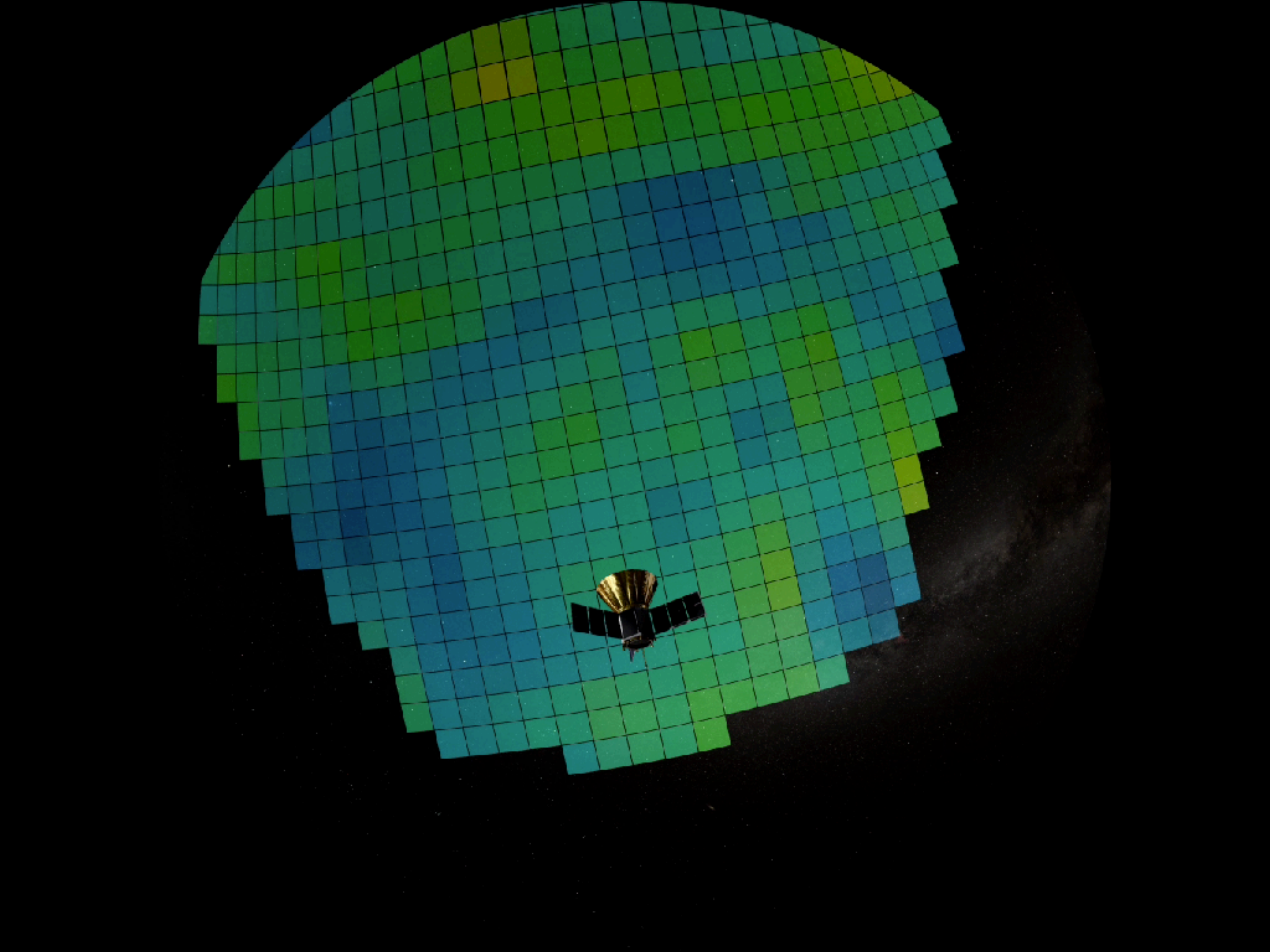
Cosmic Miso Soup

- When matter and radiation were hotter than 3000 K, matter was completely ionised. The Universe was filled with plasma, which behaves just like a soup
- Think about a Miso soup (if you know what it is). Imagine throwing Tofus into a Miso soup, while changing the density of Miso
- And imagine watching how ripples are created and propagate throughout the soup





This is a **viscous** fluid,
in which the amplitude of
sound waves **damps**
at shorter wavelength



When do sound waves become important?

- In other words, when would the Sachs-Wolfe approximation (purely gravitational effects) become invalid?
- The key to the answer: **Sound-crossing Time**
- Sound waves cannot alter temperature anisotropy at a given angular scale if there was not enough time for sound waves to propagate to the corresponding distance at the last-scattering surface
    - The distance traveled by sound waves within a given time = **The Sound Horizon**

Comoving Photon Horizon

- First, the comoving distance traveled by photons is given by setting the space-time distance to be null:

$$ds^2 = -c^2 dt^2 + a^2(t) dr^2 = 0$$

$$r_{\rm photon} = c \int_0^t \frac{dt'}{a(t')}$$

Comoving **Sound** Horizon

- Then, we replace the speed of light with a time-dependent speed of sound:

$$\Rightarrow \quad r_s = \int_0^t \frac{dt'}{a(t')} c_s(t')$$

- We cannot ignore the effects of sound waves if **$qr_s > 1$**

Sound Speed

- Sound speed of an adiabatic fluid is given by

$$c_s^2 = \delta P / \delta \rho$$

- **δP: pressure perturbation**
- **$\delta \rho$: density perturbation**

- For a baryon-photon system:

$$c_s^2 = \delta P_\gamma / (\delta \rho_\gamma + \delta \rho_B)$$

We can ignore the baryon pressure because it is much smaller than the photon pressure

Sound Speed

- Using the adiabatic relationship between photons and baryons:

$$\delta\rho_B/\bar{\rho}_B = \delta\rho_\gamma/(\bar{\rho}_\gamma + \bar{P}_\gamma) = 3\delta\rho_\gamma/4\bar{\rho}_\gamma$$

[i.e., the ratio of the number densities of baryons and photons is equal everywhere]

- and pressure-density relation of a relativistic fluid, **$\delta P_\gamma = \delta\rho_\gamma/3$**, We obtain

$$c_s^2 = \delta P_\gamma/(\delta\rho_\gamma + \delta\rho_B) = 1/3(1 + 3\bar{\rho}_B/4\bar{\rho}_\gamma)$$

- Or equivalently

$$c_s = \frac{1}{\sqrt{3(1+R)}}$$

sound speed is reduced!

where

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Value of R?

- The baryon mass density goes like a^{-3}, whereas the photon energy density goes like a^{-4}. Thus, the ratio of the two, R, goes like a.

- The proportionality constant is:

$$R = \frac{3\Omega_B}{4\Omega_\gamma}\frac{a}{a_0} = 0.6120\left(\frac{\Omega_B h^2}{0.022}\right)\frac{1091}{1+z}$$

where we used

$$\Omega_\gamma \equiv \frac{8\pi G\rho_{\gamma 0}}{3H_0^2} = 2.471\times 10^{-5}\ h^{-2} \quad \textbf{for} \quad T_0 = 2.725\ \mathrm{K}$$

For the last-scattering redshift of z_L=1090 (or last-scattering temperature of T_L=2974 K),

$$r_s = 145.3 \text{ Mpc}$$

We cannot ignore the effects of sound waves if $qr_s>1$. Since $l \sim qr_L$, this means

$$l > r_L/r_s = 96$$

where we used r_L=13.95 Gpc

Creation of Sound Waves: Basic Equations

1. Conservation equations (energy and momentum)

2. Equation of state, relating pressure to energy density
$$P = P(\rho)$$

3. General relativistic version of the "Poisson equation", relating gravitational potential to energy density
$$\nabla^2 \Phi(t, \boldsymbol{x}) = 4\pi G a^2(t) \delta\rho_M(t, \boldsymbol{x})$$

4. Evolution of the "anisotropic stress" (viscosity)

Energy Conservation

- **Total energy conservation:**

$$\sum_\alpha \left\{ \delta\dot{\rho}_\alpha + \frac{\dot{a}}{a}(3\delta\rho_\alpha + 3\delta P_\alpha + \nabla^2\pi_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} + \frac{1}{a^2}(\bar{\rho}_\alpha + \bar{P}_\alpha)\nabla^2 \delta u_\alpha \right\} = 0\,,$$

anisotropic stress:

[or, viscosity]

$$\Delta T_{ij} = a^2 \partial_i \partial_j \pi$$

velocity *potential*

$$\mathbf{v}_\alpha = \frac{1}{a}\nabla \delta u_\alpha$$

(

- C.f., Total energy conservation [unperturbed]

$$\sum_\alpha \left[\dot{\bar{\rho}}_\alpha + \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha) \right] = 0$$

)

Energy Conservation

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$$\sum_\alpha \left\{ \delta\dot{\rho}_\alpha + \frac{\dot{a}}{a}(3\delta\rho_\alpha + 3\delta P_\alpha + \nabla^2\pi_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} + \frac{1}{a^2}(\bar{\rho}_\alpha + \bar{P}_\alpha)\nabla^2\delta u_\alpha \right\} = 0\,,$$

- **Again, this is the effect of locally-defined inhomogeneous scale factor**, i.e.,
  - The spatial metric is given by $ds^2 = a^2(t)\exp(-2\Psi)d\mathbf{x}^2$
  - Thus, locally we can define a new scale factor:

$$\tilde{a}(t,\mathbf{x}) = a(t)\exp(-\Psi)$$

Energy Conservation

- **Total energy conservation:**

$$\sum_\alpha \left\{ \delta\dot{\rho}_\alpha + \frac{\dot{a}}{a}(3\delta\rho_\alpha + 3\delta P_\alpha + \nabla^2\pi_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} + \boxed{\frac{1}{a^2}(\bar{\rho}_\alpha + \bar{P}_\alpha)\nabla^2\delta u_\alpha} \right\} = 0\,,$$

- **Momentum flux going outward (inward) -> reduction (increase) in the energy density**

(**C.f., for a non-expanding medium:** $\dot{\rho} + \nabla \cdot (\rho\mathbf{v}) = 0$)

Momentum Conservation

- **Total momentum conservation**

$$\sum_\alpha \left\{ \frac{\partial}{\partial t}[(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha] + \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha + (\bar{\rho}_\alpha + \bar{P}_\alpha)\Phi + \delta P_\alpha + \nabla^2 \pi_\alpha \right\} = 0\,,$$

- **Cosmological redshift of the momentum**
- **Gravitational force given by potential gradient**
- **Force given by pressure gradient**
- **Force given by gradient of anisotropic stress**

Equation of State

- Pressure of non-relativistic species (i.e., baryons and cold dark matter) can be ignored relative to the energy density. Thus, we set them to zero: $\mathbf{P_B=0=P_D}$ and $\mathbf{\delta P_B=0=\delta P_D}$

- Unperturbed pressure of relativistic species (i.e., photons and relativistic neutrinos) is given by the third of the energy density, i.e., $\mathbf{P_\gamma=\rho_\gamma/3}$ and $\mathbf{P_\nu=\rho_\nu/3}$

- Perturbed pressure involves contributions from the bulk viscosity:

$$\delta P_\gamma = (\delta\rho_\gamma - \nabla^2\pi_\gamma)/3$$

$$\delta P_\nu = (\delta\rho_\nu - \nabla^2\pi_\nu)/3$$

The reason for this is that **trace** of the stress-energy of relativistic species **vanishes**: $\sum_{\mu=0,1,2,3} T_\mu{}^\mu = 0$

$$T_0^0 + \sum_{i=1}^{3} T_i^i = -\rho + 3P + \nabla^2\pi = 0$$

- Pres... d cold dark... ensity. Thu... D

- Unp... otons and... ene...

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Two Remarks

- In the standard scenario:
    - Energy densities are conserved separately; thus we do not need to sum over all species
    - Momentum densities of photons and baryons are NOT conserved separately but they are coupled via Thomson scattering. This must be taken into account when writing down separate conservation equations

Conservation Equations for Photons and Baryons

- Fourier transformation replaces $\nabla^2 \to -q^2$

$$X(t,\boldsymbol{x}) = (2\pi)^{-3}\int d^3q\; X_{\boldsymbol{q}}(t)\exp(i\boldsymbol{q}\cdot\boldsymbol{x})$$

$$\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma) - \frac{4q^2}{3a^2}\delta u_\gamma = 4\dot{\Psi}$$

$$\frac{\partial}{\partial t}(\delta\rho_B/\bar{\rho}_B) - \frac{q^2}{a^2}\delta u_B = 3\dot{\Psi}$$

momentum transfer via scattering

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_{\mathcal{T}}\bar{n}_e(\delta u_B - \delta u_\gamma)$$

$$\delta\dot{u}_B + \Phi = -\frac{\sigma_{\mathcal{T}}\bar{n}_e}{R}(\delta u_B - \delta u_\gamma)$$

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Conservation Equations for Photons and Baryons

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what about photon's viscosity?

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_{\mathcal{T}}\bar{n}_e(\delta u_B - \delta u_\gamma)$$

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Peebles & Yu (1970); Sunyaev & Zeldovich (1970)

Formation of a Photon-baryon Fluid

- Photons are not a fluid. Photons free-stream at the speed of light
  - The conservation equations are not enough because we need to specify the evolution of viscosity
  - Solving for viscosity requires information of the phase-space distribution function of photons: **Boltzmann equation**
- However, frequent scattering of photons with baryons* can make photons behave as a fluid: **Photon-baryon fluid**

Photons scatter with electrons via Thomson scattering. Protons scatter with electrons via Coulomb scattering. Thus we can say, effectively, photons scatter with baryons

Let's solve them!

- Fourier transformation replaces $\nabla^2 \to -q^2$

$$X(t,\boldsymbol{x}) = (2\pi)^{-3} \int d^3q \; X_{\boldsymbol{q}}(t) \exp(i\boldsymbol{q}\cdot\boldsymbol{x})$$

$$\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma) - \frac{4q^2}{3a^2}\delta u_\gamma = 4\dot{\Psi}$$

$$\frac{\partial}{\partial t}(\delta\rho_B/\bar{\rho}_B) - \frac{q^2}{a^2}\delta u_B = 3\dot{\Psi}$$

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = \sigma_{\mathcal{T}}\bar{n}_e(\delta u_B - \delta u_\gamma)$$

$$\delta\dot{u}_B + \Phi = -\frac{\sigma_{\mathcal{T}}\bar{n}_e}{R}(\delta u_B - \delta u_\gamma)$$

$$R \equiv 3\bar{\rho}_B/4\bar{\rho}_\gamma$$

Tight-coupling Approximation

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d/\sigma_{\mathcal{T}} \bar{n}_e$$

[d is an arbitrary dimensionless variable]

- And take $\sigma_{\mathcal{T}} \bar{n}_e \to \infty$ *. We obtain

$$a \frac{\partial}{\partial t}(\delta u_\gamma / a) + \Phi + \frac{\delta \rho_\gamma}{4 \bar{\rho}_\gamma} = d \,, \qquad \delta \dot{u}_\gamma + \Phi = -\frac{d}{R}$$

In this limit, viscosity π_γ is exponentially suppressed. This result comes from the Boltzmann equation but we do not derive it here. It makes sense physically.

Tight-coupling Approximation

- Eliminating *d* and using the fact that R is proportional to the scale factor, we obtain

$$a\frac{\partial}{\partial t}\left[(1+R)\delta u_\gamma/a\right]+(1+R)\Phi+\frac{\delta\rho_\gamma}{4\bar\rho_\gamma}=0$$

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain

$$\frac{1}{a(1+R)}\frac{\partial}{\partial t}\left[a(1+R)\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar\rho_\gamma-4\Psi)\right]+\frac{4q^2}{3a^2}\Phi+\frac{q^2}{a^2}\frac{\delta\rho_\gamma/\bar\rho_\gamma}{\boxed{3(1+R)}}=0$$

Wave Equation, with the speed of sound of c_s^2 = 1/3(1+R)!

Peebles & Yu (1970); Sunyaev & Zeldovich (1970)

Sound Wave!

- To simplify the equation, let's first look at the high-frequency solution
  - Specifically, we take q >> aH (the wavelength of fluctuations is much shorter than the Hubble length). Then we can ignore time derivatives of R and Ψ because they evolve in the Hubble time scale:

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

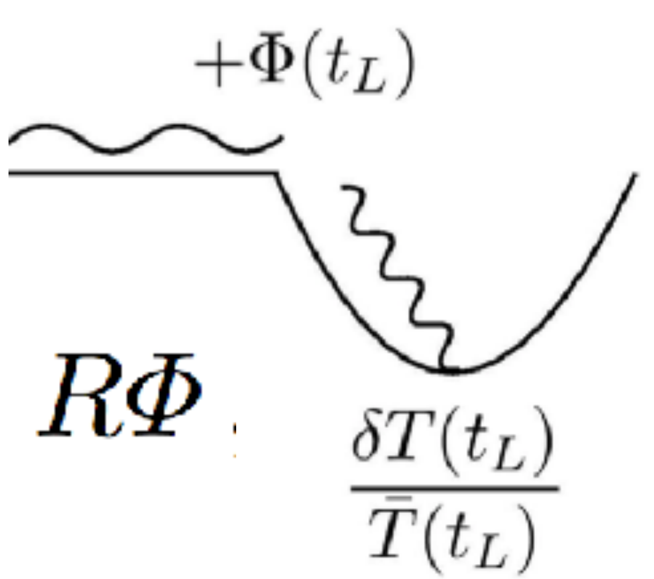
Solution: SOUND WAVE!

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=A\cos(qr_s)+B\sin(qr_s)-R\Phi$$

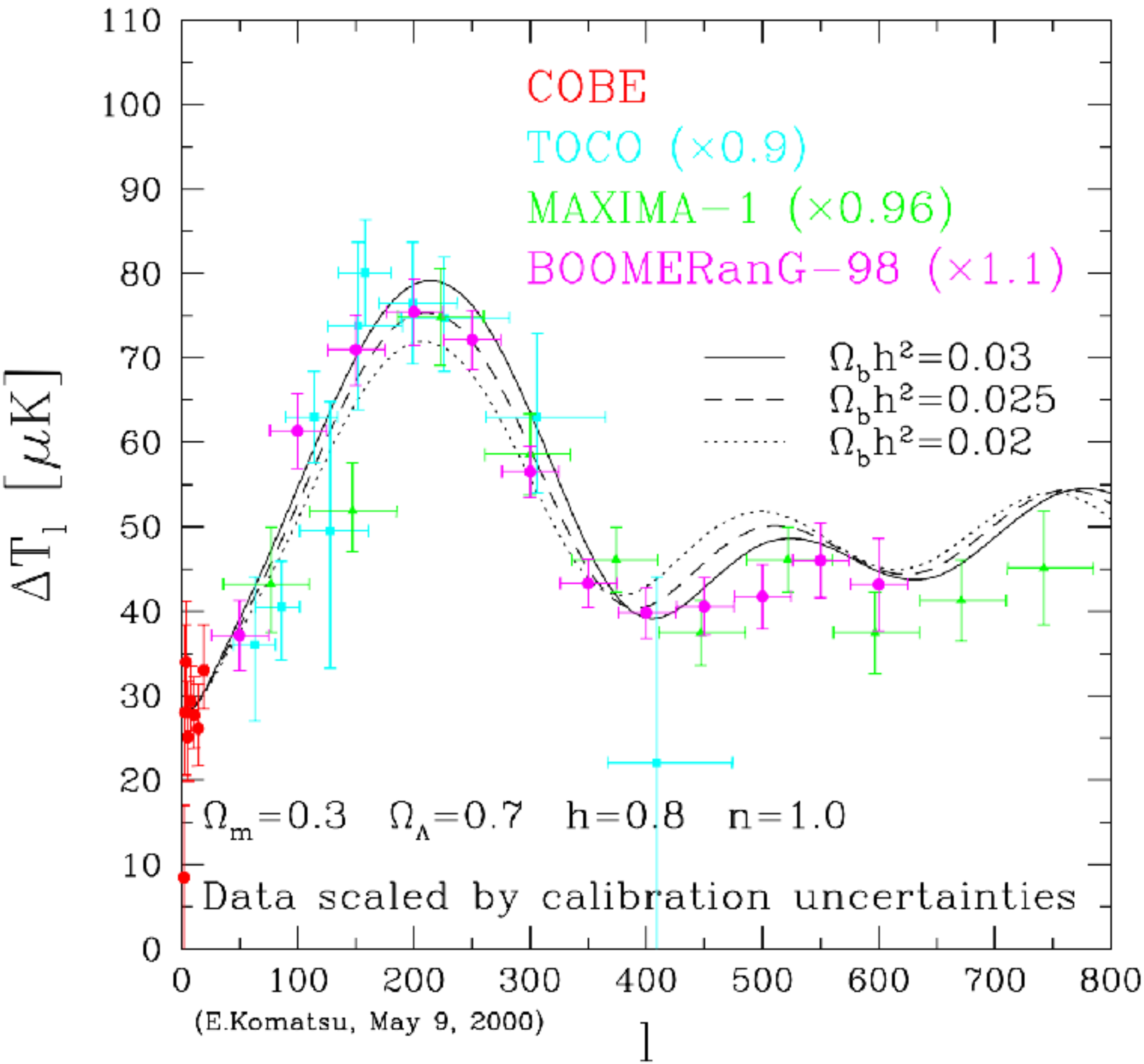
Recap

- Photons are not a fluid; but Thomson scattering couples photons to baryons, forming a **photon-baryon fluid**
- The reduced sound speed, $c_s^2=1/3(1+R)$, emerges automatically
- $\delta\rho_\gamma/4\rho_\gamma$ is the temperature anisotropy at the bottom of the potential well. Adding gravitational redshift, the observed temperature anisotropy is $\delta\rho_\gamma/4\rho_\gamma + \Phi$, which is given by

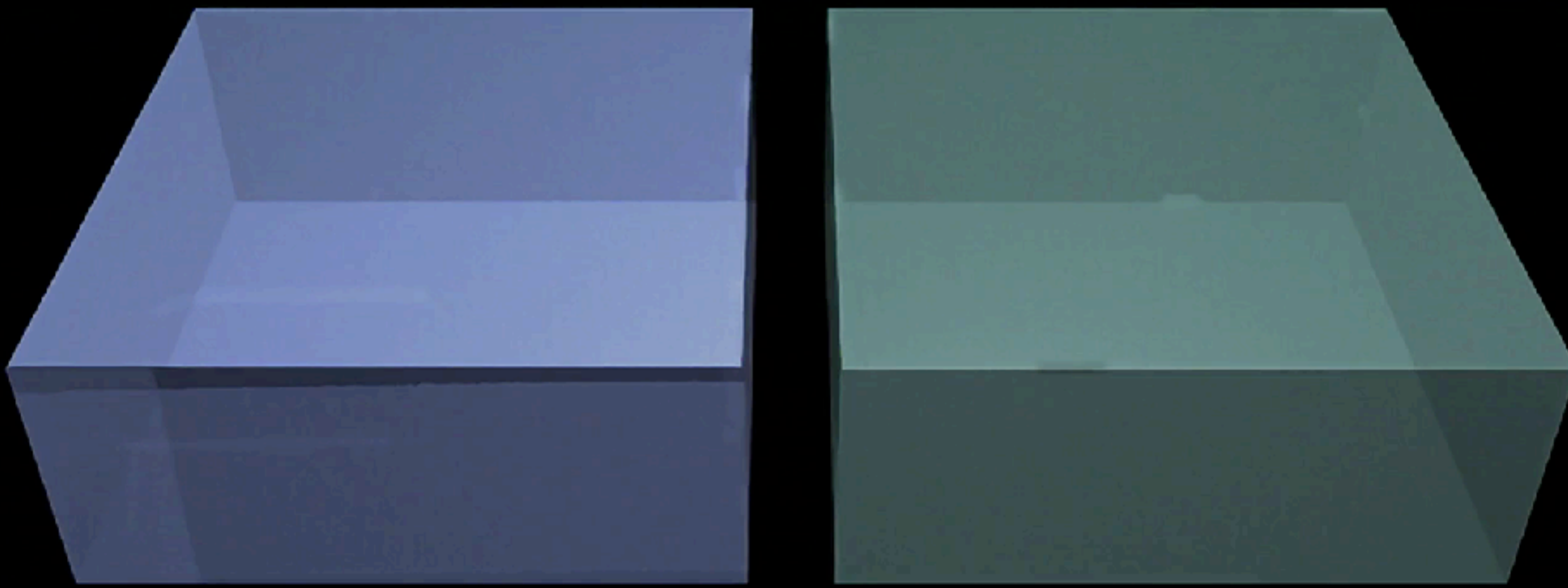
$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$



Effect of Baryon-Density

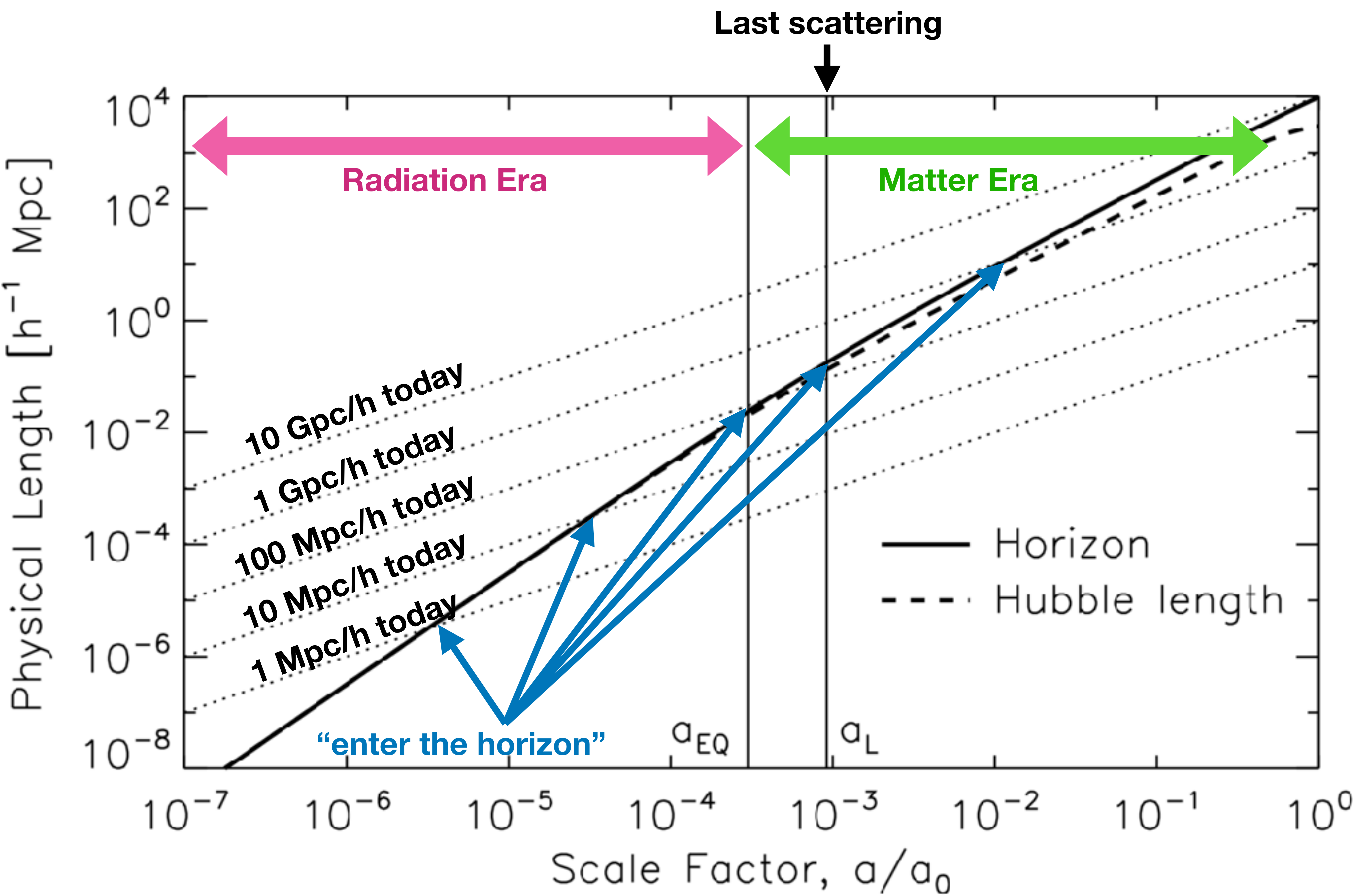


(E.Komatsu, May 9, 2000)



Stone: Fluctuations “entering the horizon”

- This is a tricky concept, but it is important
- Suppose that there are fluctuations at all wavelengths, including the ones that exceed the Hubble length (which we loosely call our “horizon”)
    - Let’s not ask the origin of these “super-horizon fluctuations”, but just assume their existence
- As the Universe expands, our horizon grows and we can see longer and longer wavelengths
    - **Fluctuations “entering the horizon”**



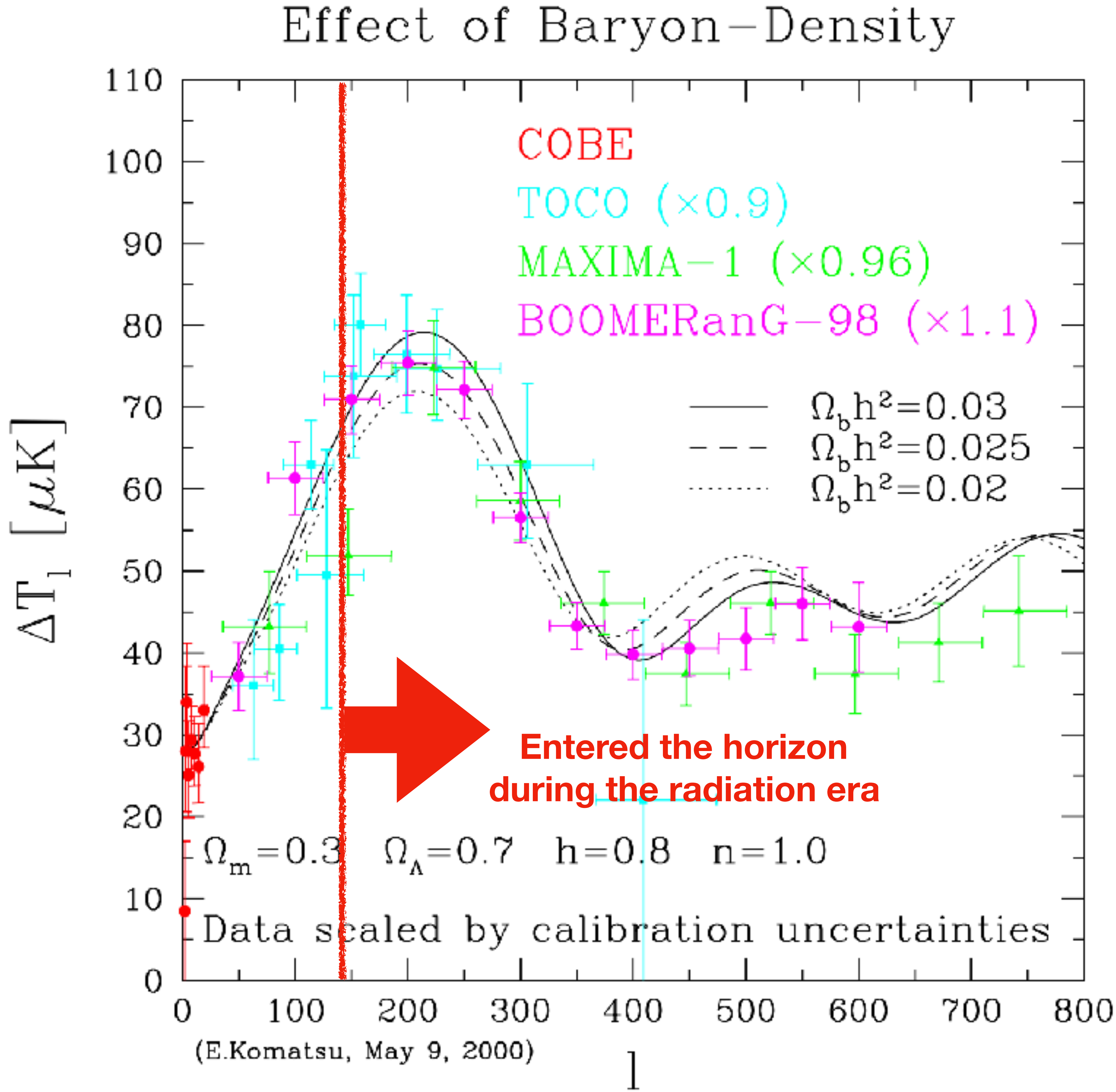
Last scattering
Radiation Era
Matter Era
10 Gpc/h today
1 Gpc/h today
100 Mpc/h today
10 Mpc/h today
1 Mpc/h today
"enter the horizon"
a_{EQ}
a_L
Horizon
Hubble length
Physical Length [h^{-1} Mpc]
Scale Factor, a/a_0
10^{4}
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}

Three Regimes

- **Super-horizon scales [q < aH]**
  - Only gravity is important
  - Evolution differs from Newtonian
- **Sub-horizon but super-sound-horizon [$aH < q < aH/c_s$]**
  - Only gravity is important
  - Evolution similar to Newtonian
- **Sub-sound-horizon scales [$q > aH/c_s$]**
  - Hydrodynamics important -> Sound waves

q_{EQ}

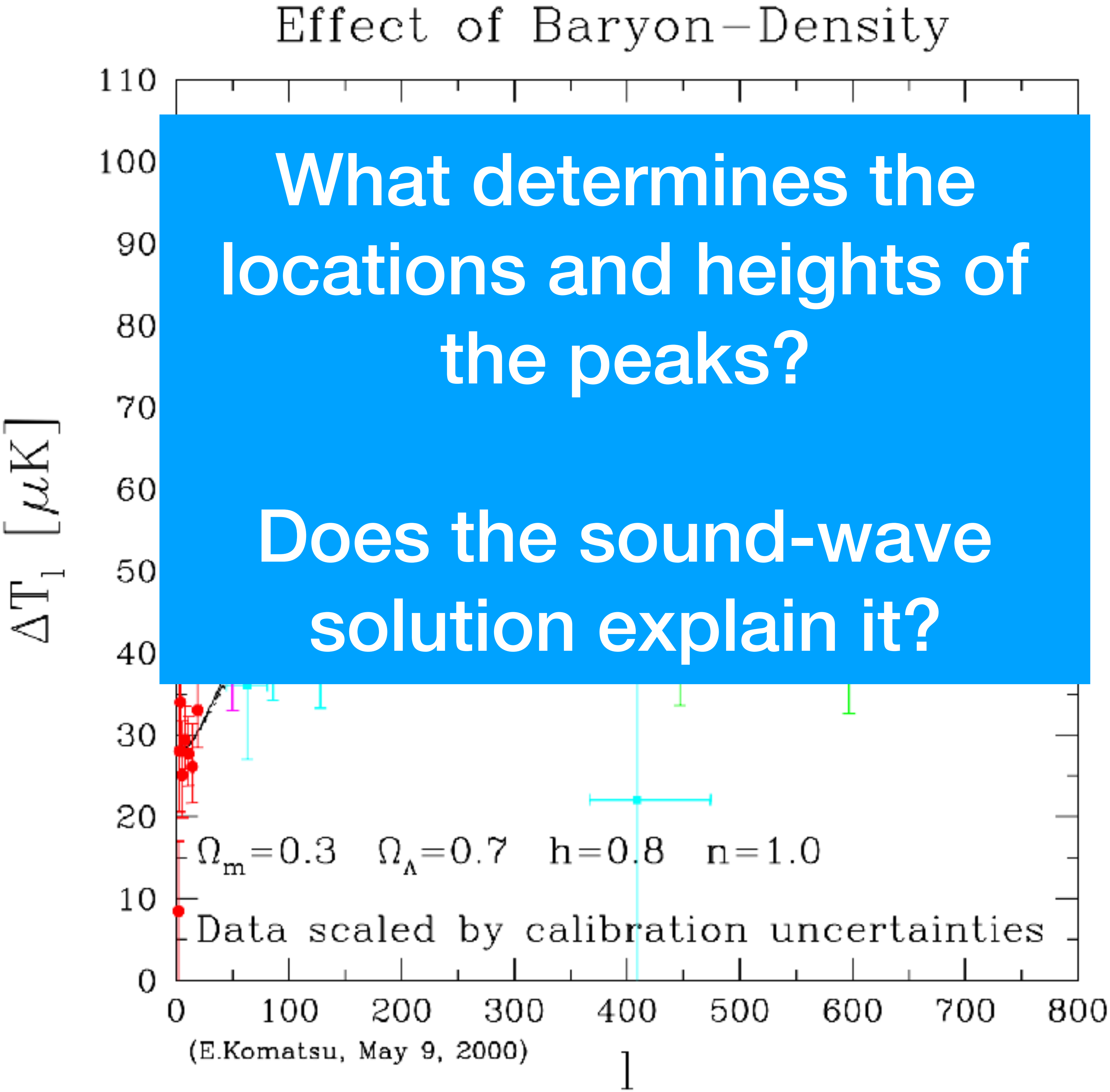
- Which fluctuation entered the horizon before the matter-radiation equality?
- $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01\ (\Omega_M h^2/0.14)\ \mathrm{Mpc}^{-1}$
- At the last scattering surface, this subtends the multipole of $l_{EQ} = q_{EQ}r_L \sim$ **140**



Effect of Baryon−Density
COBE
TOCO (×0.9)
MAXIMA−1 (×0.96)
BOOMERanG−98 (×1.1)
$\Omega_b h^2 = 0.03$
$\Omega_b h^2 = 0.025$
$\Omega_b h^2 = 0.02$
Entered the horizon during the radiation era
$\Omega_m = 0.3$ $\Omega_\Lambda = 0.7$ h=0.8 n=1.0
Data scaled by calibration uncertainties
ΔT_l [μK]
l
(E.Komatsu, May 9, 2000)

What determines the locations and heights of the peaks?

Does the sound-wave solution explain it?



Peak Locations?

High-frequency solution, for q >> aH

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$

- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi \right]^2$ with q -> l/r_L
  - Question: What are the integration constants, **A** and **B**?
  - Answer: They depend on the initial conditions; namely, adiabatic or not?
  - For adiabatic initial condition, **A >> B *when q is large***

[We will show it later.]

Peak Locations?

High-frequency solution, for q >> aH

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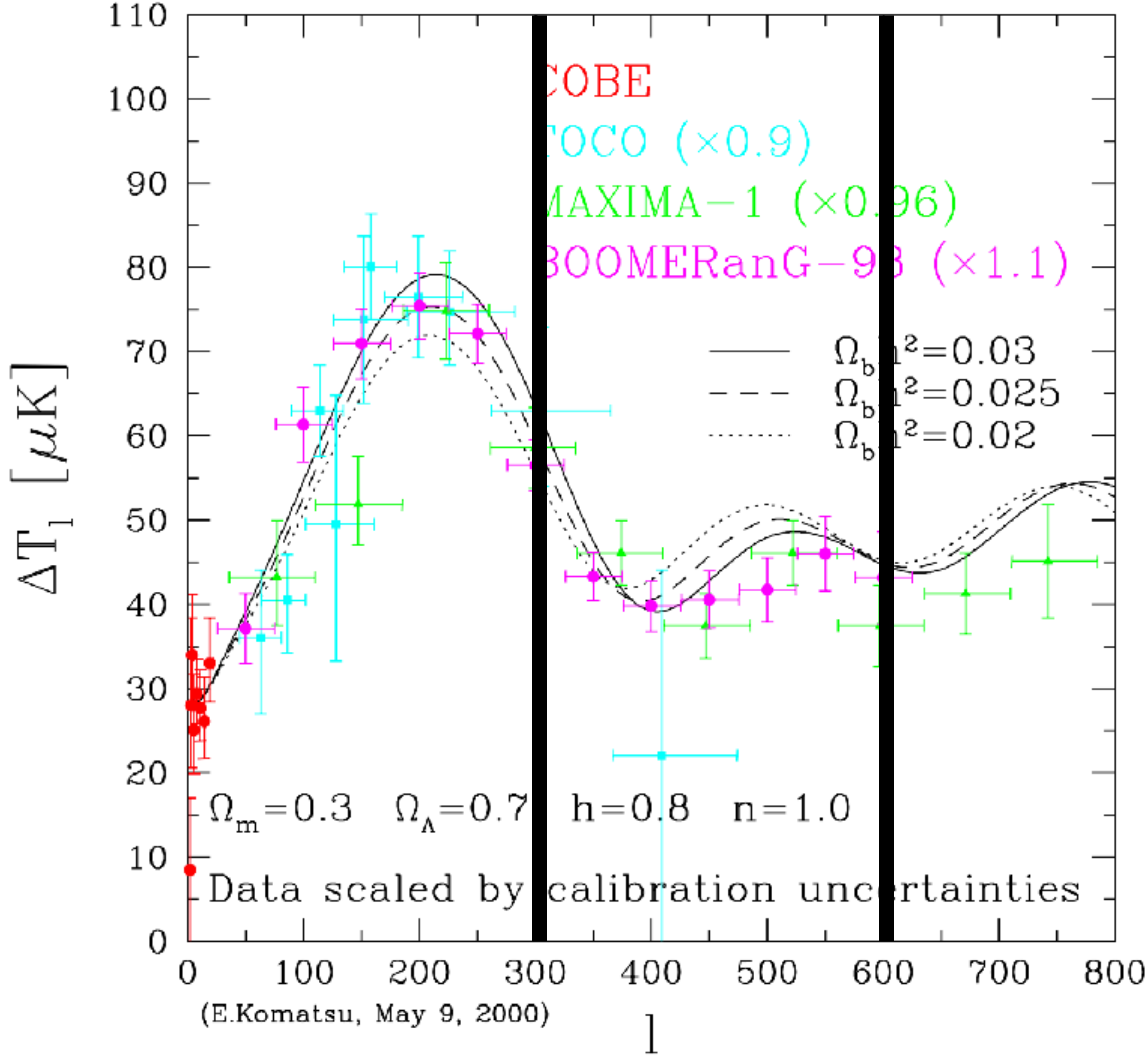
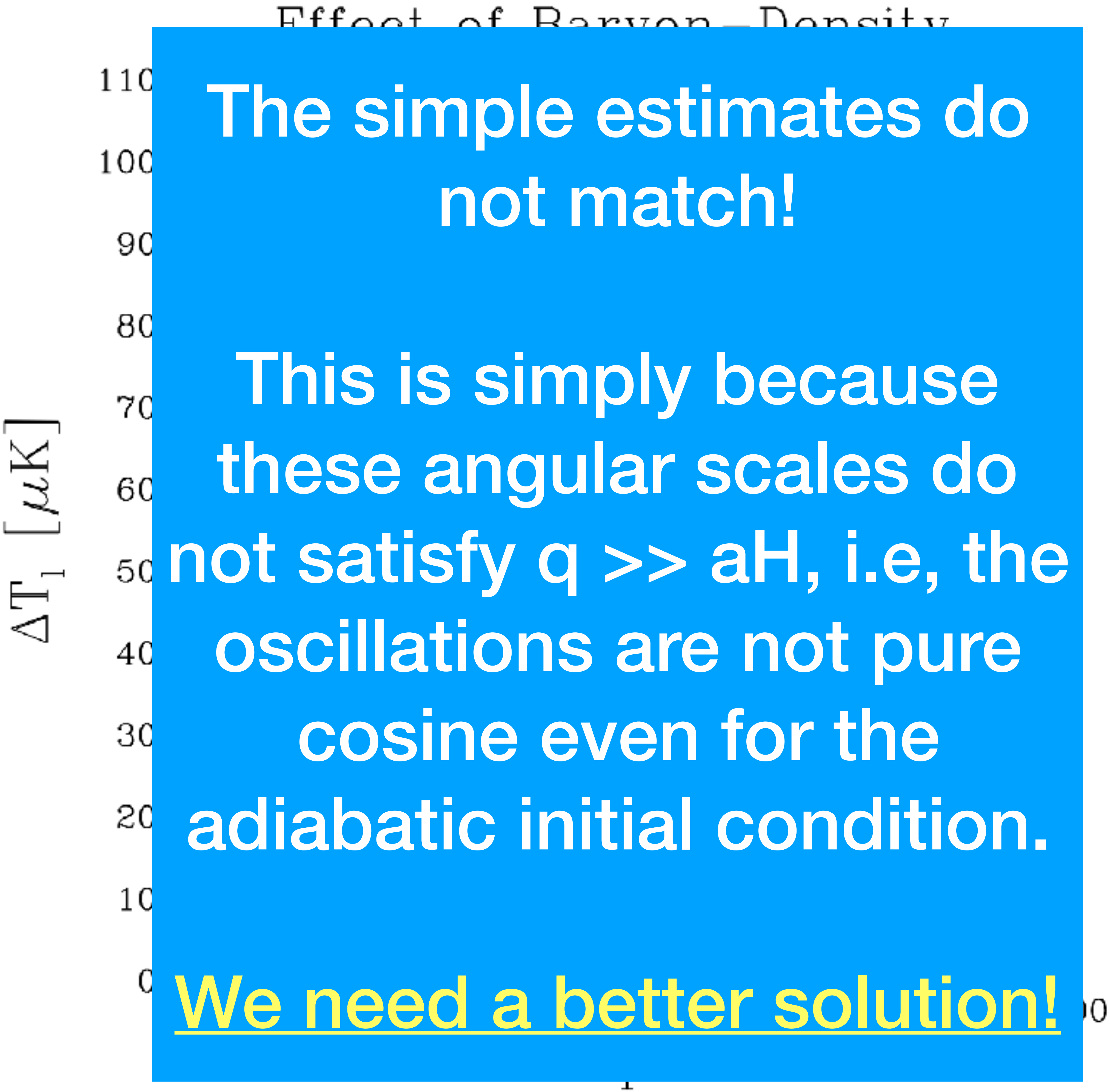
- VERY roughly speaking, the angular power spectrum C_l is given by $\left[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi\right]^2$ with q -> l/r_L

- If A>>B, the locations of peaks are

$$\ell = (1,\ 2,\ \cdots)\pi r_L/r_s(t_L) = (1,\ 2,\ \cdots) \times 302$$

Effect of Baryon-Density





The simple estimates do not match!

This is simply because these angular scales do not satisfy q >> aH, i.e, the oscillations are not pure cosine even for the adiabatic initial condition.

<u>We need a better solution!</u>

Better Solution in Radiation-dominated Era

Going back to the original tight-coupling equation..

$$\frac{1}{a(1+R)}\frac{\partial}{\partial t}\left[a(1+R)\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma - 4\Psi)\right] + \frac{4q^2}{3a^2}\Phi + \frac{q^2}{a^2}\frac{\delta\rho_\gamma/\bar{\rho}_\gamma}{3(1+R)} = 0$$

- In the radiation-dominated era, R << 1
- Change the independent variable from the time (t) to

$$\varphi \equiv qr_s = 2qt/\sqrt{3}a$$

Better Solution in Radiation-dominated Era

Then the equation simplifies to

$$\partial^2 X/\partial\varphi^2 + X + \Phi + \Psi = 0$$

where $X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$

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The solution is

$$X = \tilde{A}\cos\varphi + \tilde{B}\sin\varphi - \int_0^\varphi d\varphi' \sin(\varphi - \varphi')(\Phi + \Psi)(\varphi')$$

Better Solution in Radiation-dominated Era

Then the equation simplifies to

$$\partial^2 X/\partial\varphi^2 + X + \Phi + \Psi = 0$$

where $X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi$

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi'),$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

Einstein's Equations

- Now we need to know Newton's gravitational potential, ϕ, and the scalar curvature perturbation, ψ.
- Einstein's equations - let's look up any text books:

$$\nabla^2\Psi = 4\pi G a^2 \sum_\alpha \left[\delta\rho_\alpha - \frac{3\dot{a}}{a}(\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha\right]$$

$$\dot{\Psi} + \frac{\dot{a}}{a}\Phi = -4\pi G \sum_\alpha (\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha$$

$$\partial_i\partial_j(\Phi - \Psi) = -8\pi G a^2 \partial_i\partial_j \sum_\alpha \pi_\alpha$$

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Will come back to this later.
For now, let's ignore any viscosity.

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$$\dot{\Psi} + \frac{\dot{a}}{a}\Phi = -4\pi G \sum_\alpha (\bar{\rho}_\alpha + \bar{P}_\alpha)\delta u_\alpha$$

$$\Phi = \Psi$$

**Will come back to this later.
For now, let's ignore any viscosity.**

Einstein's Equations in Radiation-dominated Era

- Now we need to know Newton's gravitational potential, ϕ, and the scalar curvature perturbation, ψ.
- Einstein's equations - let's look up any text books:

$$\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{4}{\varphi}\frac{\partial \Phi}{\partial \varphi} + \Phi = \frac{3}{2\varphi^2}\frac{\delta \mathcal{P}}{\bar{\rho}_R}$$

$$\sum_\alpha \delta P_\alpha(t, \boldsymbol{x}) = \frac{\sum_\alpha \dot{\bar{P}}_\alpha(t)}{\sum_\alpha \dot{\bar{\rho}}_\alpha(t)} \sum_\alpha \delta\rho_\alpha(t, \boldsymbol{x}) + \delta\mathcal{P}(t, \boldsymbol{x})$$

"non-adiabatic" pressure

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We shall ignore this

$$\sum_\alpha \delta P_\alpha(t, \boldsymbol{x}) = \frac{\sum_\alpha \dot{\bar{P}}_\alpha(t)}{\sum_\alpha \dot{\bar{\rho}}_\alpha(t)} \sum_\alpha \delta\rho_\alpha(t, \boldsymbol{x}) + \delta\mathcal{P}(t, \boldsymbol{x})$$

"non-adiabatic" pressure

Kodama & Sasaki (1986, 1987)

Solution (Adiabatic) in Radiation-dominated Era

$$\Phi_{\mathrm{ADI}} = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

where

$$\varphi \equiv q r_s = 2qt/\sqrt{3}a$$

- Low-frequency limit (*super-sound-horizon scales, qr_s << 1*)
  - Φ_{ADI} -> $-2\zeta/3$ = **constant**
- High-frequency limit (*sub-sound-horizon scales, qr_s >> 1*)
  - Φ_{ADI} -> $2\zeta\cos\varphi/\varphi^2 \propto a^{-2}$ **damp**

Solution (Adiabatic) in Radiation-dominated Era

$$\Phi_{\mathrm{ADI}} = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

where

Poisson Equation

$$-q^2\Phi = 4\pi G a^2 \delta\rho$$

& oscillation solution for radiation

$$\delta\rho_R/\bar{\rho}_R \propto \cos\varphi$$

- Low-frequency ... cales, $qr_s << 1$)
  - $\Phi_{\mathrm{ADI}} \to -2\zeta/3$
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  - $\Phi_{\mathrm{ADI}} \to 2\zeta \cos\varphi/\varphi^2 \propto a^{-2}$ **damp**

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Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

ζ: Conserved on large scales

- For the adiabatic initial condition, there exists a useful quantity, **ζ**, which remains constant on large scales (*super-horizon scales, q << aH*) regardless of the contents of the Universe
  - ζ is conserved regardless of whether the Universe is radiation-dominated, matter-dominated, or whatever
- Energy conservation for q << aH:

$$\delta\dot{\rho}_\alpha + \frac{3\dot{a}}{a}(\delta\rho_\alpha + \delta P_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} = 0$$

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

ζ: Conserved on large scales

- If pressure is a function of the energy density only, i.e., $P_\alpha = P_\alpha(\rho_\alpha)$, then

$$\frac{1}{3}\frac{\delta\rho_\alpha(t,\boldsymbol{x})}{\bar{\rho}_\alpha(t)+\bar{P}_\alpha(t)} - \Psi(t,\boldsymbol{x}) = \zeta_\alpha(\boldsymbol{x})$$

integration constant

Integrate (↑)

$$\delta\dot{\rho}_\alpha + \frac{3\dot{a}}{a}(\delta\rho_\alpha + \delta P_\alpha) - 3(\bar{\rho}_\alpha + \bar{P}_\alpha)\dot{\Psi} = 0$$

Bardeen, Steinhardt & Turner (1983); Weinberg (2003); Lyth, Malik & Sasaki (2005)

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$$\frac{1}{3}\frac{\delta\rho_\alpha(t,\boldsymbol{x})}{\bar{\rho}_\alpha(t)+\bar{P}_\alpha(t)} - \Psi(t,\boldsymbol{x}) = \zeta_\alpha(\boldsymbol{x})$$

integration constant

For the adiabatic initial condition, all species share the same value of ζ_α, i.e., $\zeta_\alpha = \zeta$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2)$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2$$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

The solution is

$$X = (\tilde{A} + \Delta A)\cos\varphi + (\tilde{B} + \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma/4\bar{\rho}_\gamma - \Psi \xrightarrow{\varphi \ll 1} \zeta$$

i.e., $\tilde{A}_{\text{ADI}} = \zeta$, $\tilde{B}_{\text{ADI}} = 0$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2)$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2$$

Kodama & Sasaki (1986, 1987); Baumann, Green, Meyers & Wallisch (2016)

Sound Wave Solution in the Radiation-dominated Era

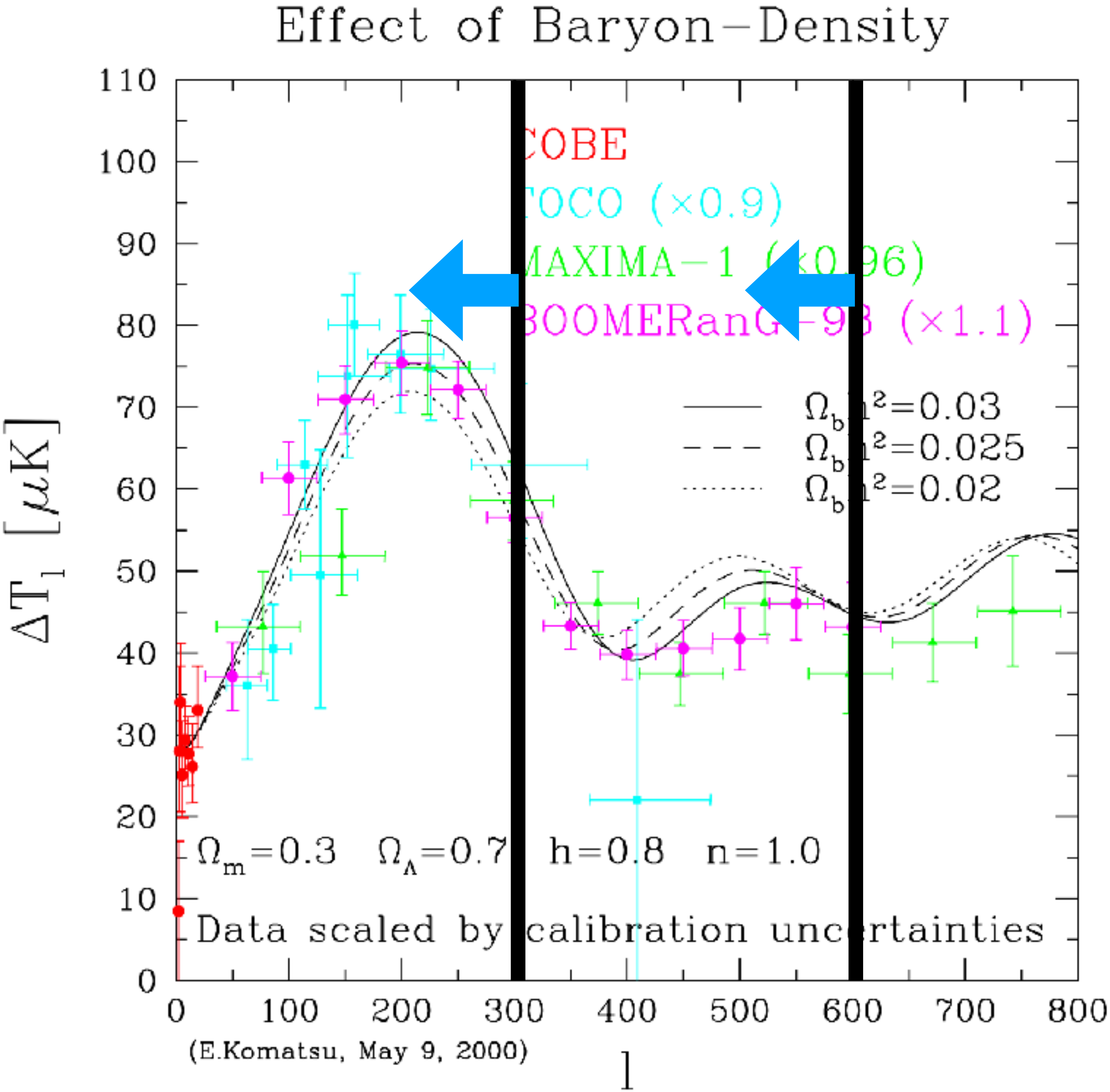
The adiabatic solution is

$$X = \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta\left(-\cos\varphi + \frac{2}{\varphi}\sin\varphi\right)$$

with

$$\Phi = \Psi = -2\zeta(\sin\varphi - \varphi\cos\varphi)/\varphi^3$$

Therefore, the solution is a pure cosine only in the high-frequency limit!



Effect of Baryon-Density
COBE
TOCO (×0.9)
MAXIMA-1 (×0.96)
BOOMERanG-98 (×1.1)
Ωbh²=0.03
Ωbh²=0.025
Ωbh²=0.02
Ωm=0.3 ΩΛ=0.7 h=0.8 n=1.0
Data scaled by calibration uncertainties
ΔTl [μK]
l
(E.Komatsu, May 9, 2000)

Roles of viscosity

- **Neutrino viscosity**
  - Modify potentials: $\partial_i\partial_j(\Phi - \Psi) = -8\pi G a^2 \partial_i\partial_j \boxed{\pi_\nu}$
- **Photon viscosity**
  - Viscous photon-baryon fluid: **damping of sound waves**

Silk (1968) "Silk damping"

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\boxed{\pi_\gamma}}{2\bar{\rho}_\gamma} = \sigma_{\mathcal{T}}\bar{n}_e(\delta u_B - \delta u_\gamma)$$

High-frequency solution without neutrino viscosity

The solution is

$$X = (\zeta + \Delta A)\cos\varphi + (\qquad \Delta B)\sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A(\varphi) \equiv \int_0^\varphi d\varphi' \sin\varphi' (\Phi + \Psi)(\varphi') = -2\zeta(1 - \sin^2\varphi/\varphi^2) \longrightarrow -2\zeta$$

$$\Delta B(\varphi) \equiv -\int_0^\varphi d\varphi' \cos\varphi' (\Phi + \Psi)(\varphi')$$

$$= 2\zeta(\varphi - \cos\varphi\sin\varphi)/\varphi^2 \longrightarrow 0$$

Chluba & Grin (2013)

High-frequency solution **with** neutrino viscosity

The solution is

$$X = (-\zeta + \Delta A_\nu)\cos\varphi + \Delta B_\nu \sin\varphi$$

where

$$X \equiv \delta\rho_\gamma / 4\bar{\rho}_\gamma - \Psi$$

$$\Delta A_\nu \longrightarrow 0.338 R_\nu \zeta$$

$$\Delta B_\nu \longrightarrow 0.418 R_\nu \zeta$$

non-zero value!

$$R_\nu \equiv \bar{\rho}_\nu / (\bar{\rho}_\gamma + \bar{\rho}_\nu) \approx 0.409$$

High-frequency solution **with** neutrino viscosity

The solution is

$$X = -C\cos(\varphi + \theta)$$

where

$$C \equiv \sqrt{(-\zeta + \Delta A_\nu)^2 + \Delta B_\nu^2}$$

$$\approx \zeta(1 + 4R_\nu/15)^{-1}$$

Hu & Sugiyama (1996)

$$\tan\theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi$$

Phase shift!

Bashinsky & Seljak (2004)

High-frequency solution

with neutrino viscosity

The s

X

where

O

Thus, the neutrino viscosity will:

(1) **Reduce the amplitude** of sound waves at large multipoles

(2) **Shift the peak positions** of the temperature power spectrum

$$\tan\theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi$$

Phase shift!

Bashinsky & Seljak (2004)

Photon Viscosity

- In the tight-coupling approximation, the photon viscosity damps exponentially
- To take into account a non-zero photon viscosity, we go to a higher order in the tight-coupling approximation

Tight-coupling Approximation (1st-order)

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d/\sigma_{\mathcal{T}}\bar{n}_e$$

[d is an arbitrary dimensionless variable]

- And take $\sigma_{\mathcal{T}}\bar{n}_e \to \infty$ *. We obtain

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} = d\,, \qquad \delta\dot{u}_\gamma + \Phi = -\frac{d}{R}$$

In this limit, viscosity π_γ is exponentially suppressed. This result comes from the Boltzmann equation but we do not derive it here. It makes sense physically.

Tight-coupling Approximation (2nd-order)

- When Thomson scattering is efficient, the relative velocity between photons and baryons is small. We write

$$\delta u_B - \delta u_\gamma = d_1/\sigma_\mathcal{T}\bar{n}_e + qd_2/(\sigma_\mathcal{T}\bar{n}_e)^2$$

where

$$d_1 = -R(\delta\dot{u}_\gamma + \Phi)$$

[d_2 is an arbitrary dimensionless variables]

- And take $\sigma_\mathcal{T}\bar{n}_e \to \infty$. We obtain

$$a\frac{\partial}{\partial t}(\delta u_\gamma/a) + \Phi + \frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma} = -R(\delta\dot{u}_\gamma + \Phi) + \frac{q}{\sigma_\mathcal{T}\bar{n}_e}d_2$$

$$\frac{\partial}{\partial t}\left[\frac{R(\delta\dot{u}_\gamma + \Phi)}{\sigma_\mathcal{T}\bar{n}_e}\right] = \frac{q}{R\sigma_\mathcal{T}\bar{n}_e}d_2$$

Tight-coupling Approximation (2nd-order)

- Eliminating d_2 and using the fact that R is proportional to the scale factor, we obtain

$$a\frac{\partial}{\partial t}\left[(1+R)\delta u_\gamma/a\right]+(1+R)\Phi+\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}-\frac{q^2\pi_\gamma}{2\bar{\rho}_\gamma}+R\frac{\partial}{\partial t}\left[\frac{R(\delta\dot{u}_\gamma+\Phi)}{\sigma_{\mathcal{T}}\bar{n}_e}\right]=0$$

- Getting π_γ requires an approximate solution of the Boltzmann equation in the 2nd-order tight coupling. We do not derive it here. The answer is

$$\pi_\gamma = -\frac{32}{45}\frac{\bar{\rho}_\gamma}{\sigma_{\mathcal{T}}\bar{n}_e}\frac{\delta u_\gamma}{a^2}$$

Kaiser (1983)

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given by the velocity potential - a well-known result in fluid dynamics

$$\pi_\gamma=-\frac{32}{45}\frac{\bar{\rho}_\gamma}{\sigma_T\bar{n}_e}\frac{\delta u_\gamma}{a^2}$$

Kaiser (1983)

Damped Oscillator

- Using the energy conservation to replace δu_γ with $\delta\rho_\gamma/\rho_\gamma$, we obtain, for q >> aH,

$$\frac{1}{a}\frac{\partial}{\partial t}\left[a\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)\right]+2\Gamma\frac{\partial}{\partial t}(\delta\rho_\gamma/\bar{\rho}_\gamma)+\frac{q^2c_s^2}{a^2}\left[\delta\rho_\gamma/\bar{\rho}_\gamma+4(1+R)\Phi\right]=0$$

New term, giving damping!

where

$$\Gamma(q,t)\equiv\frac{q^2}{6a^2\sigma_{\mathcal{T}}\bar{n}_e}\left[\frac{16}{15(1+R)}+\frac{R^2}{(1+R)^2}\right]$$

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New term, giving damping!

where

Important for high frequencies (large multipoles)

$$\Gamma(q,t)\equiv\frac{q^2}{6a^2\sigma_{\mathcal{T}}\bar{n}_e}\left[\frac{16}{15(1+R)}+\frac{R^2}{(1+R)^2}\right]$$

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New term, giving damping!

SOLUTION:

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=\left[A\cos(qr_s)+B\sin(qr_s)\right]\exp\left[-\int_0^t dt'\ \Gamma(q,t')\right]-R\Phi$$

Exponential dampling!

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New term, giving damping!

SOLUTION:

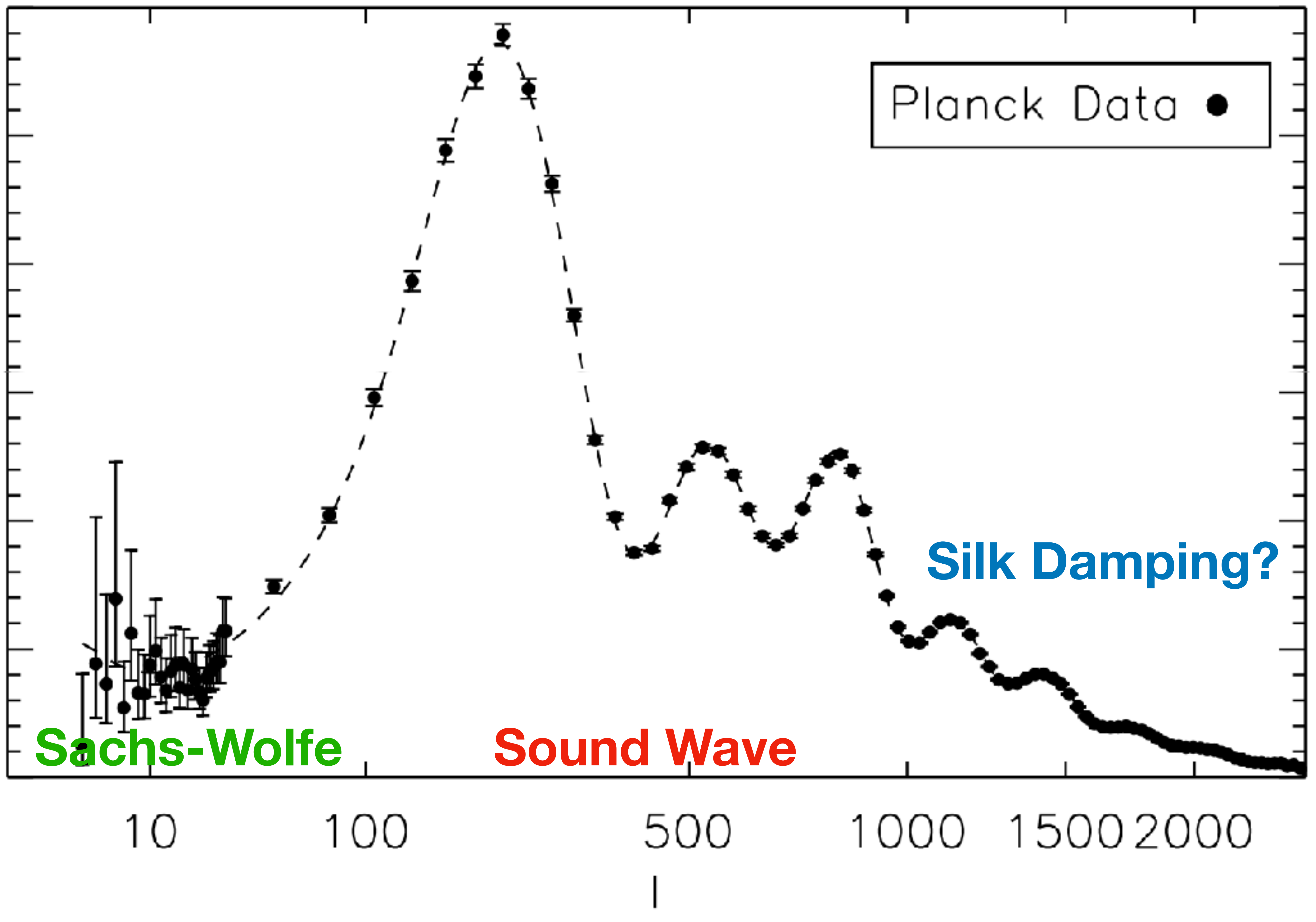
$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=[A\cos(qr_s)+B\sin(qr_s)]\exp(-q^2/q_{\rm Silk}^2)-R\Phi$$

Exponential dampling!

$$a/q_{\rm Silk}\approx(\sigma_T\bar{n}_eH)^{-1/2}$$

"diffusion length"
= length traveled by photon's random walks

Planck Collaboration (2016)

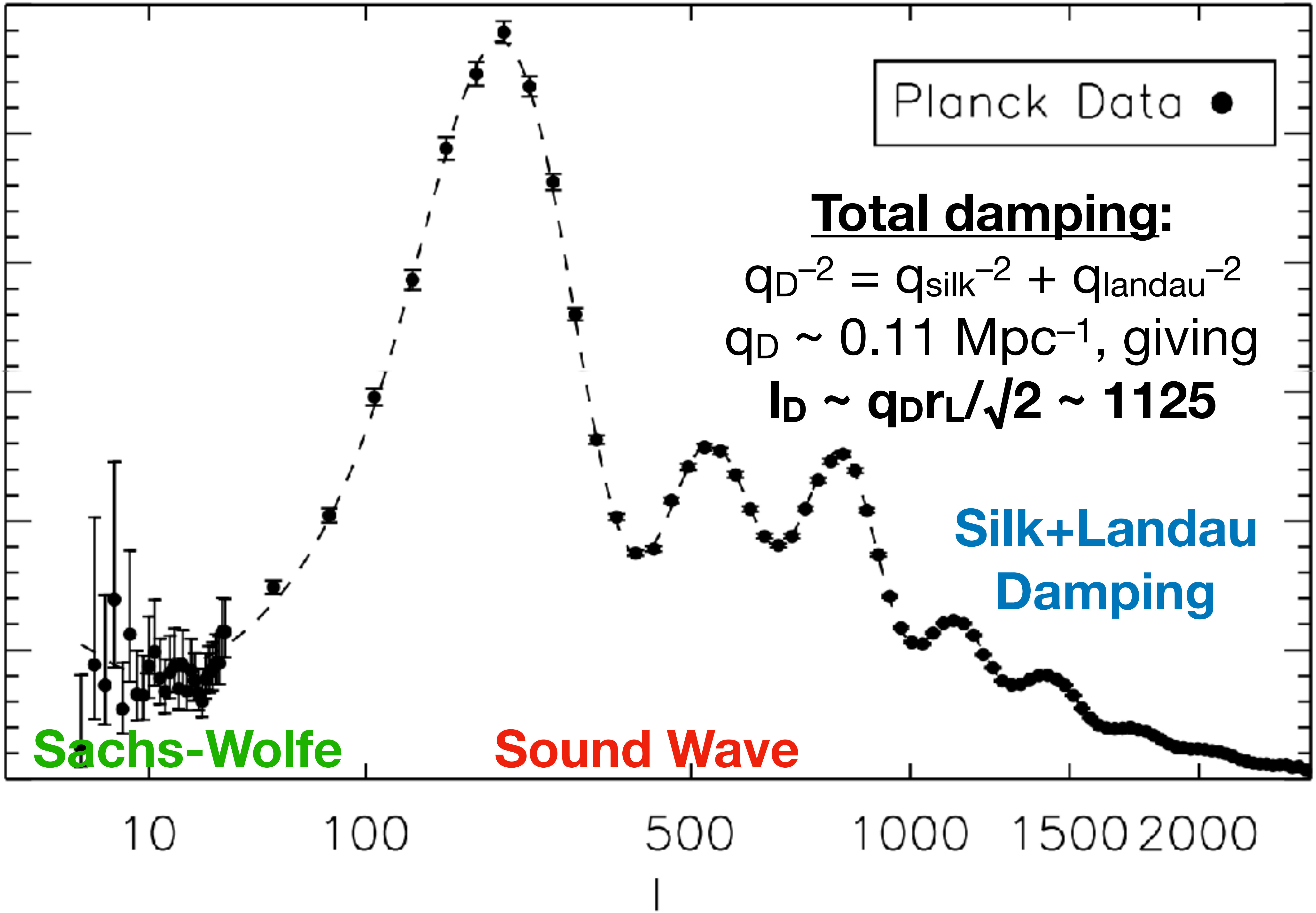


Additional Damping

- The power spectrum is $[\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi]^2$ with q -> l/r_L. The damping factor is thus $\exp(-\mathbf{2}q^2/q_{silk}^2)$
- $q_{silk}(t_L) = 0.139\ \mathrm{Mpc}^{-1}$. This corresponds to a multipole of **$l_{silk} \sim q_{silk} r_L/\sqrt{2} = 1370$.** Seems too large, compared to the exact calculation
- There is an additional damping due to a finite width of the last scattering surface, **σ~250 K**
  - "Fuzziness damping" – Bond (1996)
  - "Landau damping" - Weinberg (2001)

$$q_{\mathrm{Landau}}^{-2} = \frac{3\sigma^2 t_L^2}{8a_0^2 T_0^2 (1+R_L)} \approx \left(0.20\ \mathrm{Mpc}^{-1}\right)^{-2}$$

Planck Collaboration (2016)



Recap

- The basic structure of the temperature power spectrum is
  - The Sachs-Wolfe “plateau” at low multipoles
  - Sound waves at intermediate multipoles
    - 1st-order tight-coupling
  - Silk damping and Landau damping at high multipoles
    - 2nd-order tight-coupling

Planck Collaboration (2016)