

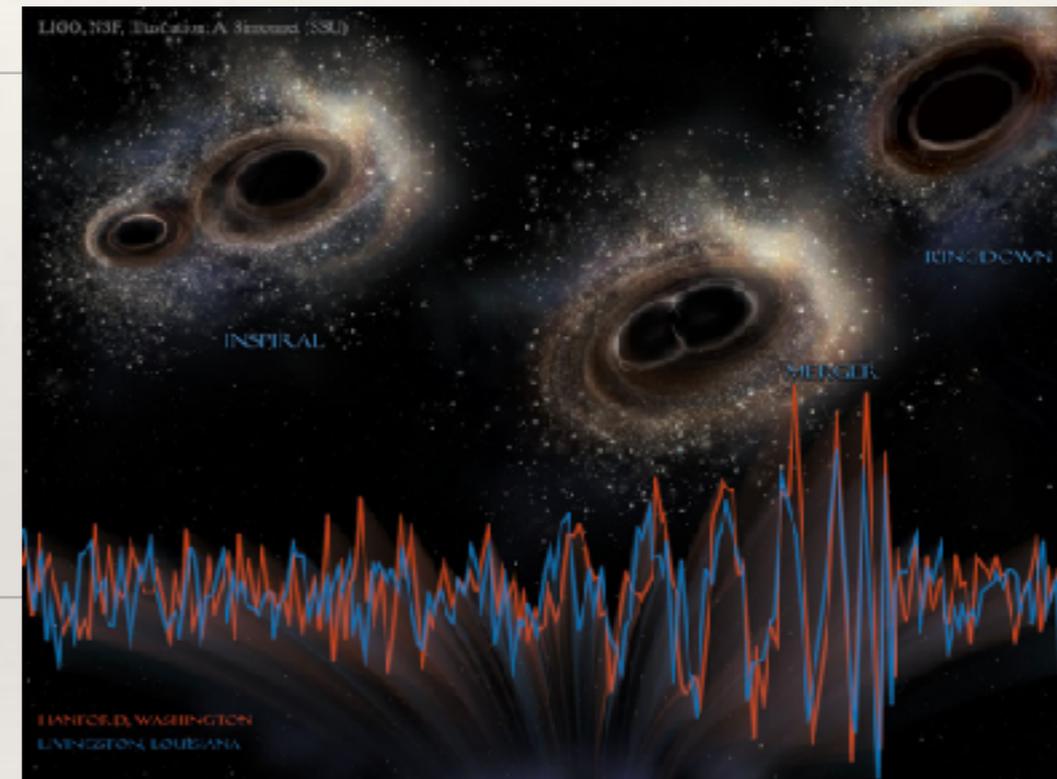
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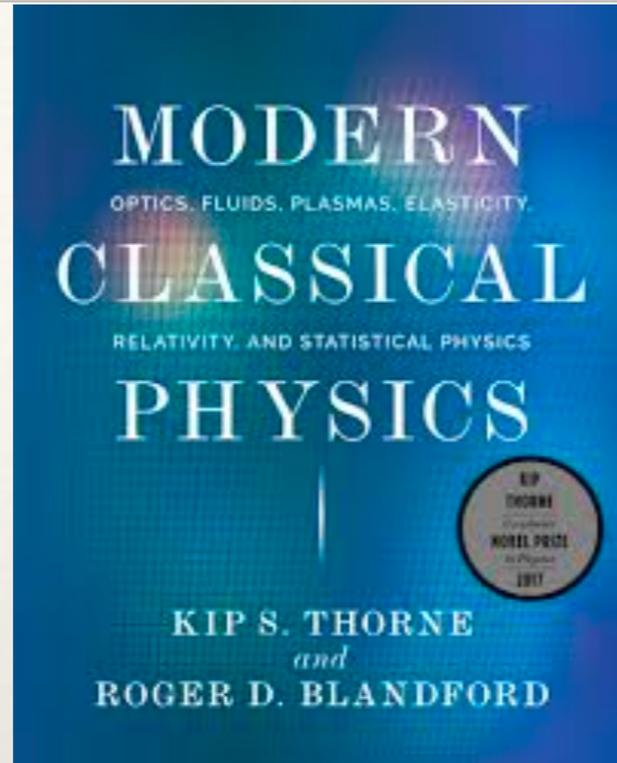
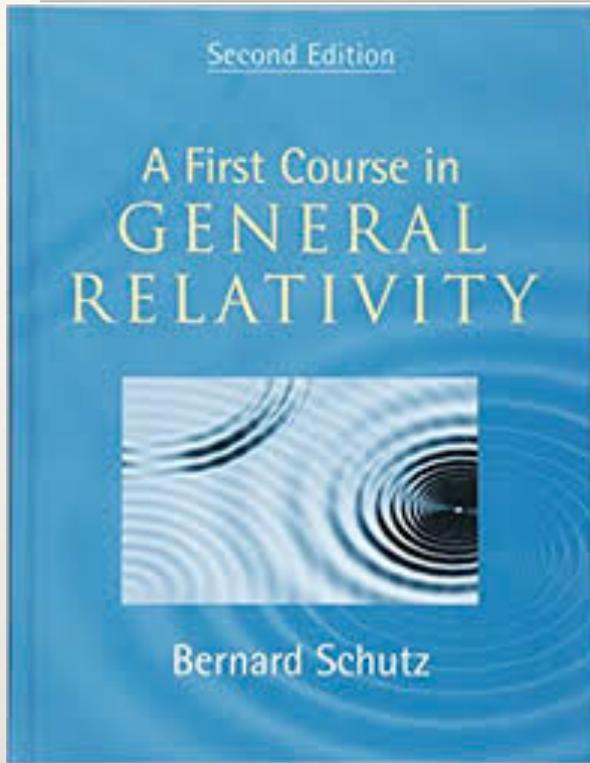
Gravitational waves

Part I



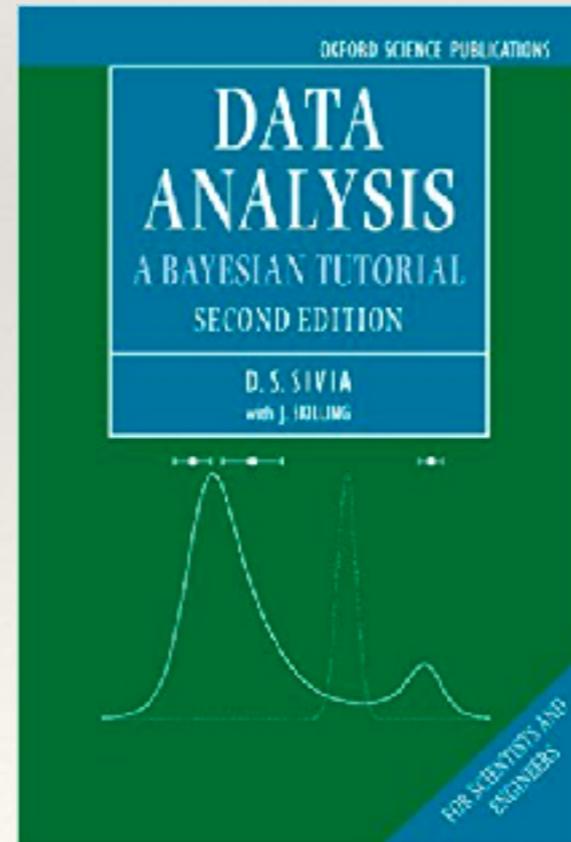
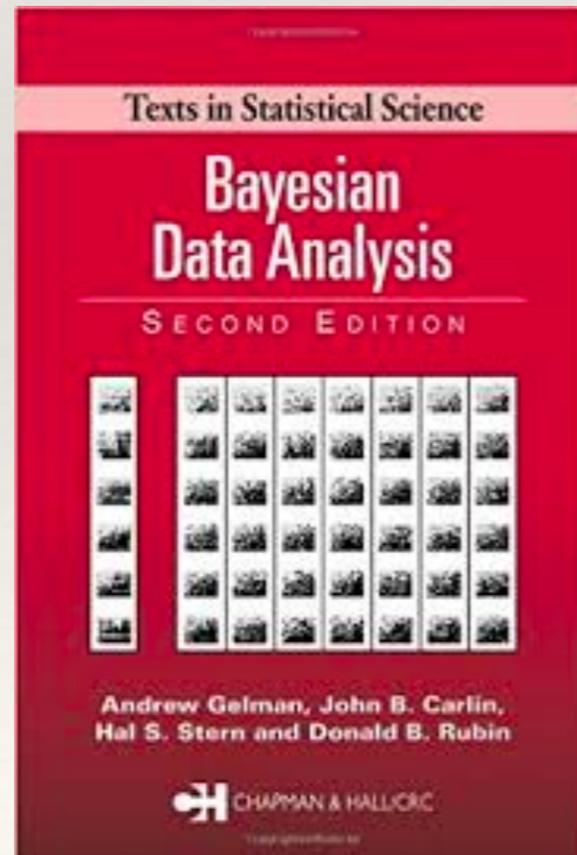
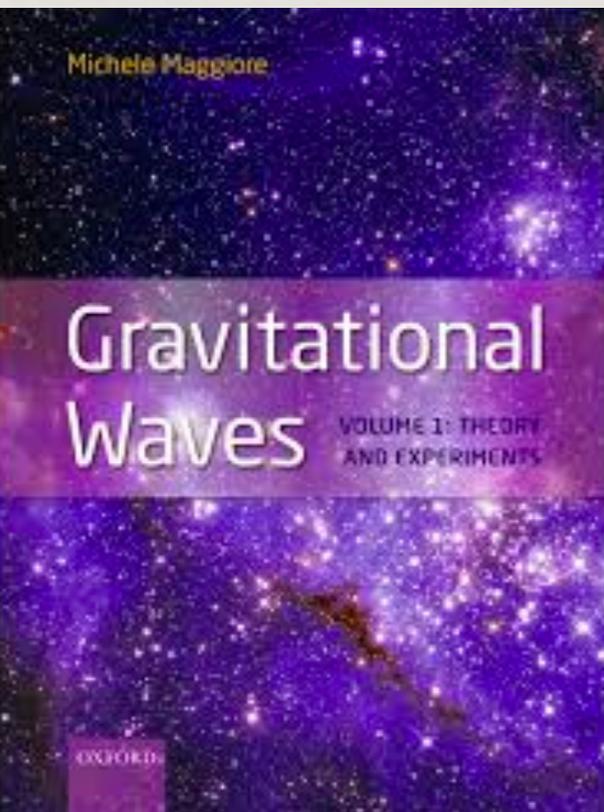
ICTP, 18-22 June 2018

Literature



Some figures in these lectures are borrowed from these books

LSC+Virgo, Phys. Rev. Lett. 116, 221101 (2016), LSC+Virgo, Phys. Rev. X 6, 041015 (2016), LSC+Virgo, Phys.Rev.Lett. 116 241102 (2016), LSC+Virgo, Phys.Rev. X6 041014 (2016), LSC+Virgo, Phys. Rev. Lett. 118, 221101 (2017), Berti et al., Class.Quantum Grav. 32, 243001 (2015), LSC+VIRGO, ArXiv: 1805.11579, S. Khan+, Phys.Rev. D93 (2016) 044007, LSC+Virgo [arXiv: 1805.11579](https://arxiv.org/abs/1805.11579) , LIGO_Virgo, Astrophys.J. 848 (2017) L12, LIGO+Virgo Phys.Rev.Lett. 119 (2017) 161101, Babak+ Phys.Rev. D95 (2017) 103012, LISA consortium [arXiv:1702.00786](https://arxiv.org/abs/1702.00786), Klein+ Phys.Rev. D93 (2016) 024003, LISA consortium, [arXiv:1305.5720](https://arxiv.org/abs/1305.5720), Amaro-Seoane+ Class.Quant.Grav. 29 (2012) 124016 .



Lecture 2

- 📍 Binary system
- 📍 Modelling GW signal from binary black holes
- 📍 Detecting GW signals with ground based observatories



Generation of GWs

Gravitational waves: can we attach stress energy tensor? Yes, but it is defined as a quantity averaged over several (GW) wavelengths.

$$T_{\alpha\beta}^{GW} = \frac{1}{16} \langle h_{+, \alpha} h_{+, \beta} + h_{\times, \alpha} h_{\times, \beta} \rangle$$

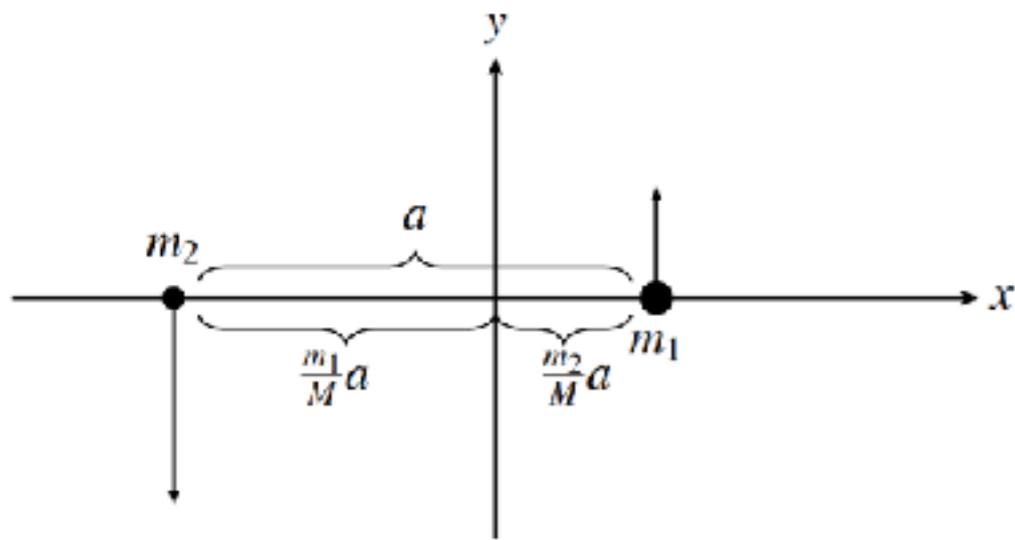
$$\frac{dE}{dt} = -\frac{1}{5} \langle \ddot{M}_{ij} \ddot{M}_{ij} \rangle \quad \text{Energy loss (energy flux): shrinking of binary orbit}$$

$$\frac{dS_j}{dt} = -\frac{2}{5} \epsilon_{jkl} \langle \ddot{M}_{ki} \ddot{M}_{li} \rangle \quad \text{Angular momentum loss: circularization of a binary}$$

Levi-Civita antisymmetric symbol



Binary system



Consider binary system on a circular orbit.

$$m_1 > m_2, \quad M = m_1 + m_2 \quad \mu = m_1 m_2 / M$$

Use Kepler's law $\omega = \sqrt{\frac{M}{a^3}}$

$$x_1 = \frac{m_2}{M} a \cos \omega t, \quad y_1 = \frac{m_2}{M} a \sin \omega t$$

$$x_2 = -\frac{m_1}{M} a \cos \omega t, \quad y_2 = -\frac{m_1}{M} a \sin \omega t$$

$$I_{jk} = \int T^{00} x_j x_k d^3 x = \int [\delta(\vec{x} - \vec{x}_1) m_1 + \delta(\vec{x} - \vec{x}_2) m_2] x_j x_k d^3 x = m_1 x_1^j x_1^k + m_2 x_2^j x_2^k$$

$$\ddot{I}_{xx} = -\ddot{I}_{yy} = -2\mu (M\omega)^{2/3} \cos 2\omega t,$$

$$\ddot{I}_{xy} = \ddot{I}_{yx} = -2\mu (M\omega)^{2/3} \sin 2\omega t,$$

polarization basis

$$\hat{e}_\theta = \hat{e}_x \cos \theta - \hat{e}_z \sin \theta, \quad \hat{e}_\phi = \hat{e}_y.$$

$$I_{\theta\theta} = I_{xx} \cos^2 \theta, \quad I_{\phi\phi} = I_{yy}, \quad I_{\theta\phi} = I_{xy} \cos \theta.$$

$$h_+ = h_{\theta\theta} = -2 (1 + \cos^2 \theta) \frac{\mu}{R} (M\omega)^{2/3} \cos [2\omega(t - R) - \phi_0]$$

$$h_\times = h_{\theta\phi} = -4 \cos \theta \frac{\mu}{R} (M\omega)^{2/3} \sin [2\omega(t - R) - \phi_0]$$



Binary system

Loss of energy due to GWs

$$L_{\text{GW}} = -\frac{dE^{\text{GW}}}{dt} = \frac{1}{5} \langle \ddot{M}_{ij} \ddot{M}_{ij} \rangle = \frac{32}{5} \eta^2 (M\omega)^{10/3}$$

Balance equation

$$\dot{E}^{\text{tot}} = \frac{m_1 m_2}{2a^2} \dot{a} = -\frac{32}{5} \eta^2 (M\omega)^{10/3}.$$

Total energy of the binary system

$$E^{\text{tot}} = \frac{m_1 m_2}{a} + \frac{m_1 (\omega r_1)^2}{2} + \frac{m_2 (\omega r_2)^2}{2} = -\frac{m_1 m_2}{2a}.$$

This equation can be easily integrated

$$a = \left[\frac{256}{5} \eta M^3 (t_c - t) \right]^{1/4},$$

$$\Delta t = \frac{5}{256 M^{5/3} \eta} (\pi f)^{8/3}$$

$$\pi f = \left[\frac{256}{5} \eta M^{5/3} (t_c - t) \right]^{-3/8}$$

$$\phi_{\text{orb}}^{(N)} = \int 2\pi f(t) dt = -2 \left[\frac{1}{5 M_c} (t_c - t) \right]^{5/8} + \phi_c$$

$$\eta = \frac{\mu}{M} = \frac{m_1 m_2}{M^2}$$



Binary system

$$a = \left[\frac{256}{5} \eta M^3 (t_c - t) \right]^{1/4}$$

○ The orbit shrinks as $t \rightarrow t_c$

$$\pi f = \left[\frac{256}{5} \eta M^{5/3} (t_c - t) \right]^{-3/8}$$

○ The frequency depend on the “chirp mass” $M_c = M\eta^{3/5}$

○ GW (and orbital) frequency grows with time and infinite at t_c (approach breaks down)

$$\phi_{orb}^{(N)} = \int 2\pi f(t) dt = -2 \left[\frac{1}{5M_c} (t_c - t) \right]^{5/8} + \phi_c$$

○ The phase depends on the chirp mass: M_c is the best measured parameter

$$\frac{df^{GW}}{dt} = \frac{96}{5\pi} M_c^{5/3} (\pi f)^{11/3}$$

○ Very strong dependance on the frequency: very slow evolution for the broad orbits

$$\text{○ If } m_1 \gg m_2, \quad M_c^{5/3} \approx \frac{m_2}{m_1} m_1^{5/3}$$

the frequency evolution could be slow even if the orbit is relativistic (extreme mass ratio inspiral EMRI)



Binary system

$$\Delta t = \frac{5}{256 M^{5/3} \eta} (\pi f)^{8/3} \quad \text{time to coalescence (merger) starting from freq. } f$$

LIGO/VIRGO: operates on the ground, freq range 30-2000 Hz.

take $f=40$ Hz, NS-NS system each mass 1.4 solar mass: $\Delta t \sim 20$ sec

take $f = 30$ Hz, BH-BH system each 30 solar mass: $\Delta t \sim 0.32$ sec

LISA (space based detector) will operate in freq. range 0.1- 100 mHz

take $f=0.1$ mHz, $M = m_1 + m_2 = 10^6 M_\odot$, $\Delta t \approx 35 \text{ days}/\eta < 1 \text{ year}$

- Post-Newtonian iterations: we plug back to Einstein equations the evolving orbit and linear solution for the GW, solve at the next order

$$\Phi = \Phi_0 + \Phi^N + \epsilon^2 \Phi^{1PN} + \epsilon^3 \Phi^{1.5PN} + \epsilon^4 \Phi^{2PN} + \dots$$

- In the near zone the curvature is comparable to the GW wavelength: scattering (backscattering) of the GWs on the Newtonian potential.



Binary system

Waveform (GW signal) in the frequency domain

GW, leading order in amplitude

$$h_+(t) = A_+ \cos \Phi(t), \quad h_\times = A_\times \sin \Phi(t)$$

Fourier transformation

$$\tilde{h}(f) = \int h(t) e^{-2\pi i f t} dt$$

If amplitude is slowly evolving,
monotoneous function of time

$$\tilde{h}_+(f) = (1 + \cos^2 \iota) \sqrt{\frac{5}{6}} \frac{1}{4\pi^{2/3}} \frac{M_c^{5/6}}{D_L} f^{-7/6} e^{i\Psi(f)},$$

$$\tilde{h}_\times(f) = 2i \cos \iota \sqrt{\frac{5}{6}} \frac{1}{4\pi^{2/3}} \frac{M_c^{5/6}}{D_L} f^{-7/6} e^{i\Psi(f)}$$

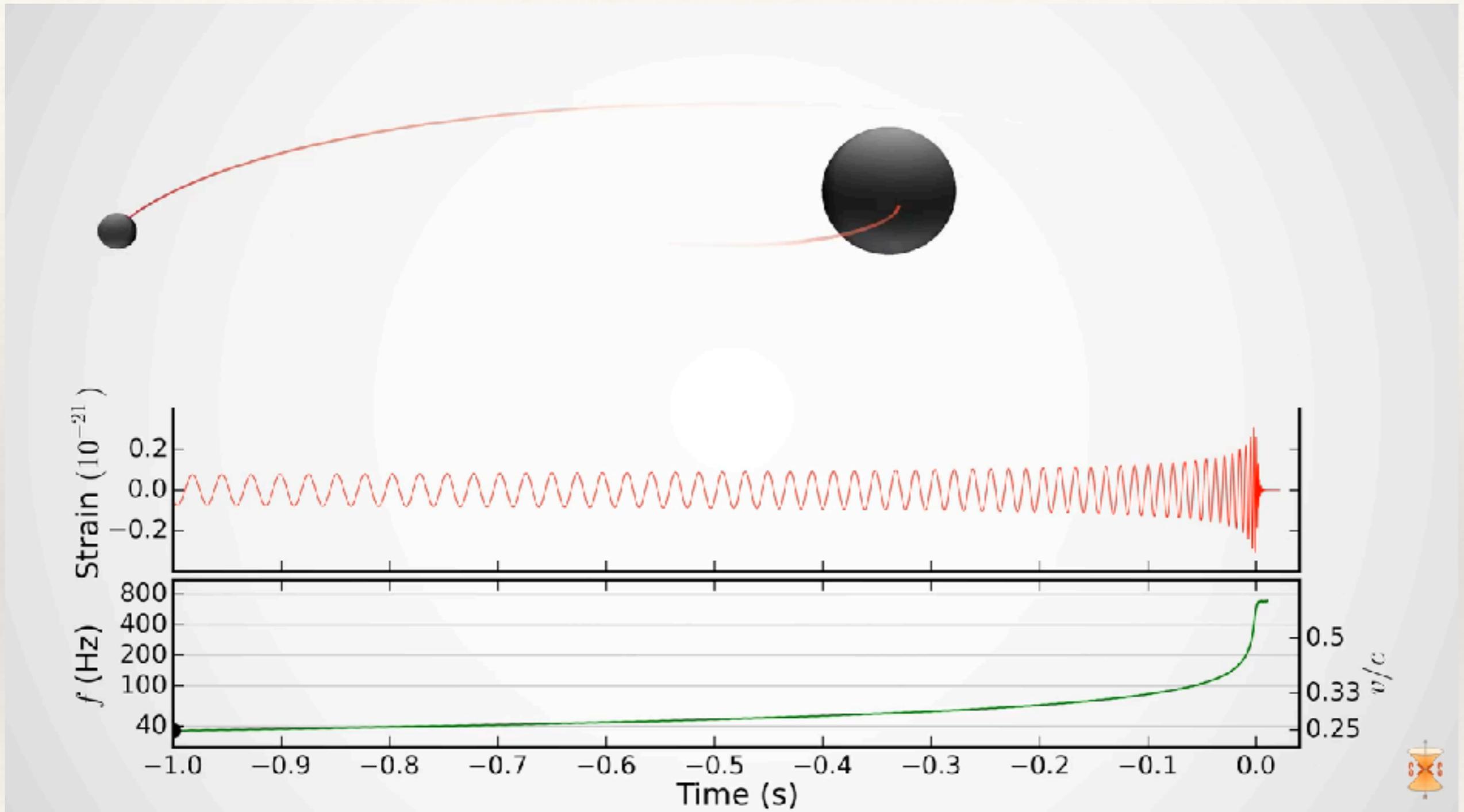
The phase in freq. domain

$$\Psi(f) = 2\pi f t_c - \phi_0 - \frac{\pi}{4} + \frac{3}{4} (8\pi M_c f)^{-5/3} + \dots (M f)^{-5/3} + \dots (M f)^{-1} + \dots (M f)^{-2/3}$$

- Amplitude and the dominant term in phase depend on M_c only (other terms depend on total mass and mass ratio).
- Amplitude is higher at low frequencies (early inspiral, slow frequency evolution, many cycles).



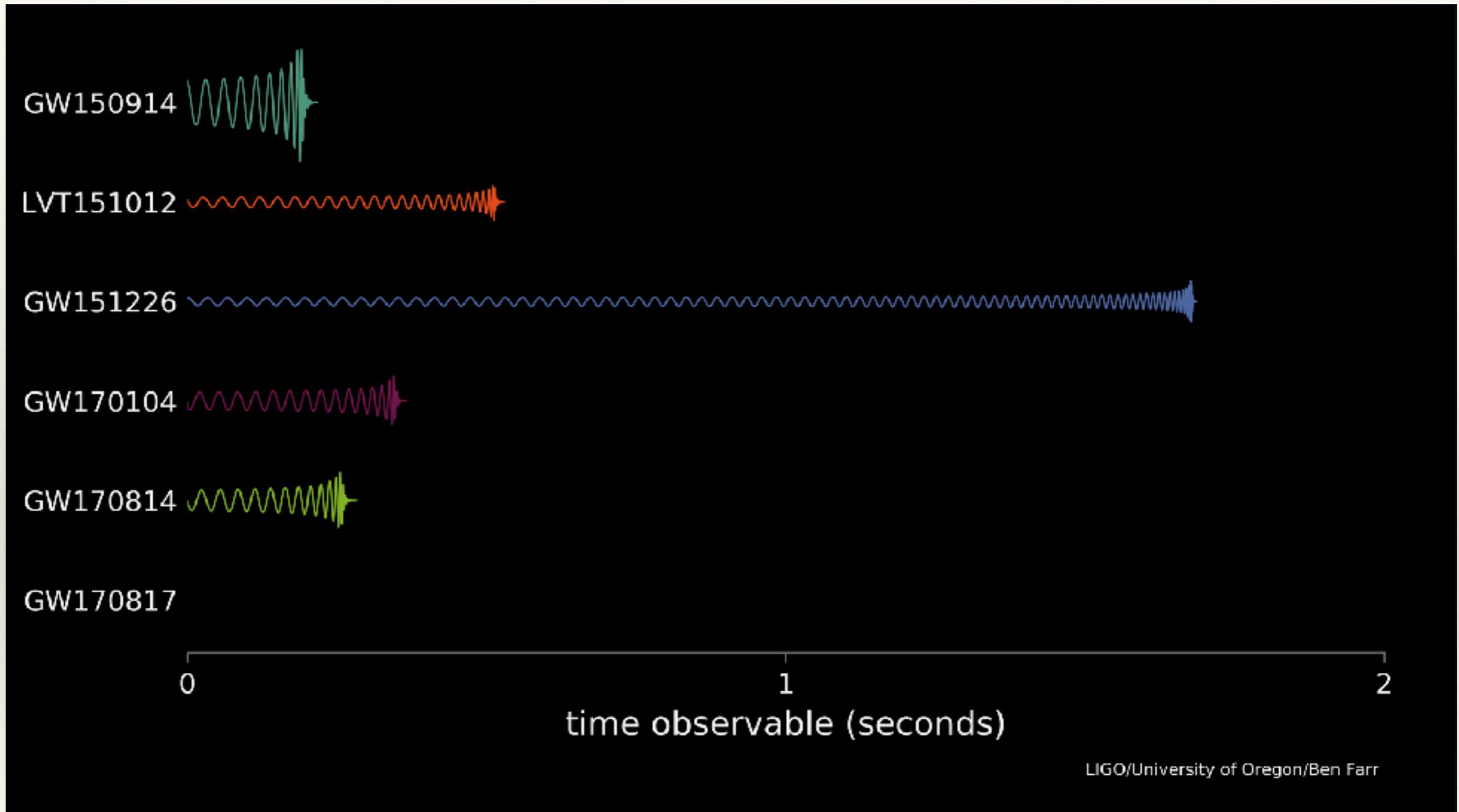
GW signal



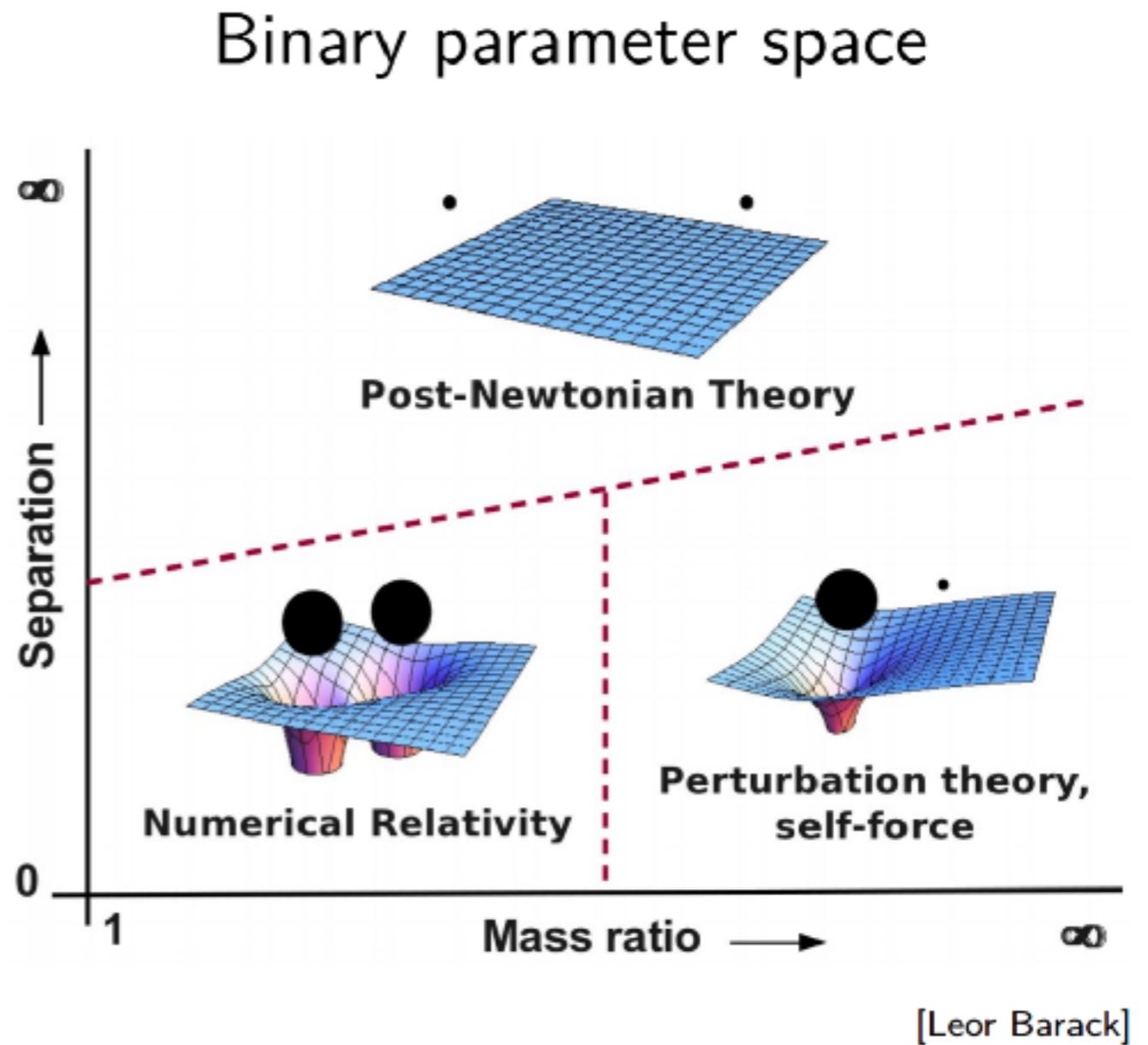
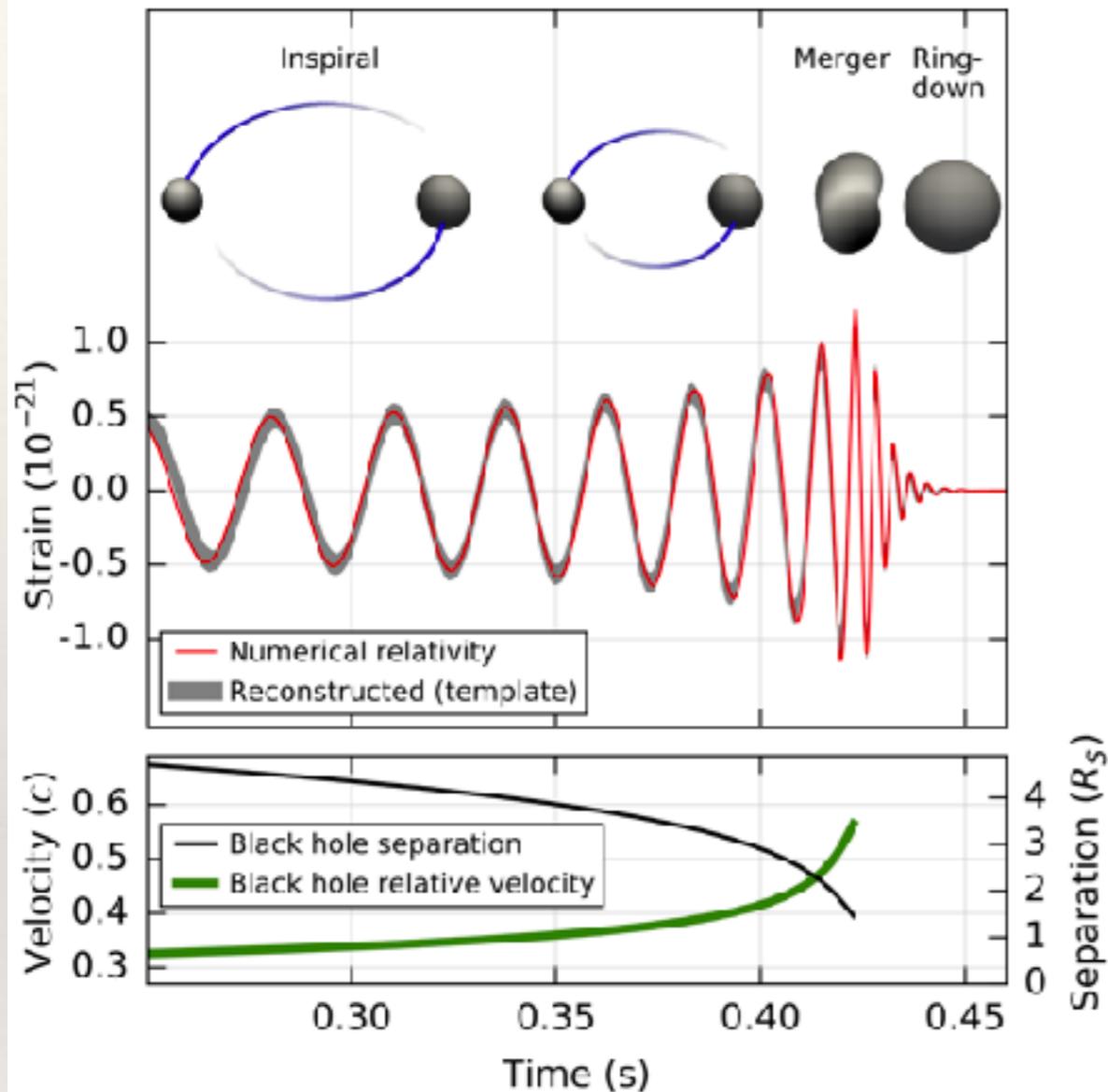
[Credits: SXS collaboration]



Detected GW signals from binaries



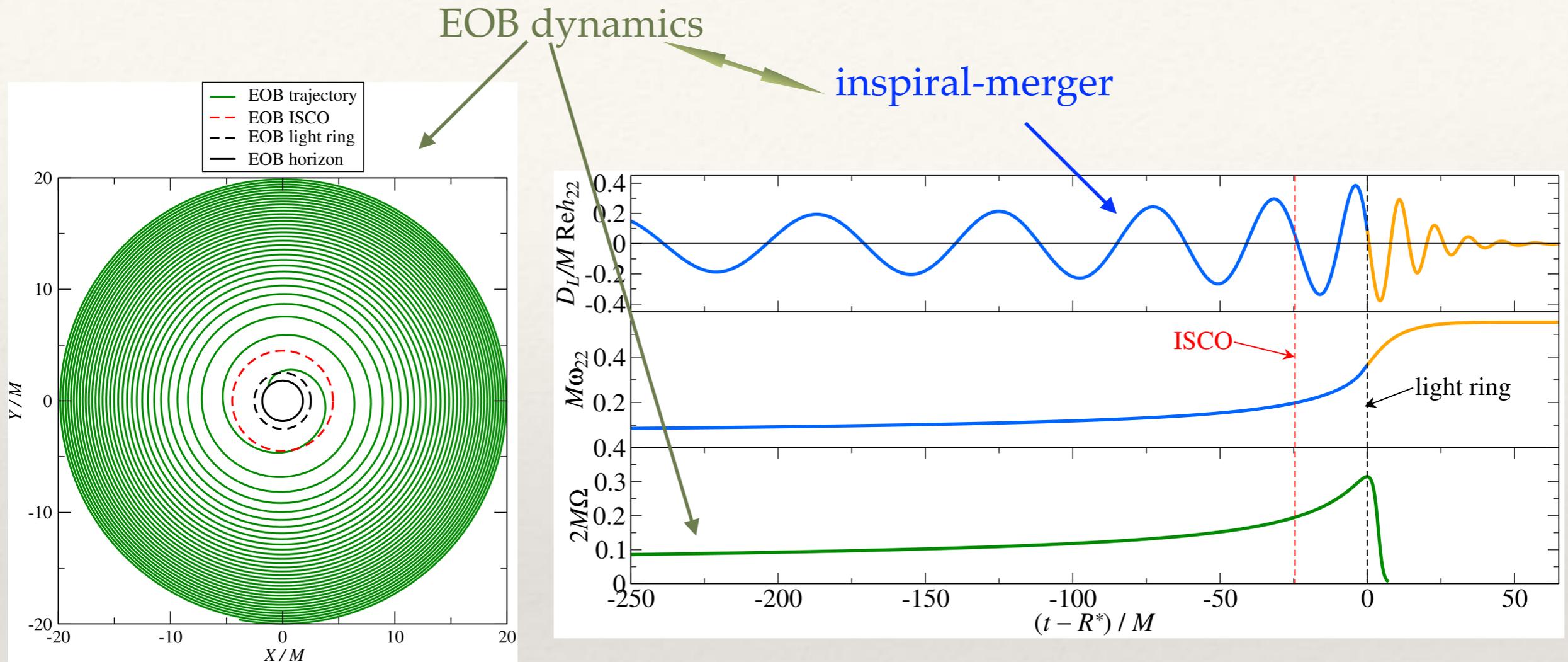
Modelling GW signal



GW signal from binary system can be conventionally split into three parts: inspiral (Post-Newtonian decomposition), merger (numerical relativity), ringdown (BH perturbation)



Modelling GW signal



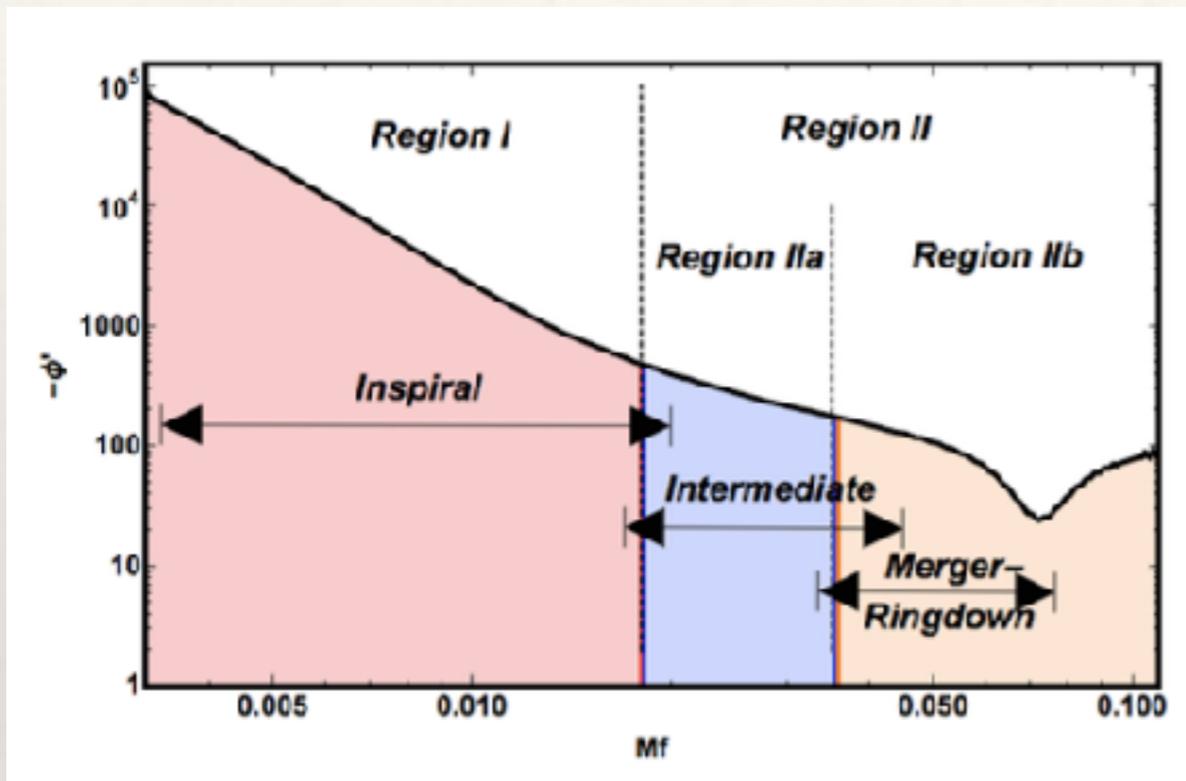
[Plot: courtesy of A. Taracchini]

The inspiral part of the signal could be pushed to smaller separations using smart resummation (Effective One Body approach) and then stitched to the ringdown



Modelling GW signal

Phenomenological model: The signal is not that complicated if separated in amplitude and phase: Can construct analytic fit to the numerical data + using analytic solution in limiting parts



$$\begin{aligned} \phi_{\text{Ins}} &= \phi_{\text{TF2}}(Mf; \Xi) \\ &+ \frac{1}{\eta} \left(\sigma_0 + \sigma_1 f + \frac{3}{4} \sigma_2 f^{4/3} + \frac{3}{5} \sigma_3 f^{5/3} + \frac{1}{2} \sigma_4 f^2 \right) \\ \phi_{\text{Int}} &= \frac{1}{\eta} \left(\beta_0 + \beta_1 f + \beta_2 \text{Log}(f) - \frac{\beta_3}{3} f^{-3} \right) \\ \phi_{\text{MR}} &= \frac{1}{\eta} \left\{ \alpha_0 + \alpha_1 f - \alpha_2 f^{-1} + \frac{4}{3} \alpha_3 f^{3/4} \right. \\ &\left. + \alpha_4 \tan^{-1} \left(\frac{f - \alpha_5 f_{\text{RD}}}{f_{\text{damp}}} \right) \right\}. \end{aligned}$$

[Khan+, PRD 2016]

- Waveform constructed in the frequency domain
- Uses Post-Newtonian results for the early evolution (inspiral) of a binary
- Precession is added by rotation taken from the Post-Newtonian evolution
- Very fast to generate

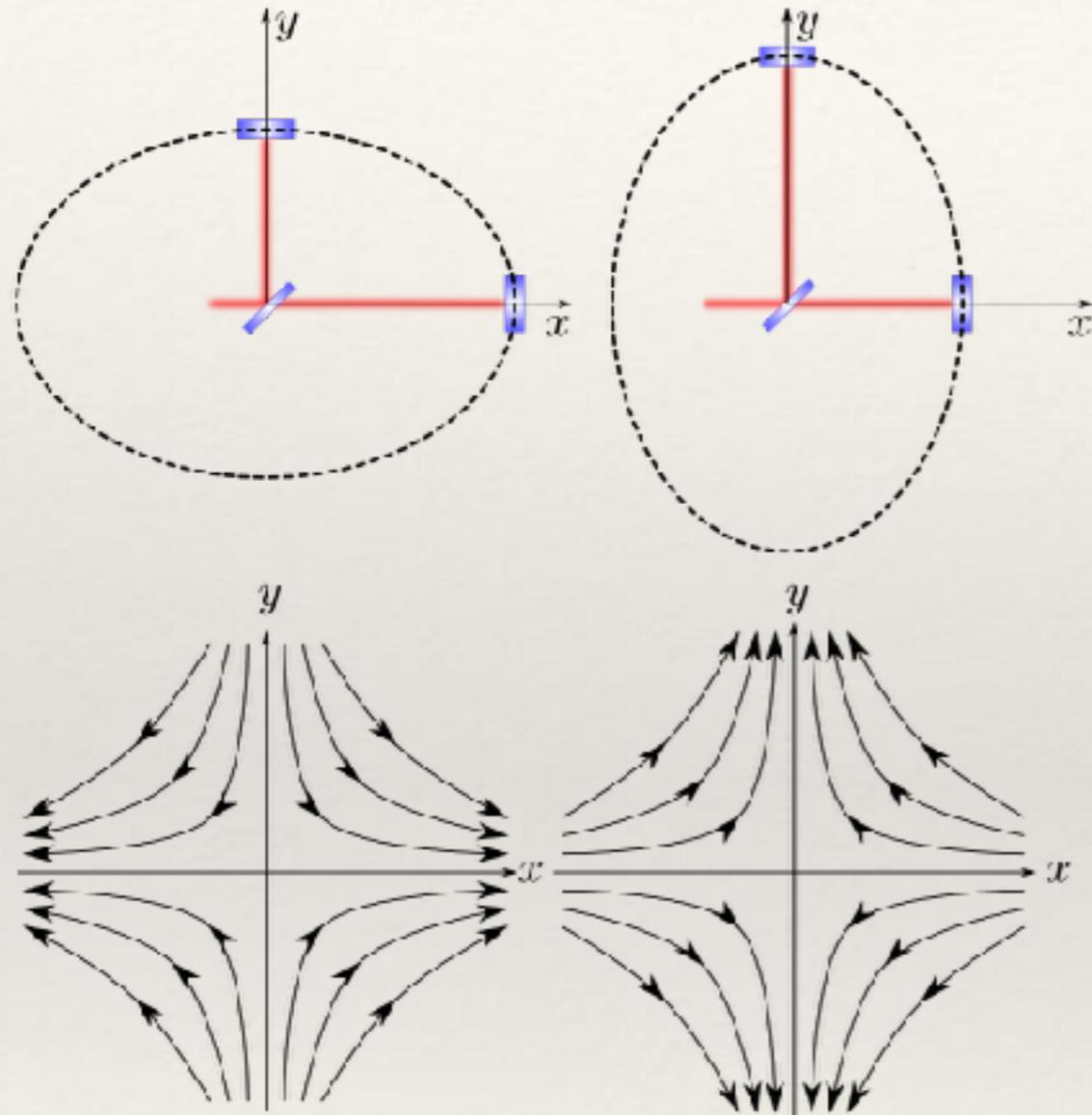


Detection of GWs

a) h_+ -polarized GW

$$h_+ > 0, t = \frac{1}{4}T_{\text{GW}}$$

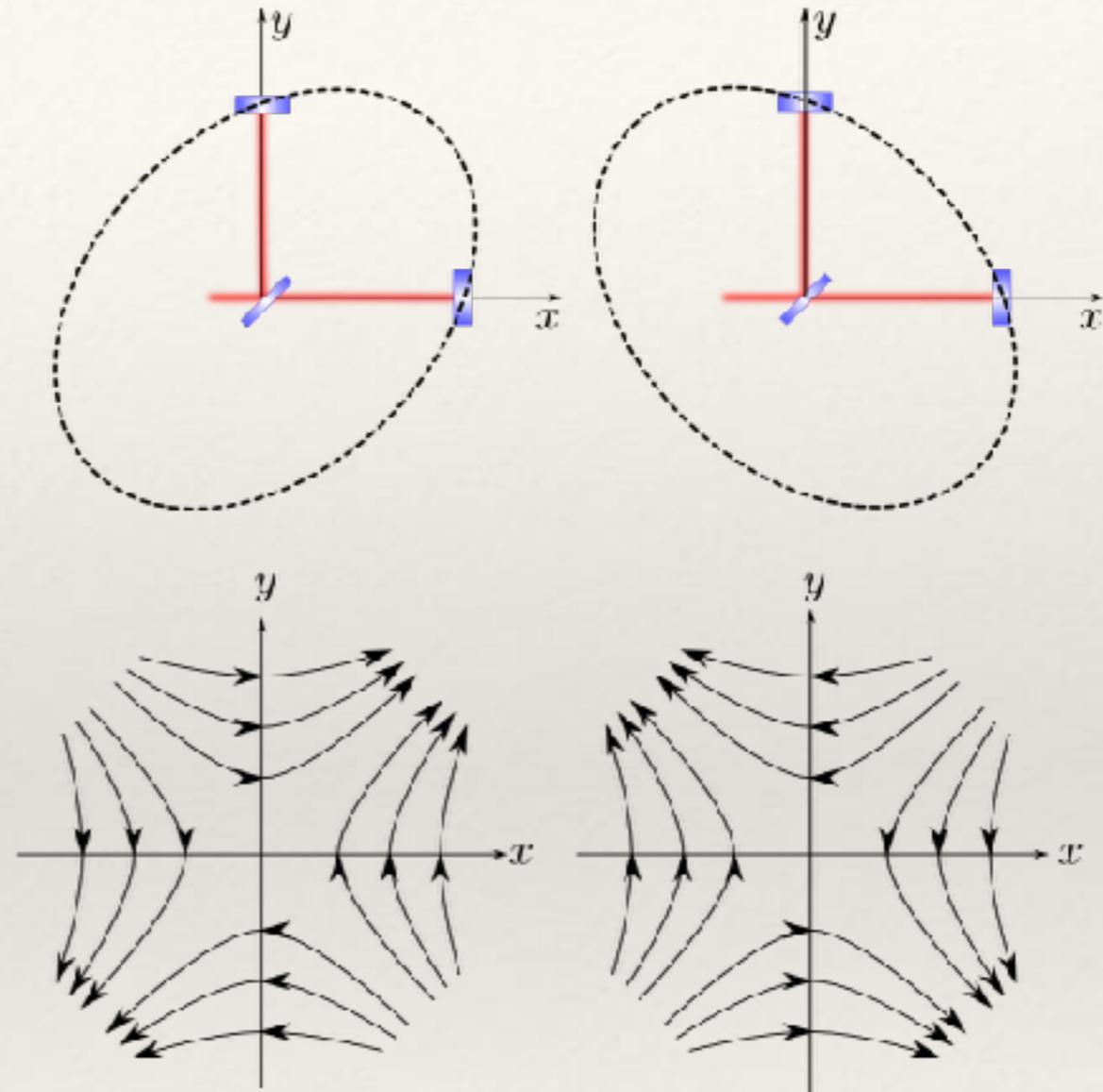
$$h_+ < 0, t = \frac{3}{4}T_{\text{GW}}$$



b) h_\times -polarized GW

$$h_\times > 0, t = \frac{1}{4}T_{\text{GW}}$$

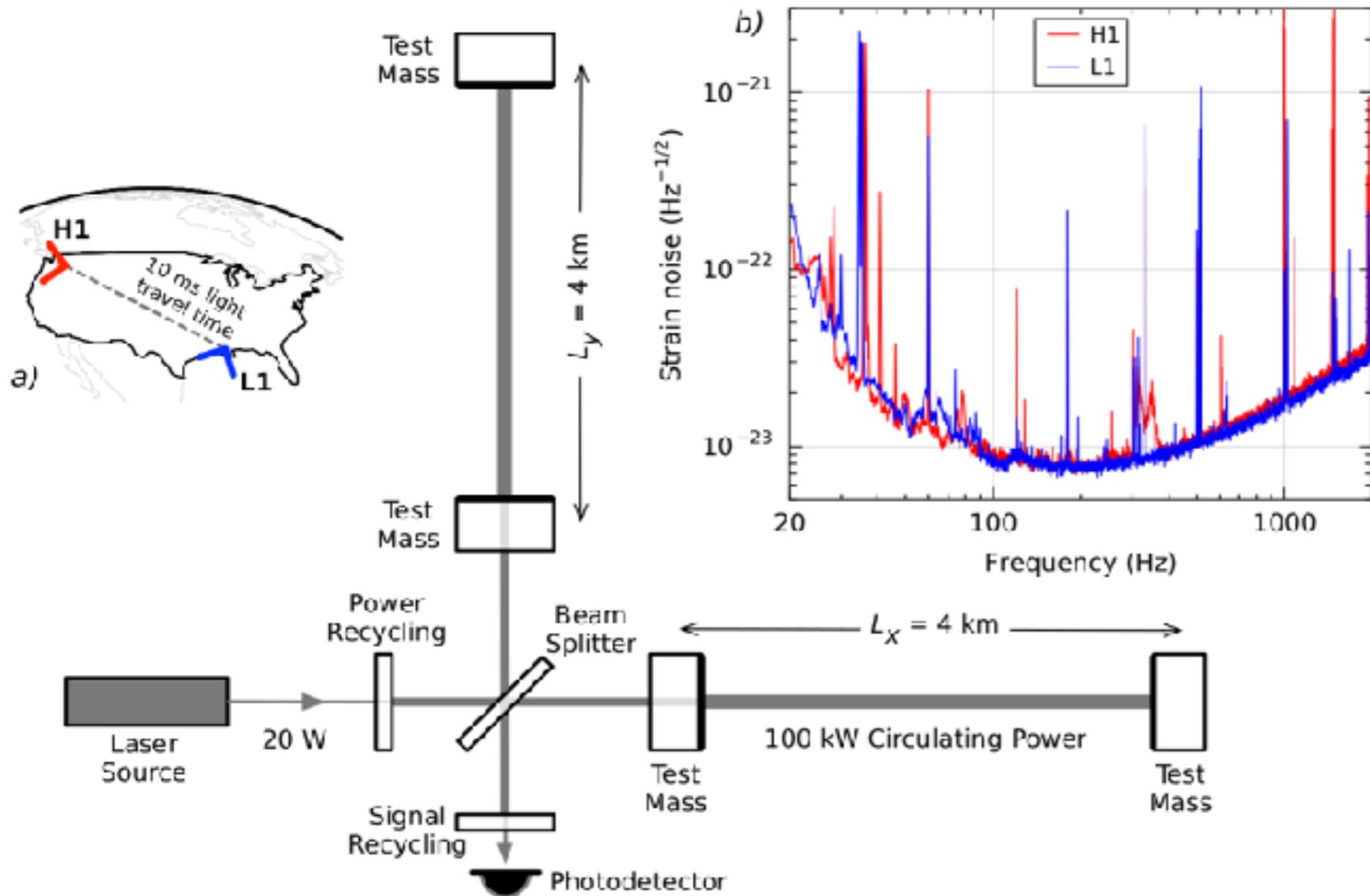
$$h_\times < 0, t = \frac{3}{4}T_{\text{GW}}$$



- The basic principle used in detecting GWs is the geodesic deviation equation in the time dependent field of GWs



Detection of GWs



We use local inertial frame located at the beam splitter

In case of LIGO/VIRGO

$$\lambda^{GW} \gg L, \quad \omega_{GW} L \ll 1$$

The whole detector can be covered (with high accuracy) by the LIF

In this frame (LIF of the beam splitter) the coordinate distance is the proper distance:

$$\hat{x} = L_x(t) = \left(1 + \frac{1}{2}h_+(t)\right) L_x(t_0)$$

similar for x polarization

$$\hat{y} = L_y(t) = \left(1 - \frac{1}{2}h_+(t)\right) L_y(t_0)$$



Detection of GWs

Since we have covered the whole detector by LIF, the propagation of the laser light as it was in the flat spacetime

$$\phi_x^l = -\nu_l(\hat{t} - \hat{x}) + O(\nu_l L_0 h(\omega L_0)^2) \quad \text{phase of the laser}$$

↙
Frequency of the laser

The phase shift between round trip along x and y

$$\Delta\phi^l = \phi_x^l - \phi_y^l = -2\nu_l(L_y - L_x) = 2\nu_l(L_x^{(0)} - L_y^{(0)} + L^{(0)} h_+(t))$$

- For the ground based detectors we can choose the LIF which covers the whole detector, so that the spacetime is flat with high accuracy.
- In this frame the laser light propagation is as in flat s/t, the mirrors are moving under time varying tidal forces of GWs → change in the optical path.

$$\lambda^{GW} \gg L, \quad \omega_{GW} L \ll 1$$



Detection of GWs

$$\lambda^{GW} \approx L, \quad \omega_{GW} L \approx 1$$

We cannot set the LIF covering the whole detector, but we can introduce LIF for the background curvature. This is the case for LISA and PTA. We use TT-gauge:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{TT} + O(h^2)$$

Consider the wave propagating in z-direction

$$ds^2 = dt^2 + [1 + h_+(t - z)]dx^2 + [1 - h_+(t - z)]dy^2 + dz^2$$

Mirrors are moving along geodesics. In TT frame the coordinate distance does not change (but the proper distance does) $d\hat{x}^2 = g_{xx}dx^2$

Need to consider equation of propagation of e/m signal $g^{\alpha\beta} \phi_{,\alpha}^l \phi_{,\beta}^l = 0$

The phase difference for the round trip is

$$\Delta\phi_l = \phi_l|_x - \phi_l|_y = -\nu_l \left[-2(L_x^{(0)} - L_y^{(0)}) + \frac{1}{2}H(t - 2L_x^{(0)}) + \frac{1}{2}H(t - 2L_y^{(0)}) - H(t) \right]$$

$$H(t) = \int_0^t h(t') dt'$$

$$\omega_{GW} L \ll 1 \longrightarrow \Delta\phi^l \approx 2\nu_l (L_x^{(0)} - L_y^{(0)} + L^{(0)} h_+(t))$$



LIGO/VIRGO

Consider the long wavelength approximation: suitable for LIGO/VIRGO. The response of a single detector:

$$h(t) = \frac{1}{2} (\hat{n}_i^{(1)} h^{ij} \hat{n}_j^{(1)} - \hat{n}_i^{(2)} h^{ij} \hat{n}_j^{(2)})$$

strain: differential change in the arm length

unit vectors along arms

We can decompose the GW signal in the polarization basis

$$h^{ij} = \epsilon_+^{ij} h_+ + \epsilon_\times^{ij} h_\times \longrightarrow h(t) = F_+ h_+ + F_\times h_\times$$

$$F_+ = \frac{1}{2} (\hat{n}_i^{(1)} \epsilon_+^{ij} \hat{n}_j^{(1)} - \hat{n}_i^{(2)} \epsilon_+^{ij} \hat{n}_j^{(2)})$$

similar for x polarization

F_+, F_\times antenna beam function, it depends on the sky positions of the source and polarization angle

$$F_+ = \frac{1}{2} \cos(2\psi) (1 + \cos^2 \theta_s) \cos(2\phi_s) - \sin(2\psi) \cos \theta_s \sin(2\phi_s),$$

polarization angle

$$F_\times = \frac{1}{2} \sin(2\psi) (1 + \cos^2 \theta_s) \cos(2\phi_s) + \cos(2\psi) \cos \theta_s \sin(2\phi_s)$$

sky position of GW source



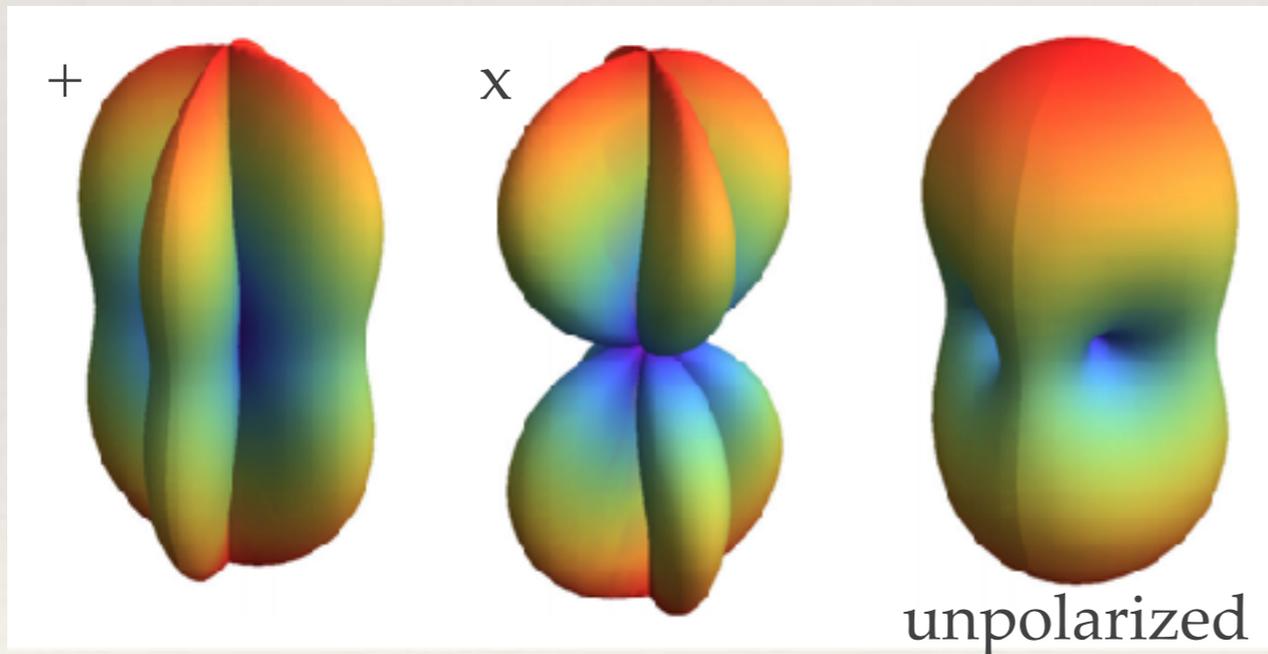
LIGO/VIRGO

Substitute the GW signal from a binary system we get for the strain:

$$h = 2\mu \frac{A_I}{D_L} (M\omega_{orb})^{2/3} \cos(2\phi_{orb}(t) + \phi_I)$$

↙
detector index

$$A = \sqrt{(1 + \cos^2 \iota)^2 F_+^2 + 4 \cos^2 \iota F_\times^2}$$



If the GW signal is short (in time), $A \sim \text{const}$:
degeneracy in the parameter space.

$$\frac{A}{D_L} = \frac{1}{D_{eff}} \quad \text{Only effective distance can be measured.}$$

Sky position of a short signal (transient) can be measured only by a network of detectors.

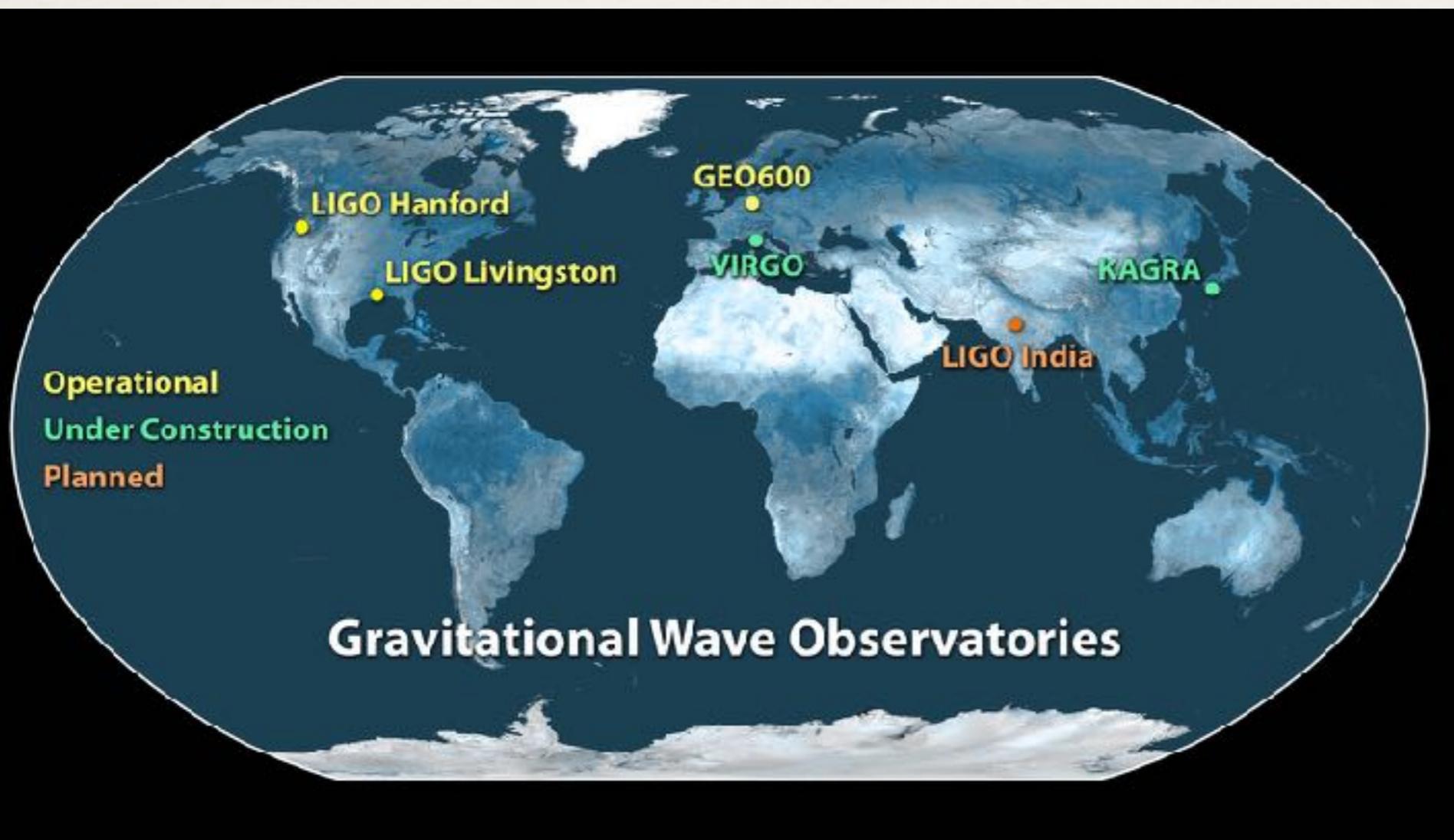


LIGO/VIRGO

Triangulation: measuring the time of arrival of the signal at each interferometer - depends on the sky position

- 2 detectors - circle in the sky
- 3 detectors - two points in the sky

Amplitude and phase consistency - helps!

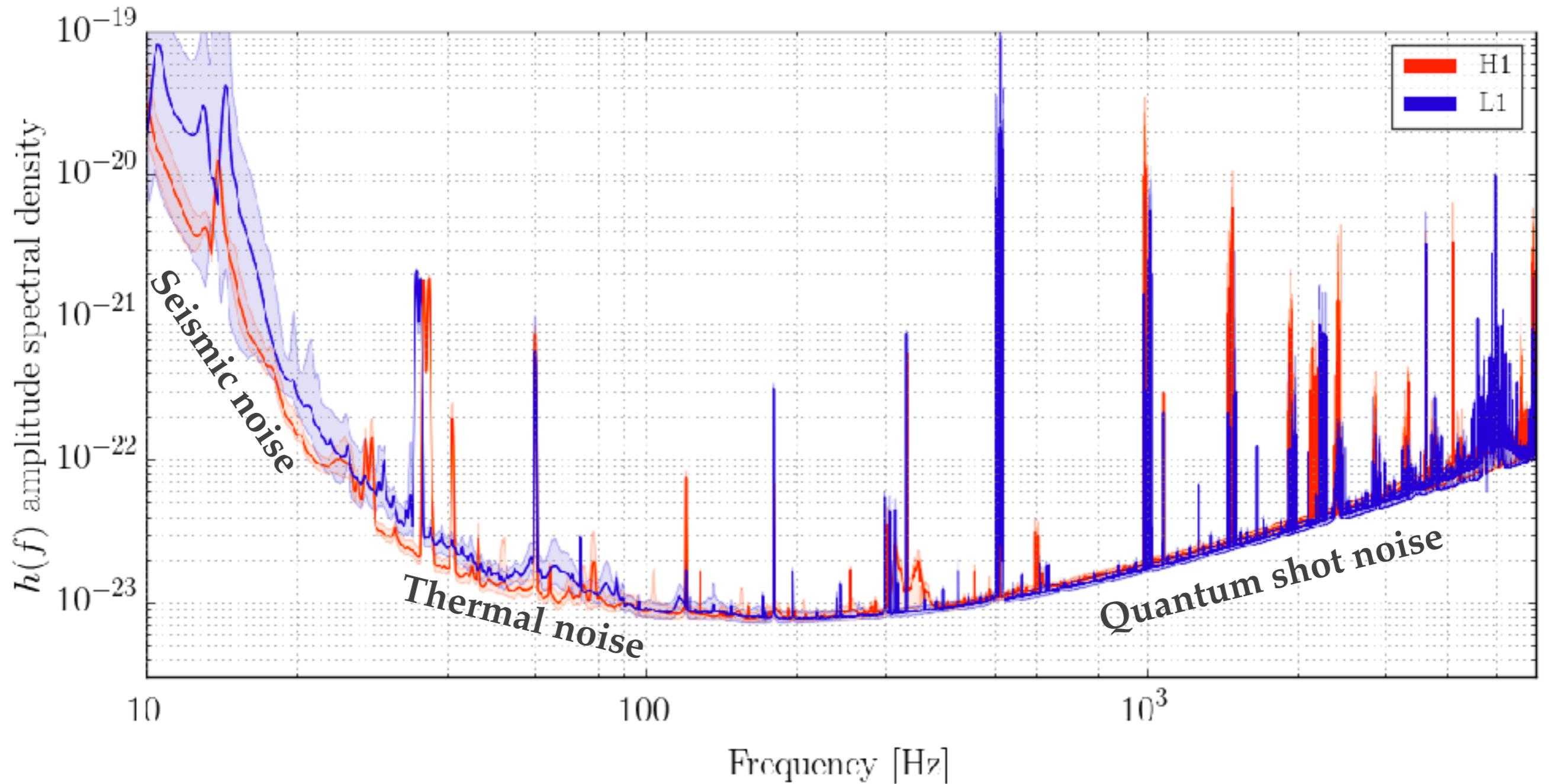


Global network of GW observatories.



LIGO/VIRGO

Sensitivity



LIGO/VIRGO

$$\Delta L = \frac{1}{2} h^{GW} L$$

$$h \approx 10^{-22}, \rightarrow L = 4\text{km}, \rightarrow \Delta L \approx 10^{-16}\text{cm}$$

