The Spherical Evolution model

Beyond linear theory:

Spherical collapse of ‘halos’

Spherical expansion of ‘voids’

Recall linear theory:

- When radiation dominated ($H = 1/2t$):

$$(d^2\delta/dt^2) + 2H\,(d\delta/dt) = (d^2\delta/dt^2) + (d\delta/dt)/t = 0$$

$$\delta(t) = C_1 + C_2 \ln(t) \quad \text{(weak growth)}$$

- In distant future (H = constant):

$$(d^2\delta/dt^2) + 2H_\Lambda(d\delta/dt) = 0$$

$$\delta(t) = C_1 + C_2 \exp(-2H_\Lambda t)$$

- If flat matter dominated ($H = 2/3t$):

$$\delta(t) = \delta_+ t^{2/3} + \delta_- t^{-1} \propto a(t) \quad \text{at late times}$$

- Linear growth just multiplicative factor, so if initial conditions Gaussian, linearly evolved field is too, with

$$P(k,t) = D^2(t)/D^2(t_{init})\, P(k,t_{init})$$

Growing mode in ΛCDM

$\delta(x,t) = D(t)\, \delta_{init}(x)$

Conventional to express as function of z:

$D(z) = (5/2)\, \Omega_0 H_0^2\, H(z) \int_z^\infty dz'\, (1+z')/H^3(z')$

$$g(z) \equiv (1+z)\, D(z) \approx \frac{5\Omega(z)}{2} \frac{1}{\Omega^{4/7}(z) - \Omega_\Lambda(z) + [1+\Omega_m(z)/2][1+\Omega_\Lambda(z)/70]}$$

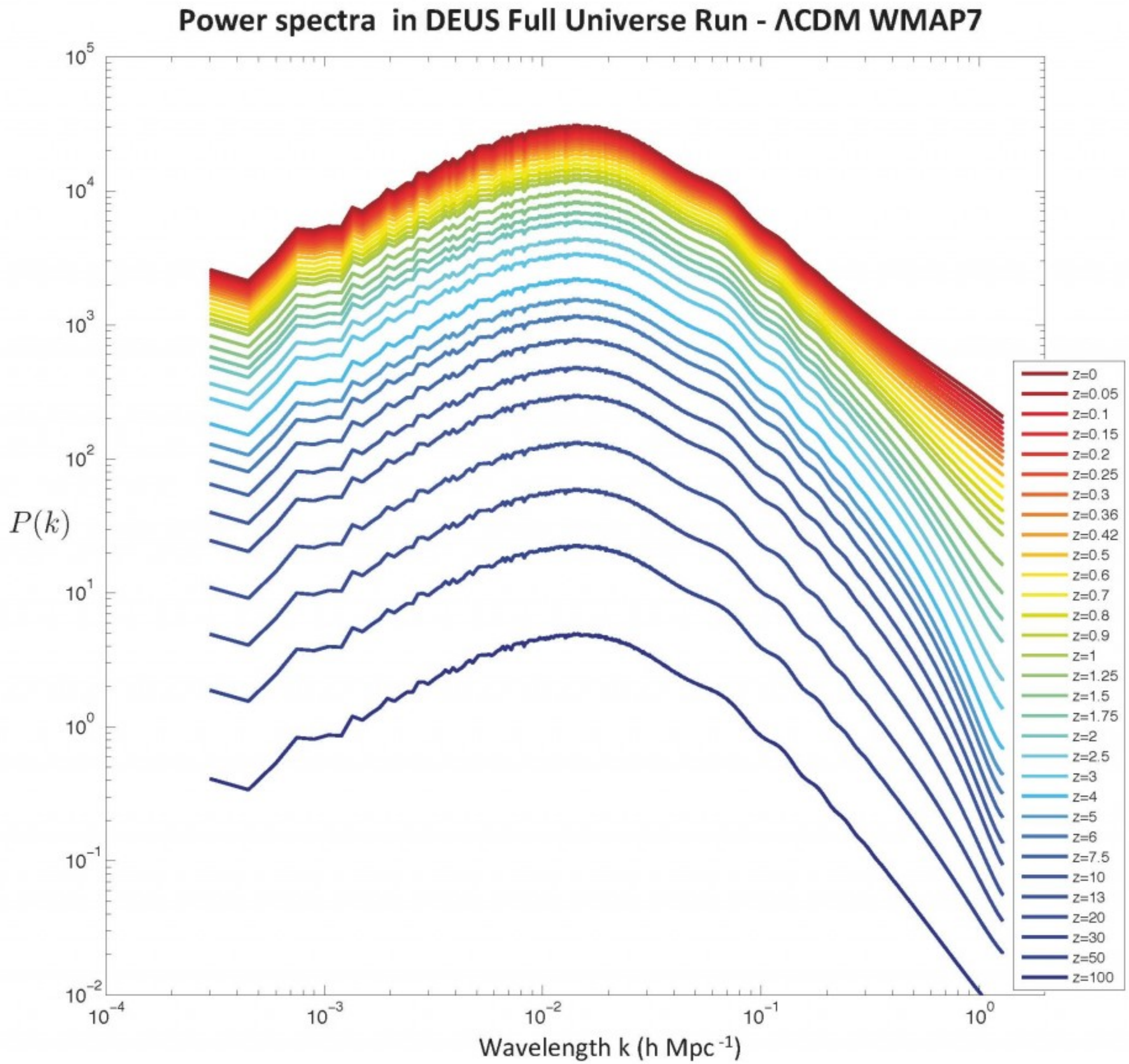
This is normalized so that D(z) → 1/(1+z) at high z, since early universe is matter dominated (EdS). Growth slows down as Λ dominates.

At small k, only amplitude grows

At high k, shape also changes

Bumps and wiggles = cosmic variance

Ratio of Pk not sensitive to cosmic variance



Power spectra in DEUS Full Universe Run: Non-linear imprint - ΛCDM WMAP7

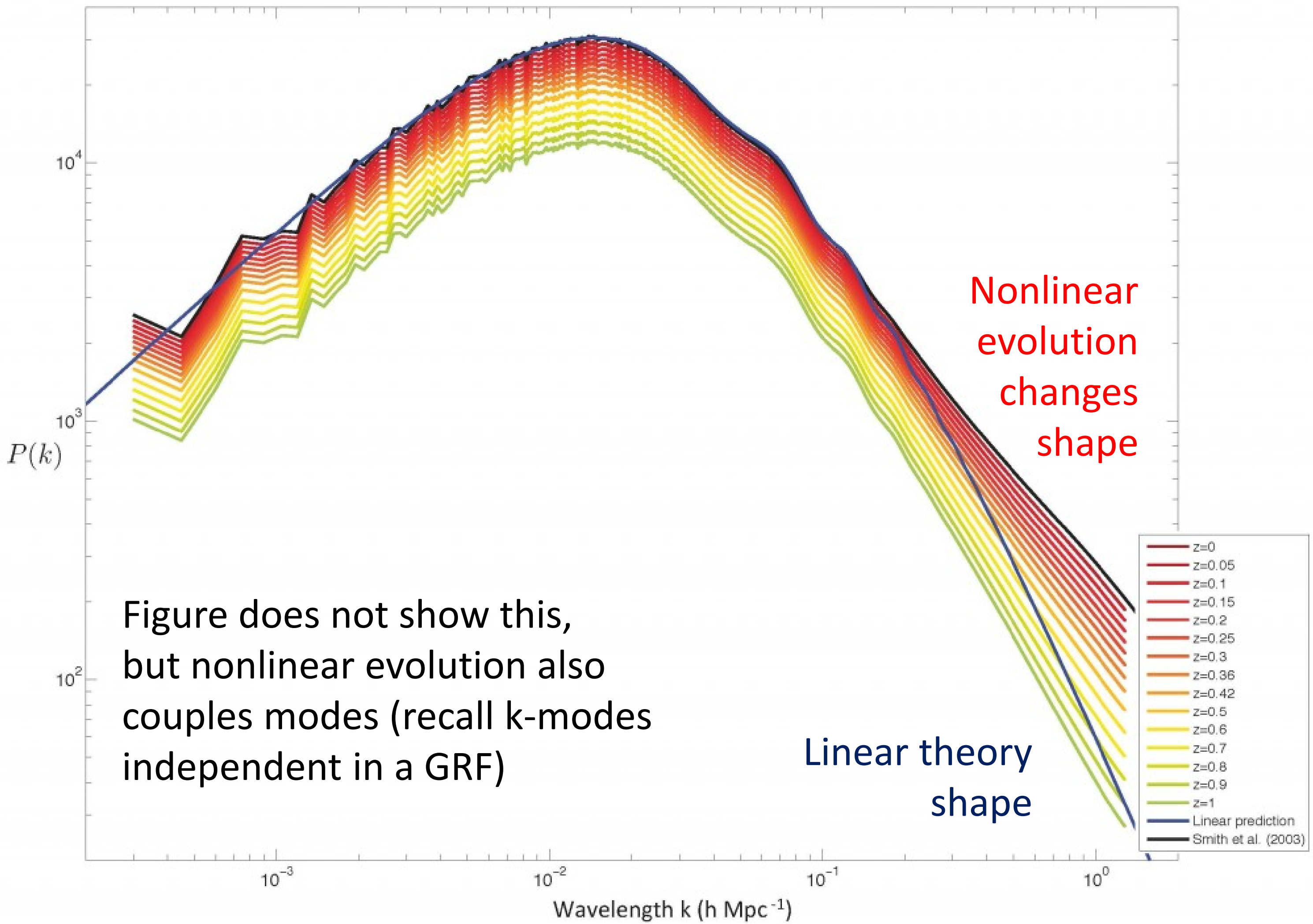
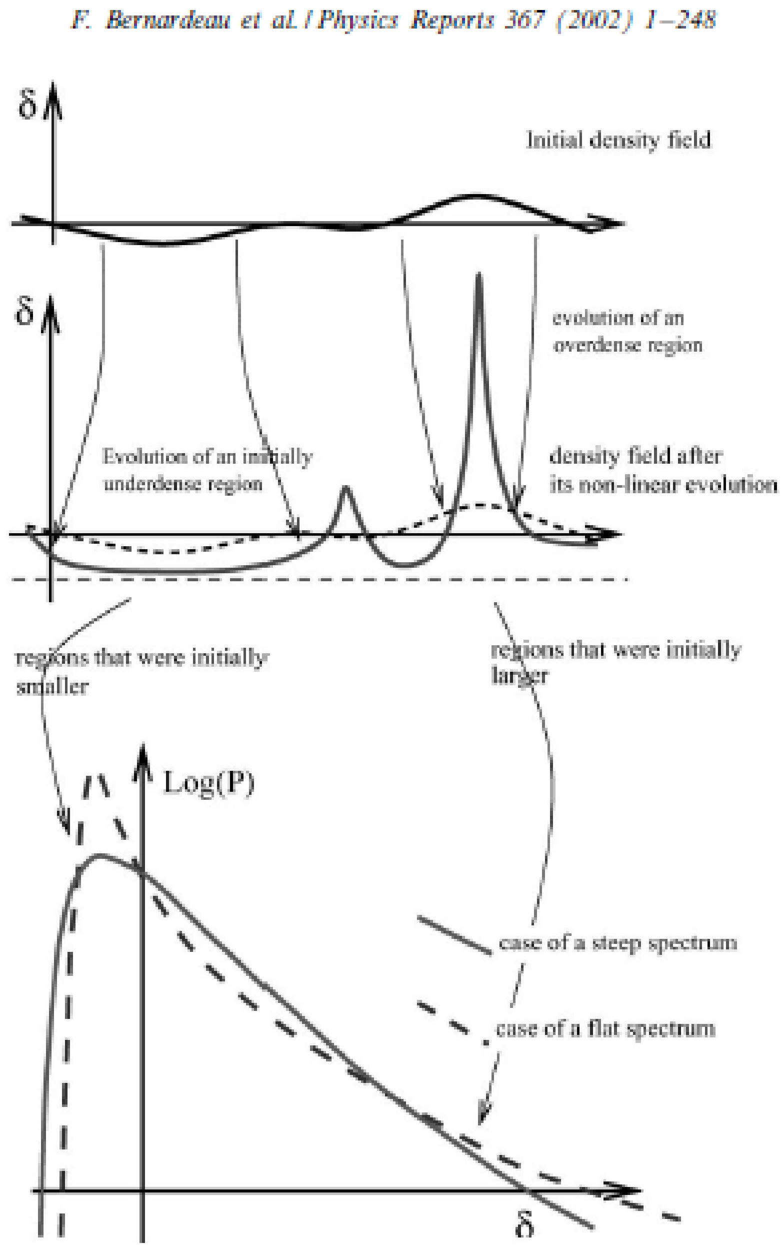


Figure does not show this, but nonlinear evolution also couples modes (recall k-modes independent in a GRF)

Initially Gaussian fluctuation field becomes very non-Gaussian

Linear growth just multiplicative factor, so cannot explain non-Gaussianity at late times

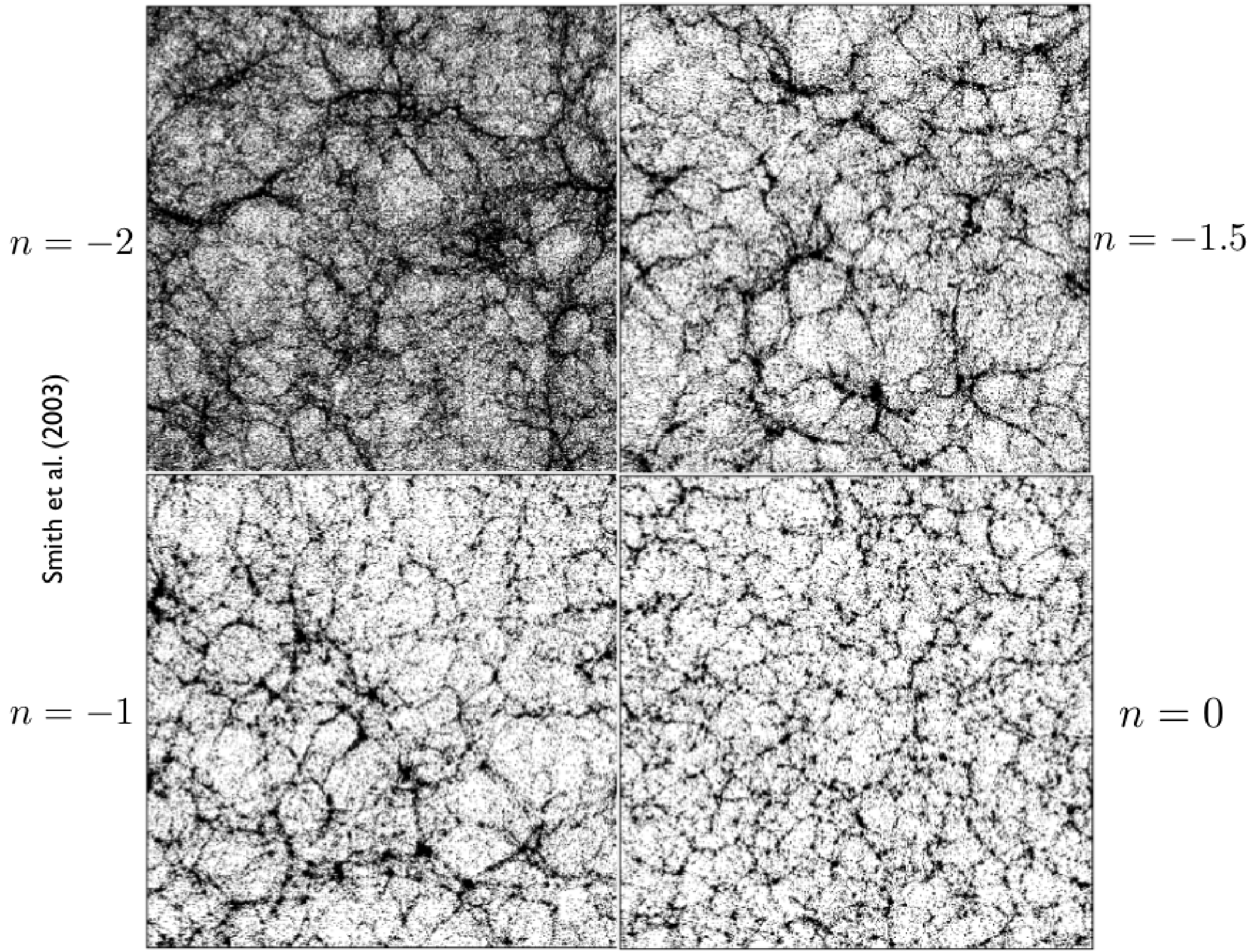


Estimate of 'nonlinear' scale

- $\sigma^2(r) = \langle\delta^2(t)\rangle = \int dk/k\ 4\pi\ k^3\ P(k,t)\ W^2(kR)$
- If $P(k) = Ak^n$ then $\langle\delta^2(t)\rangle \sim R^{-(3+n)} \sim M^{-(3+n)/3}$ (converges only for n>-3).
  – Convergence of potential fluctuations only if n=1.
- Note: $P(k,t) = D_+^2(t)\ P(k)$, so $\langle\delta^2(t)\rangle \sim 1$ means nonlinear structure on scales smaller than $R_{nl} \sim D_+^{2/(3+n)} \sim t^{(4/3)/(3+n)}$

Hierarchical structure formation for -3<n<1

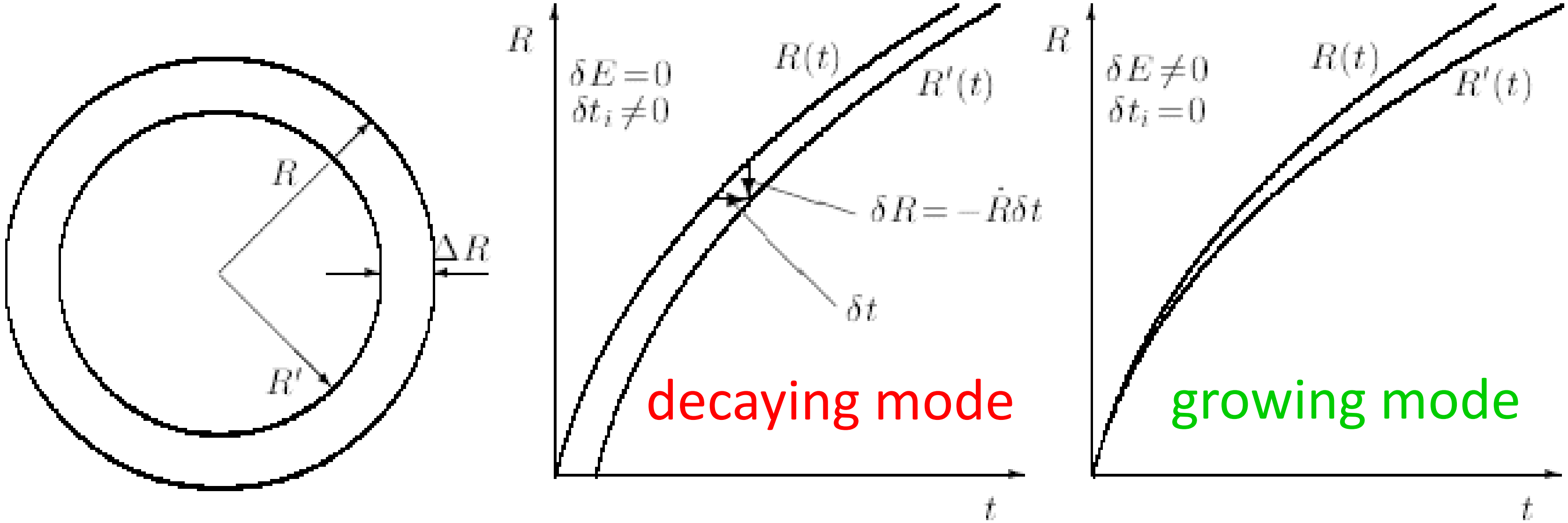
Structure formation for $P(k) \propto k^n$



Spherical evolution model

$$d^2R/dt^2 = -\,GM/R^2 + \Lambda R$$
$$= -\,\rho\,(4\pi G/3H^2)\,H^2 R + \Lambda R$$
$$= -\,\tfrac{1}{2}\,\Omega(t) H(t)^2 R + \Lambda R$$

- Note: currently fashionable to modify gravity. Should we care that only $1/R^2$ or R give stable circular orbits?



- Excise a sphere and replace with smaller one of same mass
- Perturbing time of 'big bang' → decaying mode
- Perturbing energy of sphere → growing mode

Spherical evolution model

- Initially, $E_i = -GM/R_i + (H_iR_i)^2/2 \qquad (\Lambda = 0)$
- Shells remain concentric as object evolves; if denser than background, object pulls itself together as background expands around it
- At 'turnaround': $E = -GM/r_{max} = E_i$
- So $-GM/r_{max} = -GM/R_i + (H_iR_i)^2/2$
- Hence $(R_i/r) = 1 - H_i^2R_i^3/2GM$

$$= 1 - (3H_i^2/8\pi G)\,(4\pi R_i^3/3)/M$$

$$= 1 - 1/(1+\Delta_i) = \Delta_i/(1+\Delta_i) \approx \Delta_i$$

To match to ‘growing mode’

$E_i = -G\,\rho_i\,(4\pi R_i^{\,3}/3)(1+\Delta_i)/R_i + (H_iR_i)^2(1 - \Delta_i/3)^2/2$

$= [-(\rho_i/\rho_{ci})(1+\Delta_i) + (1 - \Delta_i/3)^2]\,(H_iR_i)^2/2$

$\approx [-1-\Delta_i + 1 - 2\Delta_i/3]\,(H_iR_i)^2/2 = -(5\Delta_i/3)\,(H_iR_i)^2/2$

$= -GM/r_{\rm max} = -GM/R_i\,(R_i/r_{\rm max})$

$= -(R_i/r_{\rm max})\,(\rho_i/\rho_{ci})(1+\Delta_i)\,(H_iR_i)^2/2$

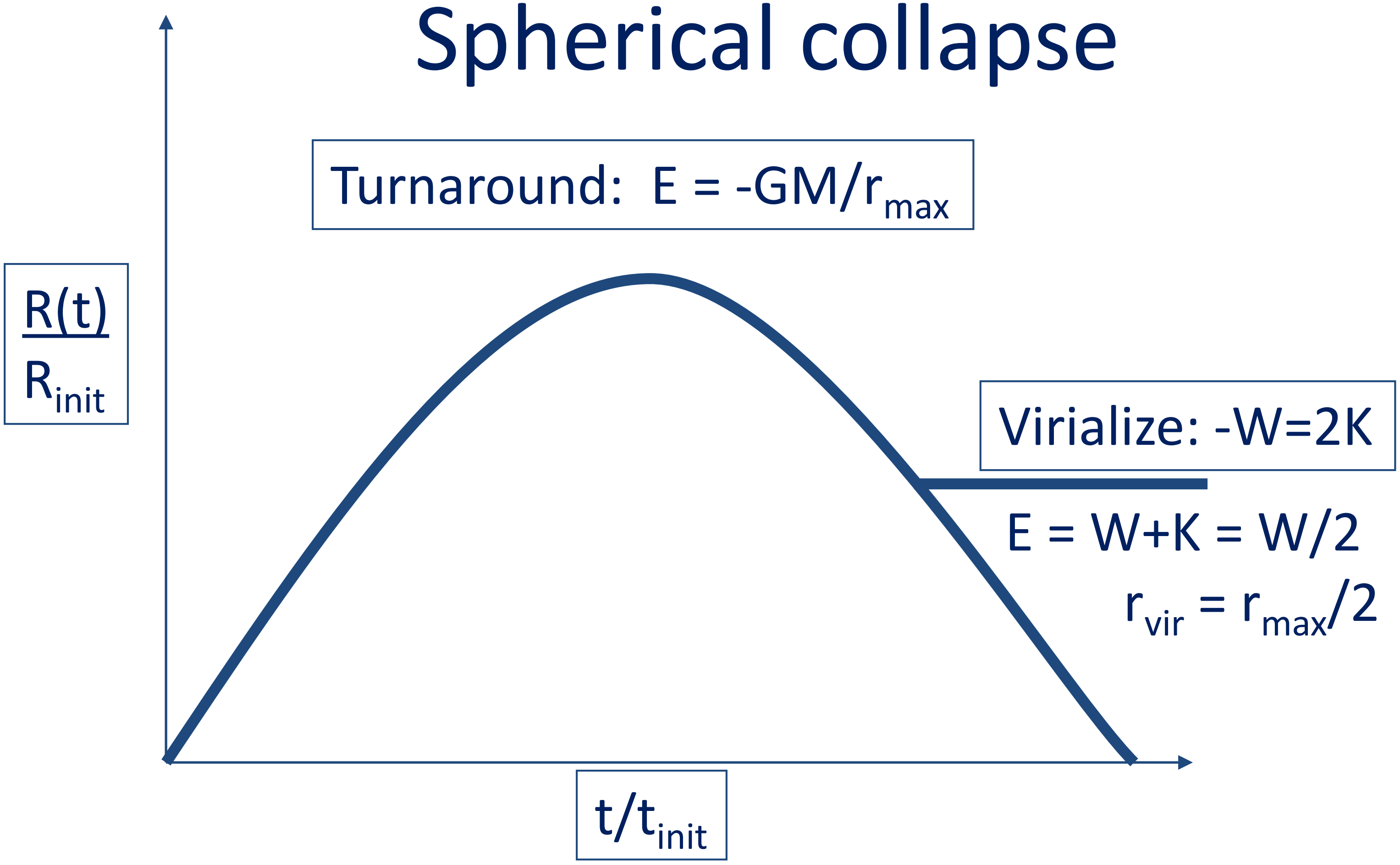
$= -(R_i/r_{\rm max})\,(1+\Delta_i)\,(H_iR_i)^2/2$

- Hence $(R_i/r_{max}) = (5\Delta_i/3)/(1+\Delta_i) \approx 5\Delta_i/3$

Virialization

- Final object virializes: $-W = 2K$
- $E_{vir} = W+K = W/2 = -GM/2r_{vir} = -GM/r_{max}$
  - so $r_{vir} = r_{max}/2$:
- Ratio of initial to final size = (density)$^{1/3}$
  - final density determined by initial overdensity
- To form an object at present time, must have had a critical over-density initially
- Critical density same for all objects!
- To form objects at high redshift, must have been even more over-dense initially

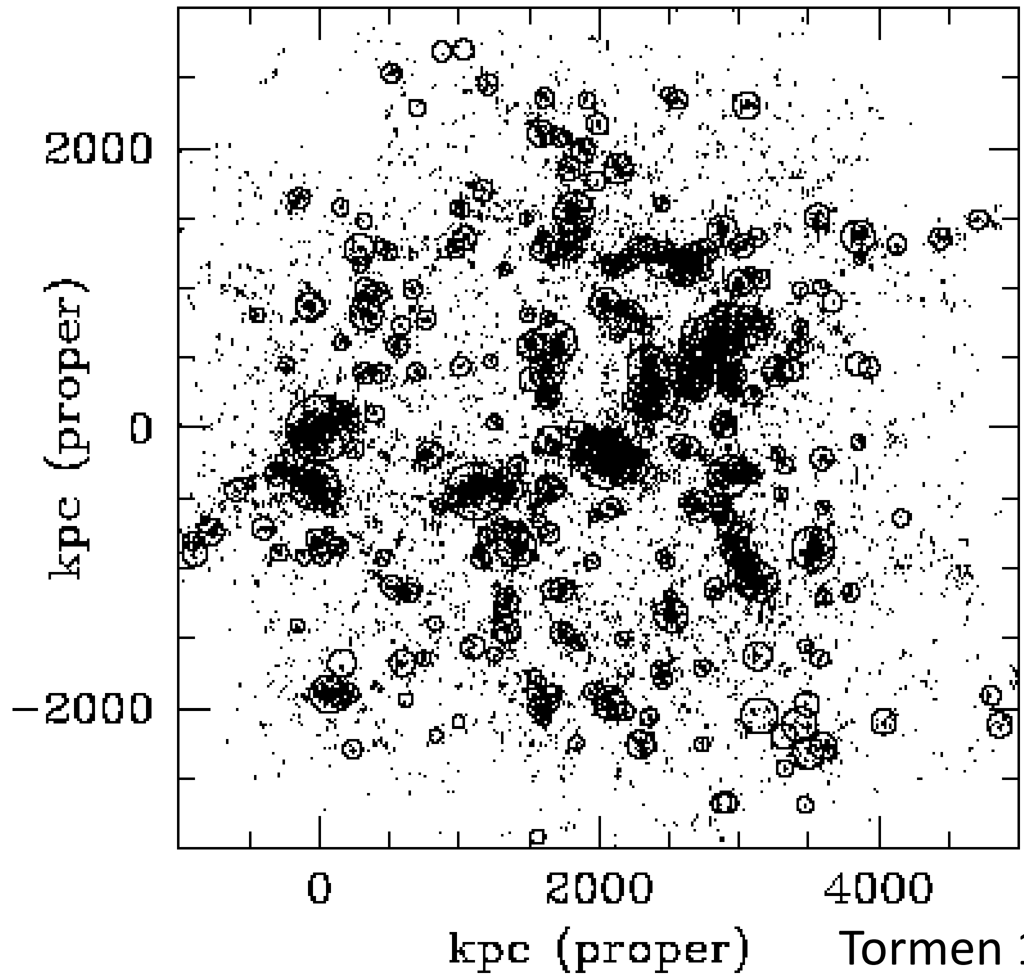
Nonlinear evolution: Spherical collapse



Modify gravity → modify collapse

Spherical evolution model

- 'Collapse' depends on initial over-density Δ_i; same for all initial sizes
- Critical density depends on cosmology
- Final objects all have same density, whatever their initial sizes
- Collapsed objects called halos are ~ 200× denser than critical (background?!), whatever their mass



Tormen 1997

(Figure shows particles at z~2 which, at z~0, are in a cluster)

Exact Parametric Solution
(R_i/R) vs. θ and (t/t_i) vs. θ

$$1 + \delta(t) = \text{Mass}/(\rho_{com}\text{Volume})$$
$$= (R_{initial}/R)^3$$
$$= (9/2)\,[\theta - \sin(\theta)]^2/\,[1 - \cos(\theta)]^3$$

And

$$\delta_L(t) = (3/10)\,(9/2)^{1/3}\,[\theta - \sin(\theta)]^{2/3}$$

This is for EdS, but cosmology dependence weak.

For underdensities:

$$\theta - \sin(\theta) \rightarrow \sinh(\theta) - \theta$$
$$1 - \cos(\theta) \rightarrow \cosh(\theta) - 1$$

Nonlinear over-density

- Turnaround at $\theta=\pi$, so nonlinear density at turnaround is $(9/2)(\pi^2/8) = 5.55$x background
- Subsequent collapse and virialization on a scale that is 2x smaller → density 8x larger.
- This happens at time $2t_{ta}$ so background density has decreased by $(2^{2/3})^3 = 4$.
- As a result final object is $(9/2)(\pi^2/8) \times 8 \times 4 =$ $18\pi^2$ x background density.
- This factor depends on cosmology: For LCDM, $18\pi^2 [1 + (\Omega_{vir} - 1)\, 82/178 - (\Omega_{vir} - 1)^2\, 39/178]/\Omega_{vir}$

Exact Parametric Solution

(R_i/R) vs. θ and (t/t_i) vs. θ

very well approximated by…

$$(R_{\rm initial}/R)^3$$

$$= \text{Mass}/(\rho_{\rm com}\text{Volume})$$

$$= 1 + \delta \approx (1 - D_{\rm Linear}(t)\, \delta_i/\delta_{sc})^{-\delta sc}$$

Dependence on cosmology from $\delta_{sc}(\Omega,\Lambda)$, but this is rather weak

Also works for underdensities!

Exact Parametric Solution
(R_i/R) vs. θ and (t/t_i) vs. θ

Now, $1+\delta$ vs δ_L is monotonic:

$\delta = \delta_L + (17/21)\,\delta_L^2 + (341/567)\,\delta_L^3 + \ldots$

These coefficients are *exactly* the same as the monopole in PT

Terms like δ_L^2 are convolutions in k-space

Therefore k-modes of nonlinear δ are coupled

So it can be inverted:

$\delta_L = \delta - (17/21)\,\delta^2 + (2815/3969)\,\delta^3 + \ldots$

This is for EdS, but in practice, approximately cosmology independent.

Exact Parametric Solution

(R_i/R) vs. θ and (t/t_i) vs. θ

very well approximated by…

$$(R_{\text{initial}}/R_t)^3$$

$$= \text{Mass}/(\rho_{\text{com}}\text{Volume})$$

$$= 1 + \delta(t) \approx (1 - D_{\text{Linear}}(t)\ \delta_i/\delta_{sc})^{-\delta sc}$$

Dependence on cosmology from $\delta_{sc}(\Omega,\Lambda)$, but this is rather weak

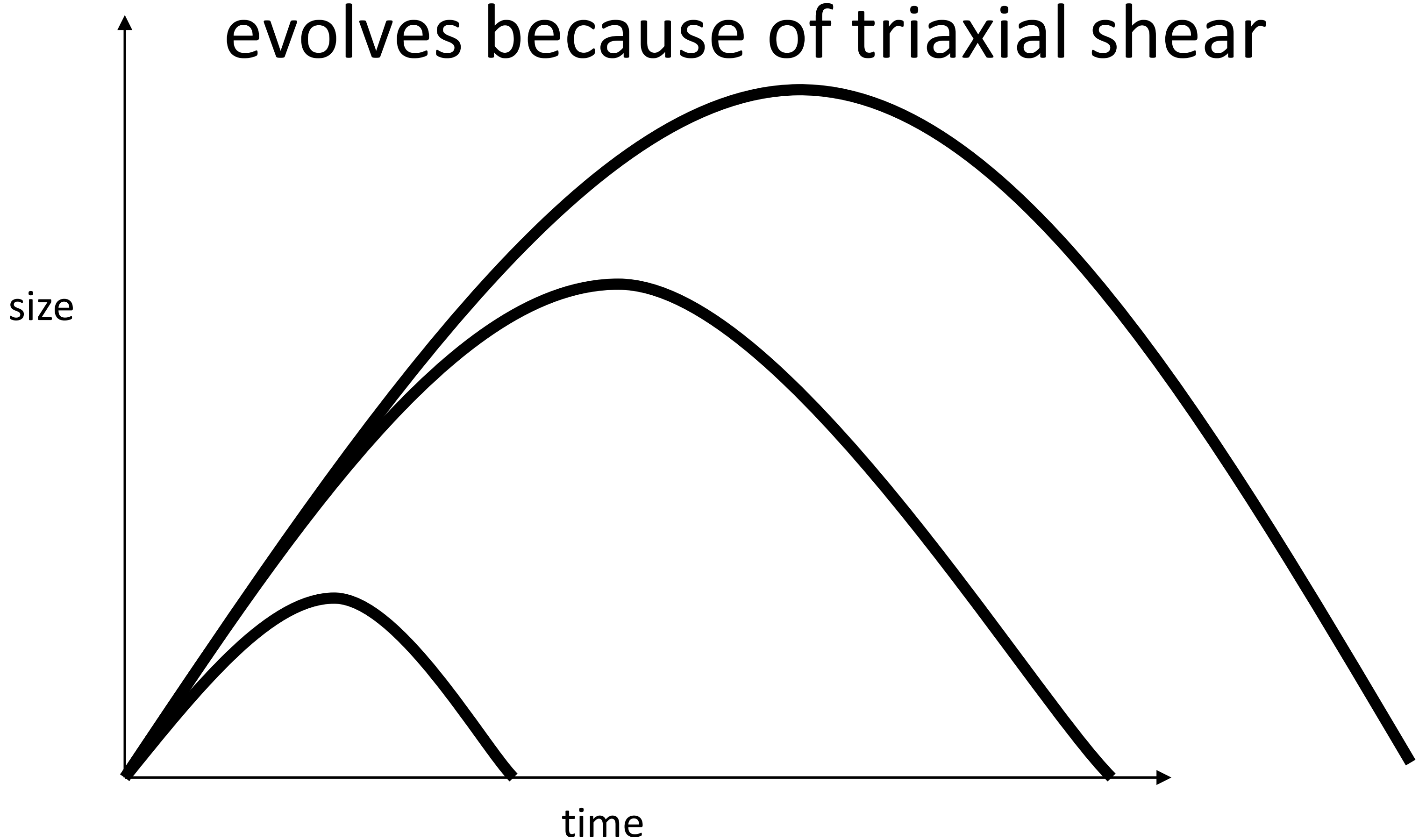
$1 + \delta \approx (1 - \delta_{Linear}/\delta_{sc})^{-\delta sc}$

- As $\delta_{Linear} \rightarrow \delta_{sc}$ ($\approx$ 1.686), $\delta \rightarrow$ infinity
  - This is virialization limit
  - Zeldovich (approximation) has $\delta_{sc} = 3$
  - Standard perturbation theory has $\delta_{sc} = 21/13 = 1.61$
- As $\delta_{Linear} \rightarrow 0$, $\delta \approx \delta_{Linear}$
- If $\delta_{Linear} = 0$ then $\delta = 0$
  - This does not happen in modified gravity models where $D(t) \rightarrow D(k,t)$
  - Related to loss of Birkhoff's theorem when r^{-2} lost?
- Note $1+\delta \rightarrow 0$ as $\delta_{Linear} \rightarrow -\infty$
  - Why is $\delta_{Linear} < -1$ sensible?

Only very fat cows are spherical….

(Lin, Mestel & Shu 1963; Icke 1973; White & Silk 1978; Bond & Myers 1996; Sheth, Mo & Tormen 2001; Ludlow, Boryazinski, Porciani 2014)

Triaxial collapse: initial sphere evolves because of triaxial shear

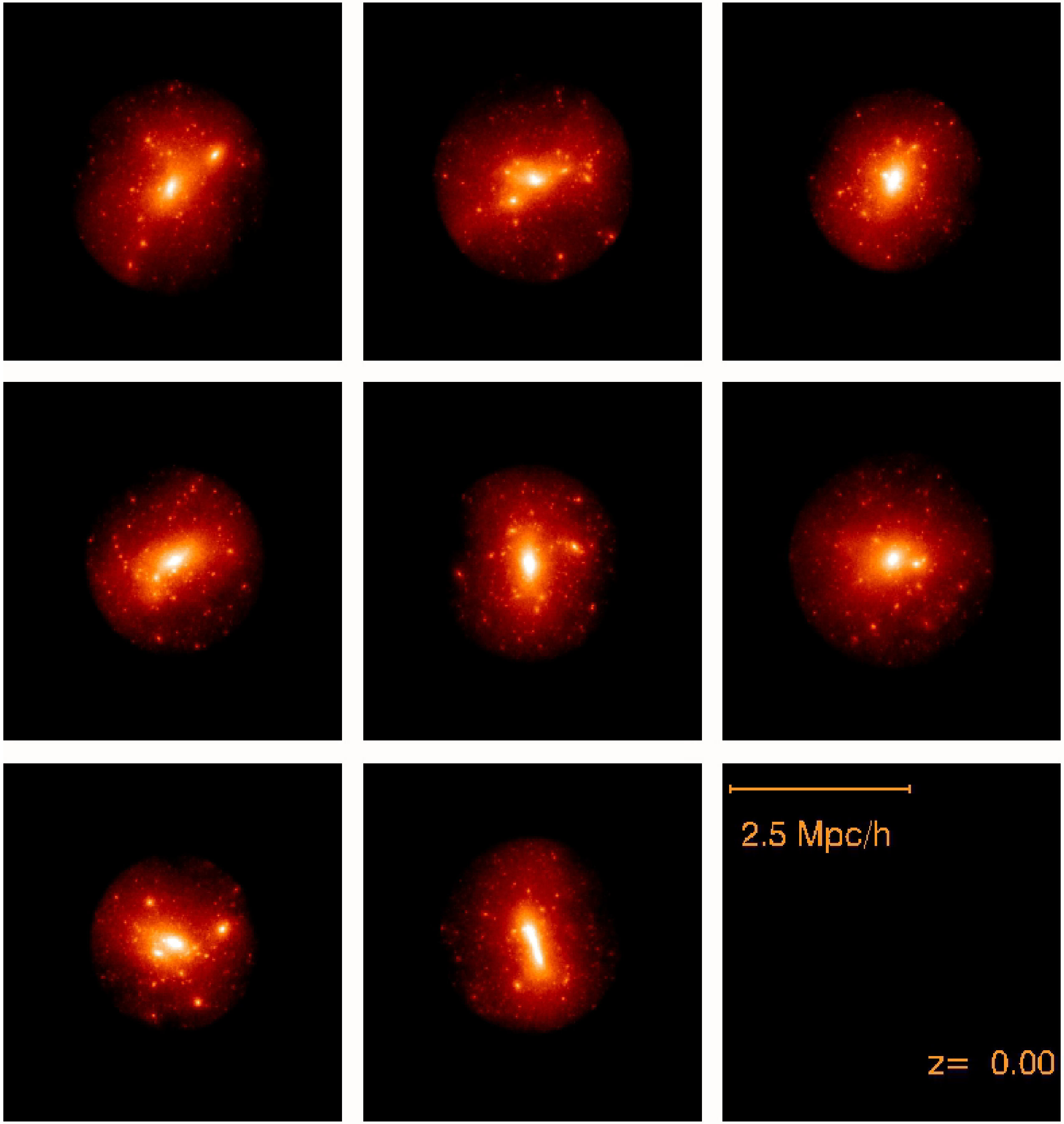


Evolution of 2^{nd} axis very similar to spherical model of same initial density

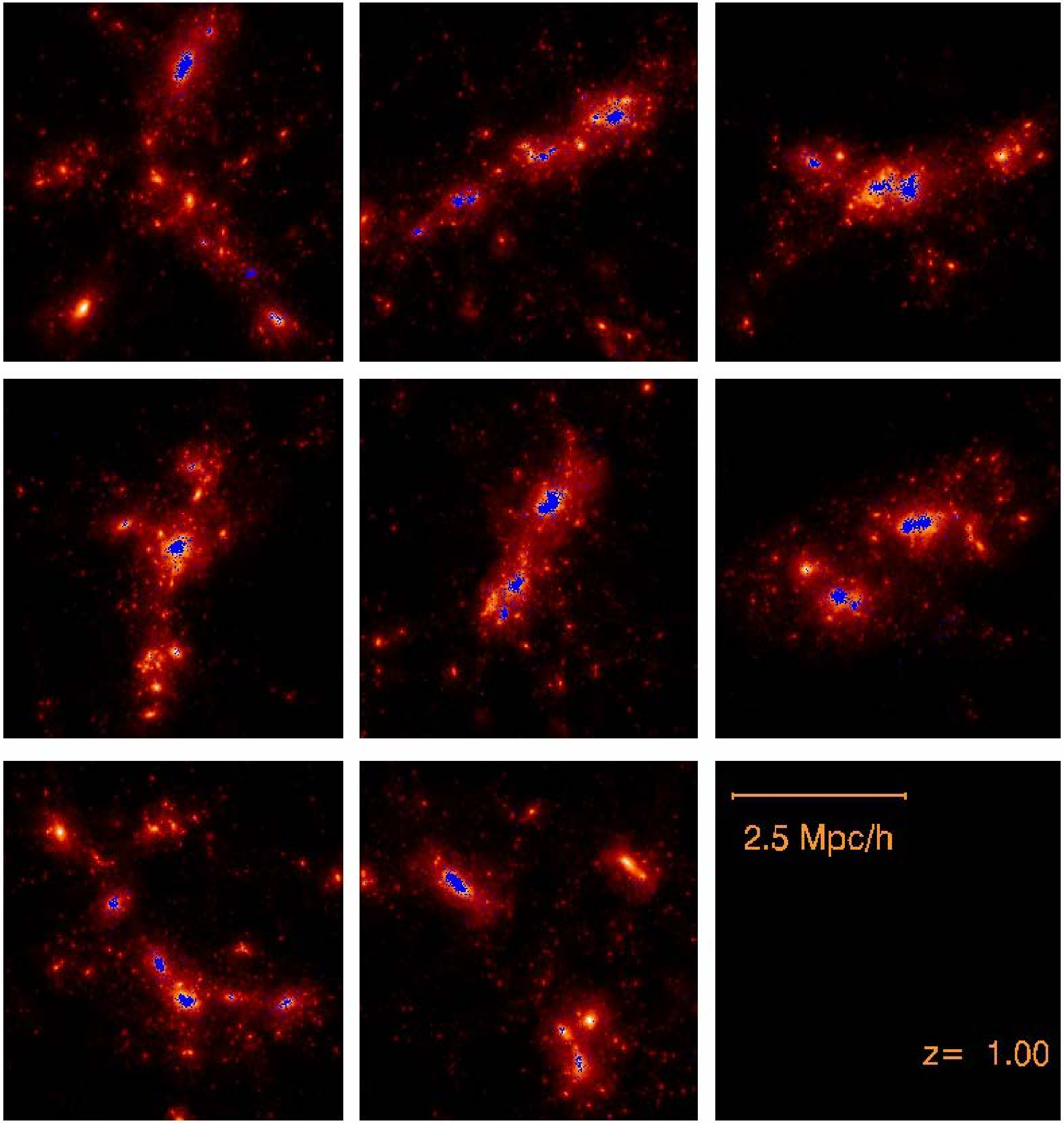
Collapse of 1^{st} axis sooner than in spherical model; collapse of all 3 axes takes longer (Bond & Myers 1996; Sheth, Mo, Tormen 2001)

Why does this work at all?

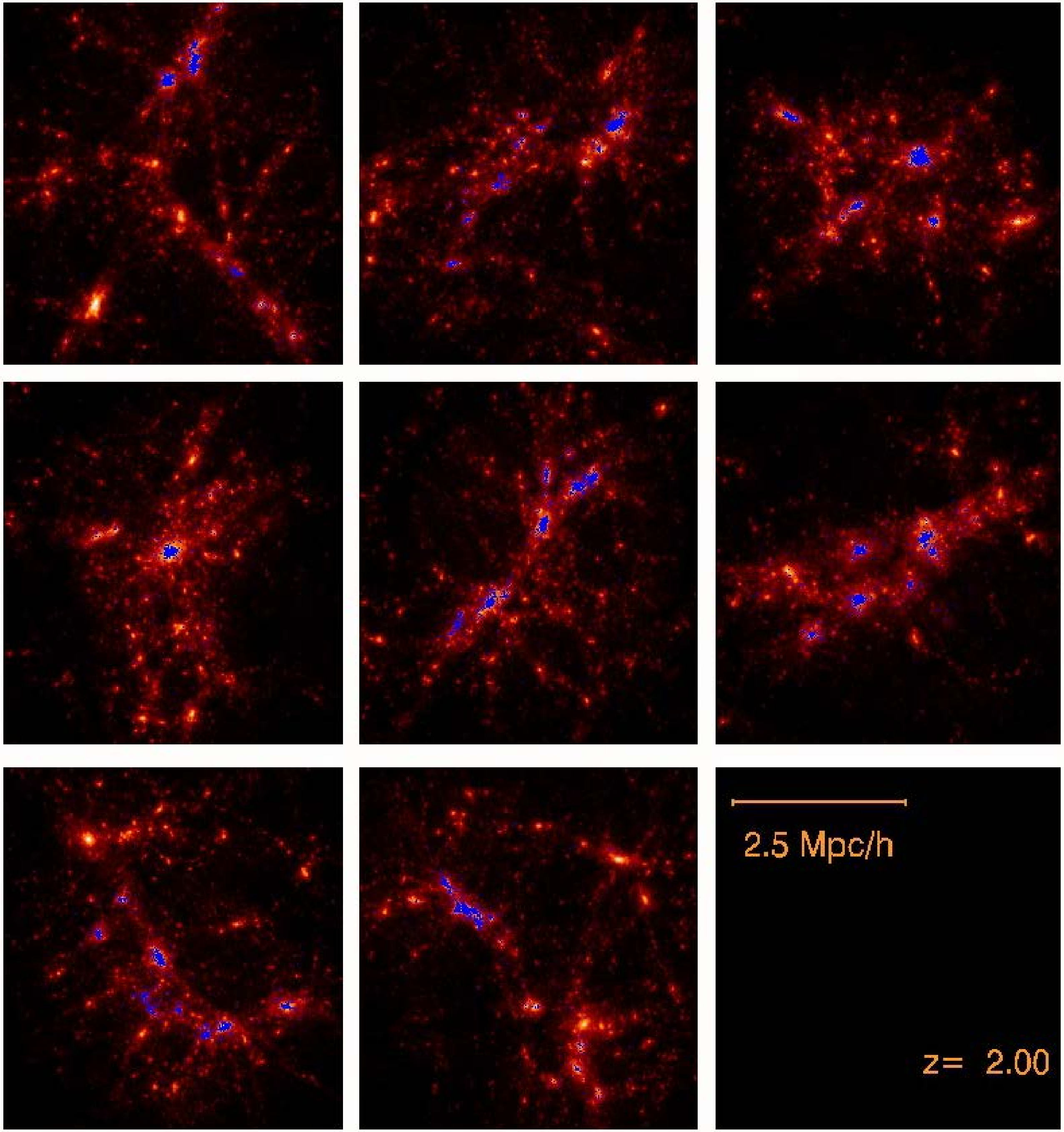
- Collapse is lumpy, not smooth
- Collapse is anisotropic, not spherical



- 8 halos, $10^{15} M_{sun}$ at z=0 in ΛCDM
- Only dark matter particles within R_{200} shown



- Same objects at z=1
- Blue shows dark matter within 20kpc at z=0



- Same objects at z=2
- Blue shows dark matter within 20kpc at z=0

Why does this work at all?

- Collapse is lumpy, not smooth
  - Centers of virialized subclumps at early time end up in center of virialized halo at later time
  - Spherical collapse has rank ordering in binding energy 'built-in'
- Collapse is anisotropic, not spherical
  - Monopole of full anisotropic solution is given by SC at <u>all</u> orders

Virial Motions (within 'halos')

- $(R_i/r_{\rm vir}) \sim f(\Delta_i)$: ratio of initial and final sizes depends on initial overdensity
- Mass $M \sim R_i^3$ (since initial overdensity « 1)
- So final virial density $\sim M/r_{\rm vir}^3 \sim (R_i/r_{\rm vir})^3 \sim$ function of critical density: Hence, all virialized objects have the same density, $\Delta_{\rm vir}\,\rho_{\rm crit}(z)$, whatever their mass
- $V^2 \sim GM/r_{\rm vir} \sim (Hr_{\rm vir})^2 \Delta_{\rm vir} \sim (HGM/V^2)^2\,\Delta_{\rm vir} \sim (HM)^{2/3}$: massive objects have larger internal velocities or temperatures; H decreases with time, so, for a given mass, virial motions (or temperature) higher at high z

Hydrostatic equilibrium

$$P = \left(\frac{N}{V}\right) kT = \frac{\rho_b kT}{\mu m_H}$$

$$\frac{dP}{dr} = \frac{k_B}{\mu m_H}\left[T\frac{d\rho_b}{dr} + \rho_b\frac{dT}{dr}\right] = -\frac{GM\rho_b}{r^2}$$

$$\frac{k_B T}{\mu m_H}\left[\frac{1}{\rho_b}\frac{d\rho_b}{dr} + \frac{1}{T}\frac{dT}{dr}\right] = -\frac{GM}{r^2}$$

$$\frac{k_B T}{\mu m_H}\left[\frac{r}{\rho_b}\frac{d\rho_b}{dr} + \frac{r}{T}\frac{dT}{dr}\right] = -\frac{GM}{r}$$

$$M_{tot}(< r) = \frac{-kTr}{G\mu m_H}\left[\frac{dln\rho_b}{dlnr} + \frac{dlnT}{dlnr}\right]$$

V
O
I
D
S