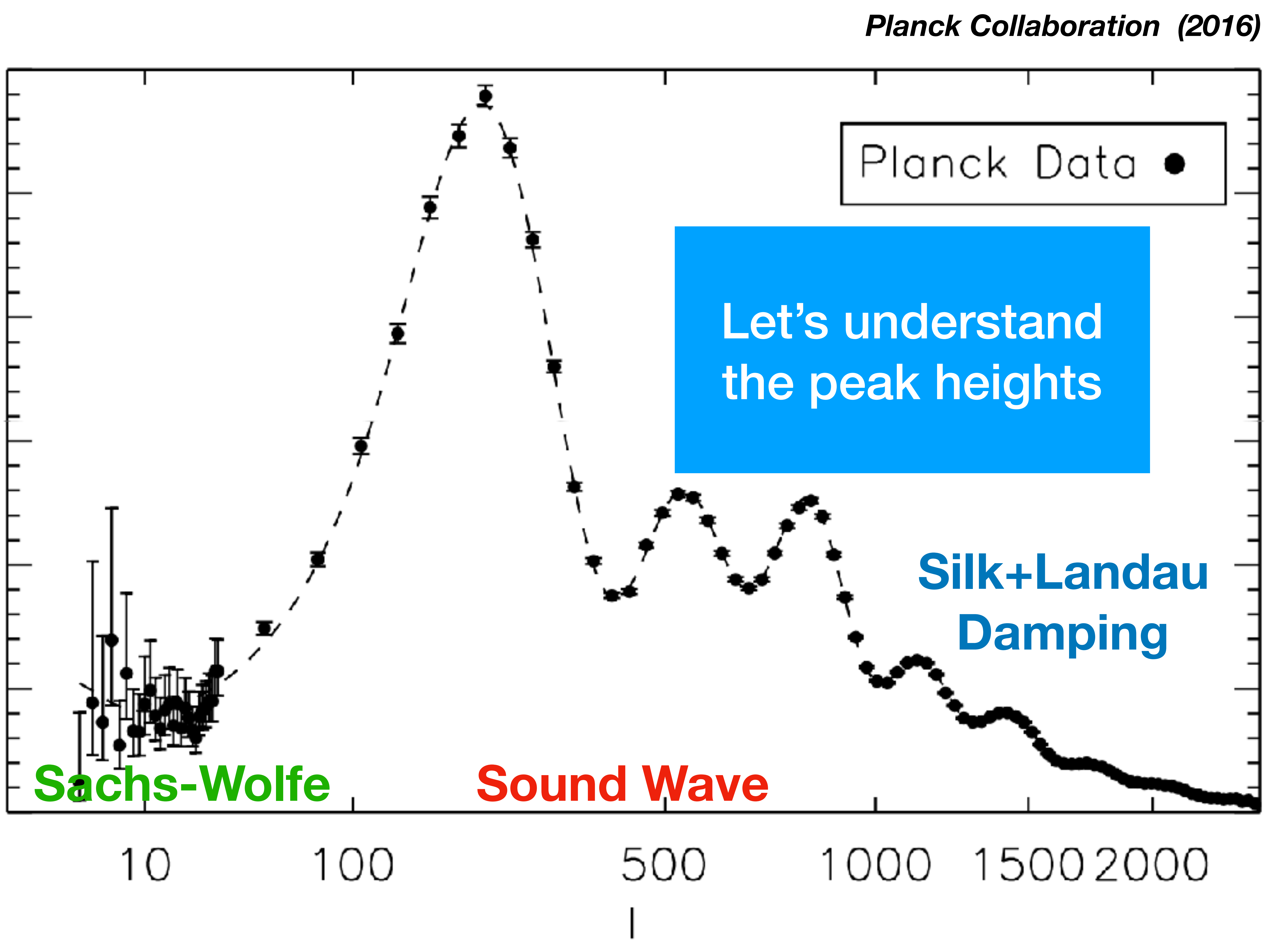
Lecture 3

- Cosmological parameter dependence of the temperature power spectrum

- Polarisation of the CMB


Planck Collaboration (2016)
Planck Data
Let's understand the peak heights
Silk+Landau Damping
Sachs-Wolfe
Sound Wave
10
100
500
1000
1500
2000
l

Matching Solutions

- We have a very good analytical solution valid at low and high frequencies during the radiation era:

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta\left(-\cos\varphi + \frac{2}{\varphi}\sin\varphi\right)$$

- Now, match this to a high-frequency solution valid at the last-scattering surface (when R is no longer small)

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = A\cos(qr_s) + B\sin(qr_s) - R\Phi$$

Matching Solutions

- We have a very good analytical solution valid at low and high frequencies during the radiation era:

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} - \Psi = \zeta\left(-\cos\varphi + \frac{2}{\varphi}\sin\varphi\right)$$

- Now, match this to a high-frequency solution valid at the last-scattering surface (when R is no longer small)

Slightly improved solution, with a weak time dependence of R using the WKB method [*Peebles & Yu (1970)*]

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = (1+R)^{-1/4}[A\cos(qr_s) + B\sin(qr_s)] - R\Phi$$

Weinberg "Cosmology", Eq. (6.5.7)

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=\frac{\zeta}{5}\left\{3R\mathcal{T}(q)-(1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s+\theta(q)]\right\}$$

where *T*(q), *S*(q), *θ*(q) are "transfer functions" that smoothly interpolate two limits as

$q \ll q_{EQ}$: $\mathcal{S}\to 1,\ \mathcal{T}\to 1,\ \theta\to 0$

$q \gg q_{EQ}$: $\mathcal{S}\to 5,\ \mathcal{T}\propto \ln q/q^2,\ \theta\to 0.062\pi$

"EQ" for "matter-radiation Equality epoch"

with $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01\ \mathrm{Mpc}^{-1}$, giving $l_{EQ}=q_{EQ}r_L \sim 140$

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales

Weinberg "Cosmology", Eq. (6.5.7)

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

where *T*(q), *S*(q), *θ*(q) are "transfer functions" that smoothly interpolate two limits as

q << q_{EQ}: $\mathcal{S} \to 1,\ \mathcal{T} \to 1,\ \theta \to 0$

q >> q_{EQ}: $\mathcal{S} \to 5,\ \mathcal{T} \propto \ln q/q^2,\ \theta \to 0.062\pi$

"EQ" for "matter radiation Equality epoch"

with q_{EQ} ... $_Q r_L$ ~ 140

- (*) To a ... solution is given b... not importa...

Due to the decay of gravitational potential during the radiation dominated era

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma}+\Phi=\frac{\zeta}{5}\left\{3R\mathcal{T}(q)-(1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s+\theta(q)]\right\}$$

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"EQ" for "matter-radiation Equality epoch"

with $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01\ \mathrm{Mpc}^{-1}$, giv

- (*) To a good approximation, the low- given by setting R=0 because sound important at large scales

Due to the neutrino anisotropic stress

High-frequency Solution(*) at the Last Scattering Surface

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

q -> 0(*) $\longrightarrow \; -\dfrac{\zeta}{5}$

This should agree with the Sachs-Wolfe result: Φ/3; thus,

$$\Phi = -3\zeta/5$$ **in the matter-dominated era**

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales

Weinberg "Cosmology", Eq. (6.5.7)

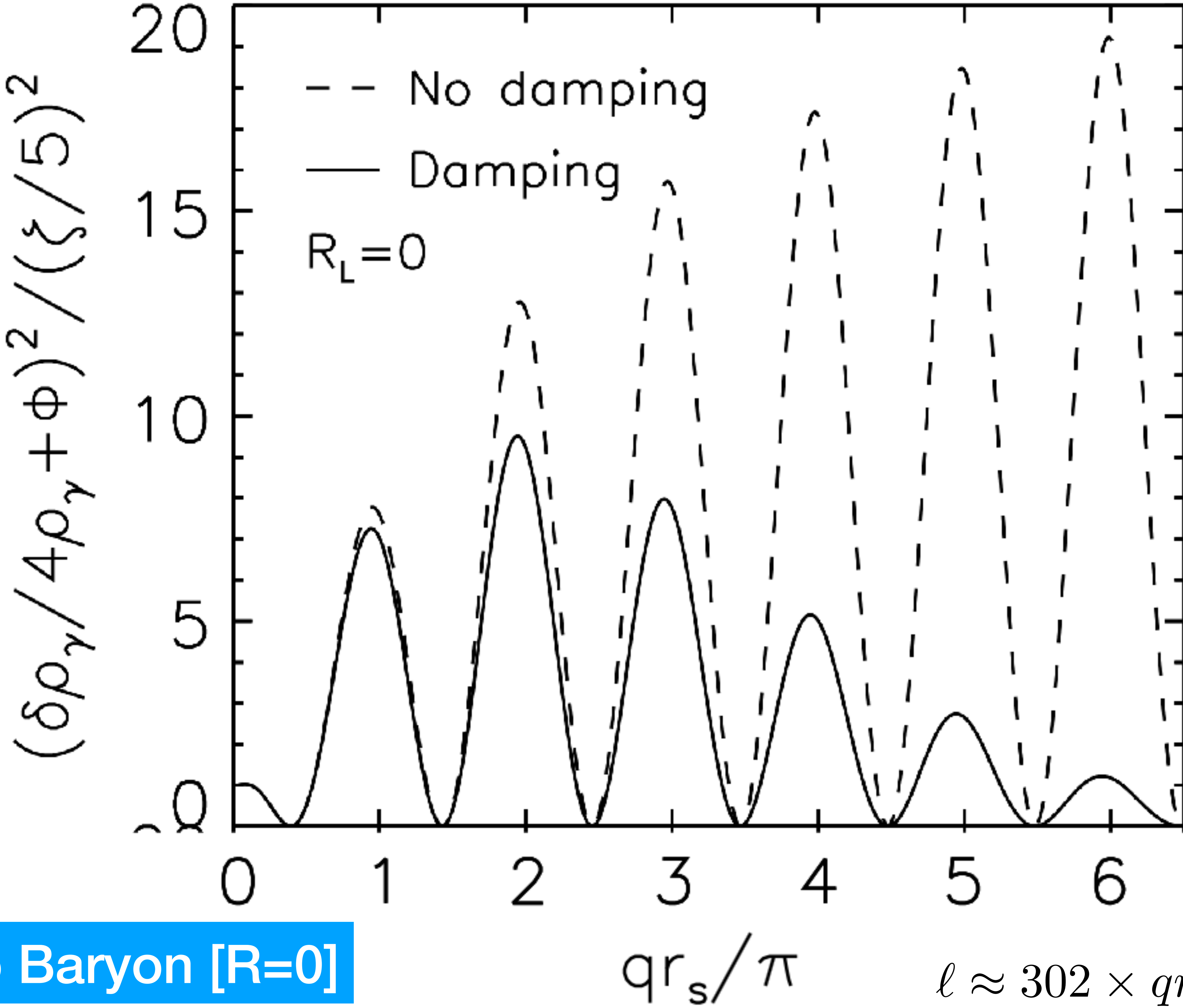
Effect of Baryons

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{ \boxed{3R\mathcal{T}(q)} - \boxed{(1+R)^{-1/4}} \mathcal{S}(q)\cos[qr_s + \theta(q)] \right\}$$

Shift the zero-point of oscillations

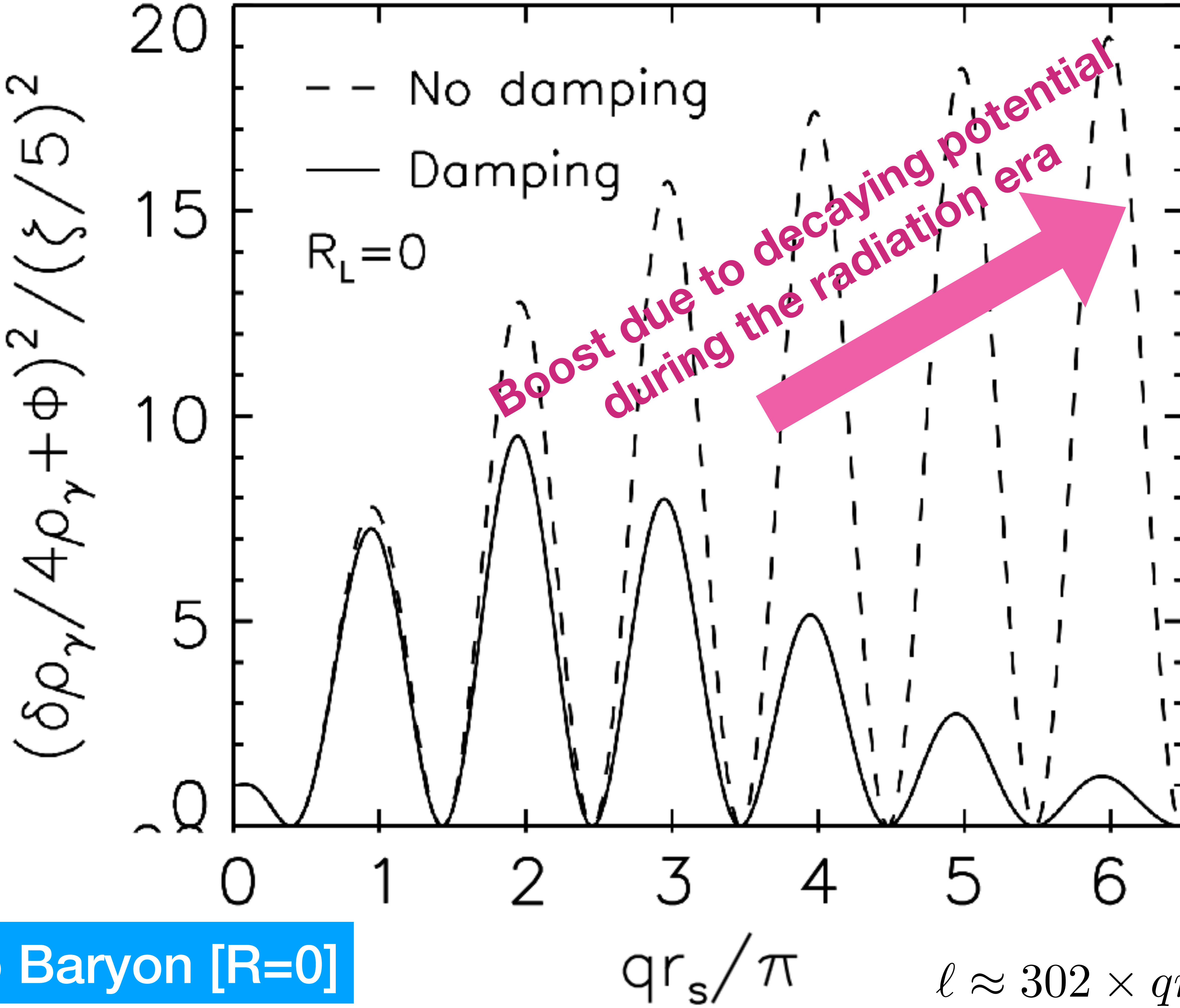
Reduce the amplitude of oscillations

- (*) To a good approximation, the low-frequency solution is given by setting R=0 because sound waves are not important at large scales



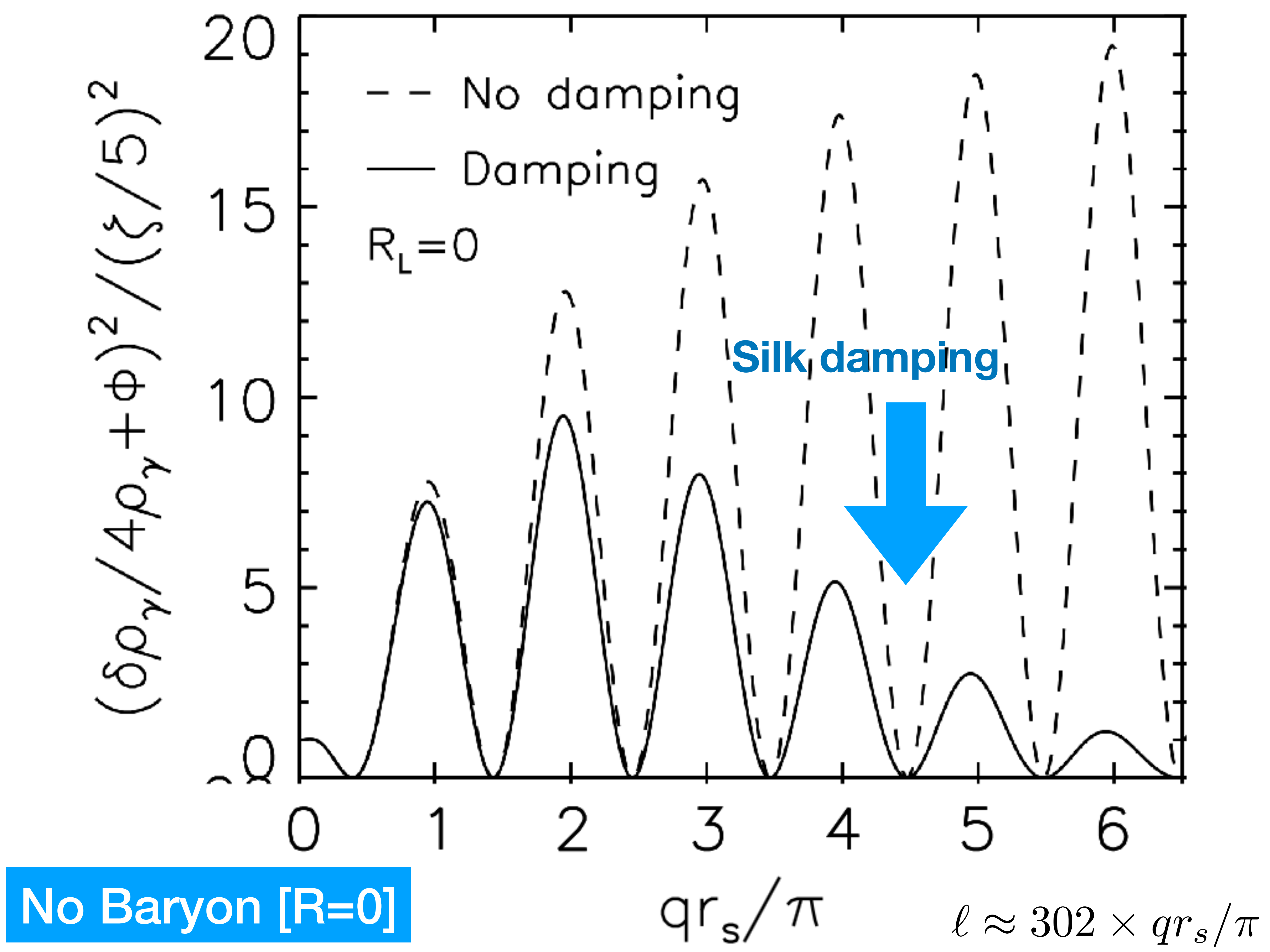
No Baryon [R=0]

$\ell \approx 302 \times qr_s/\pi$

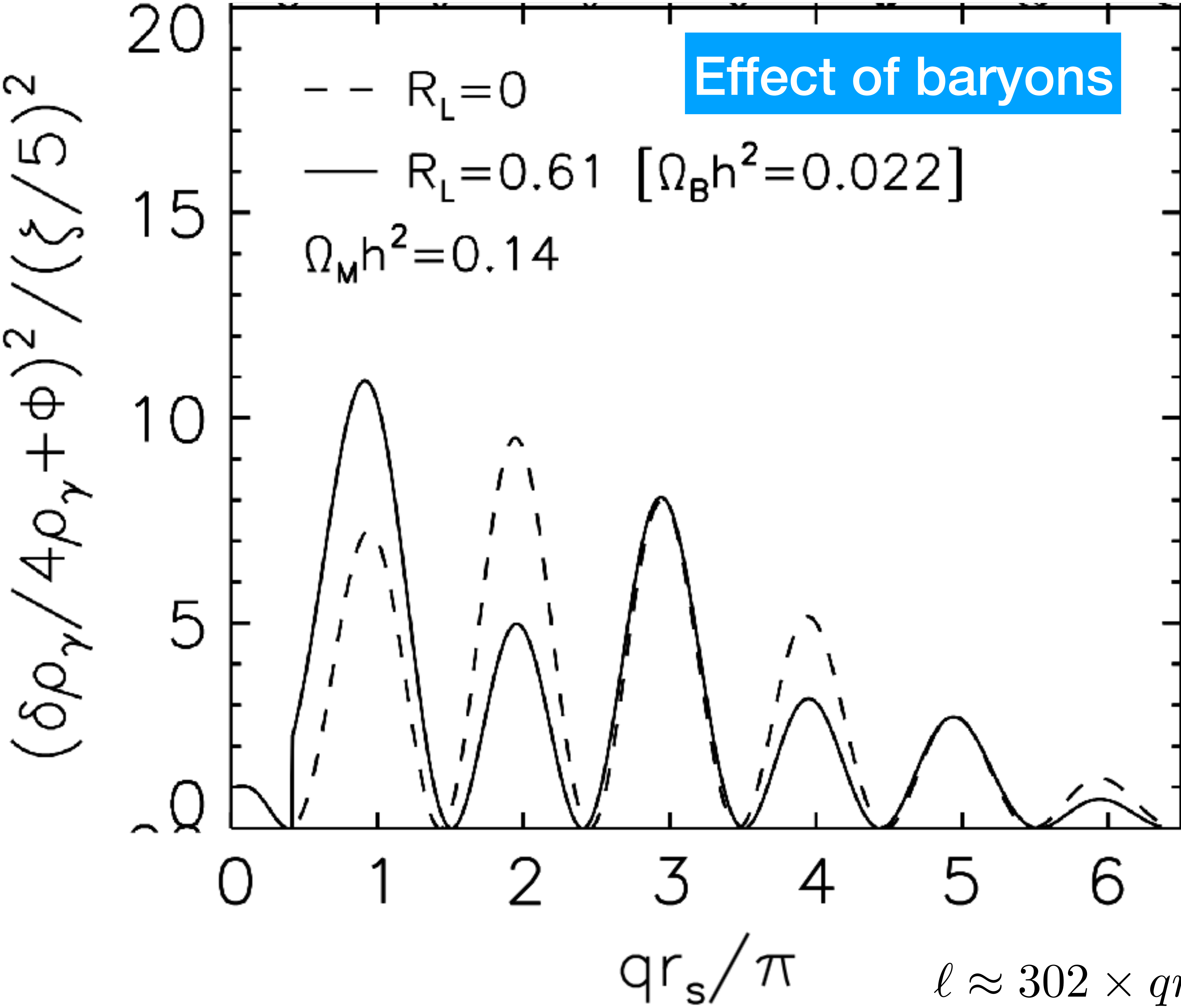


No Baryon [R=0]

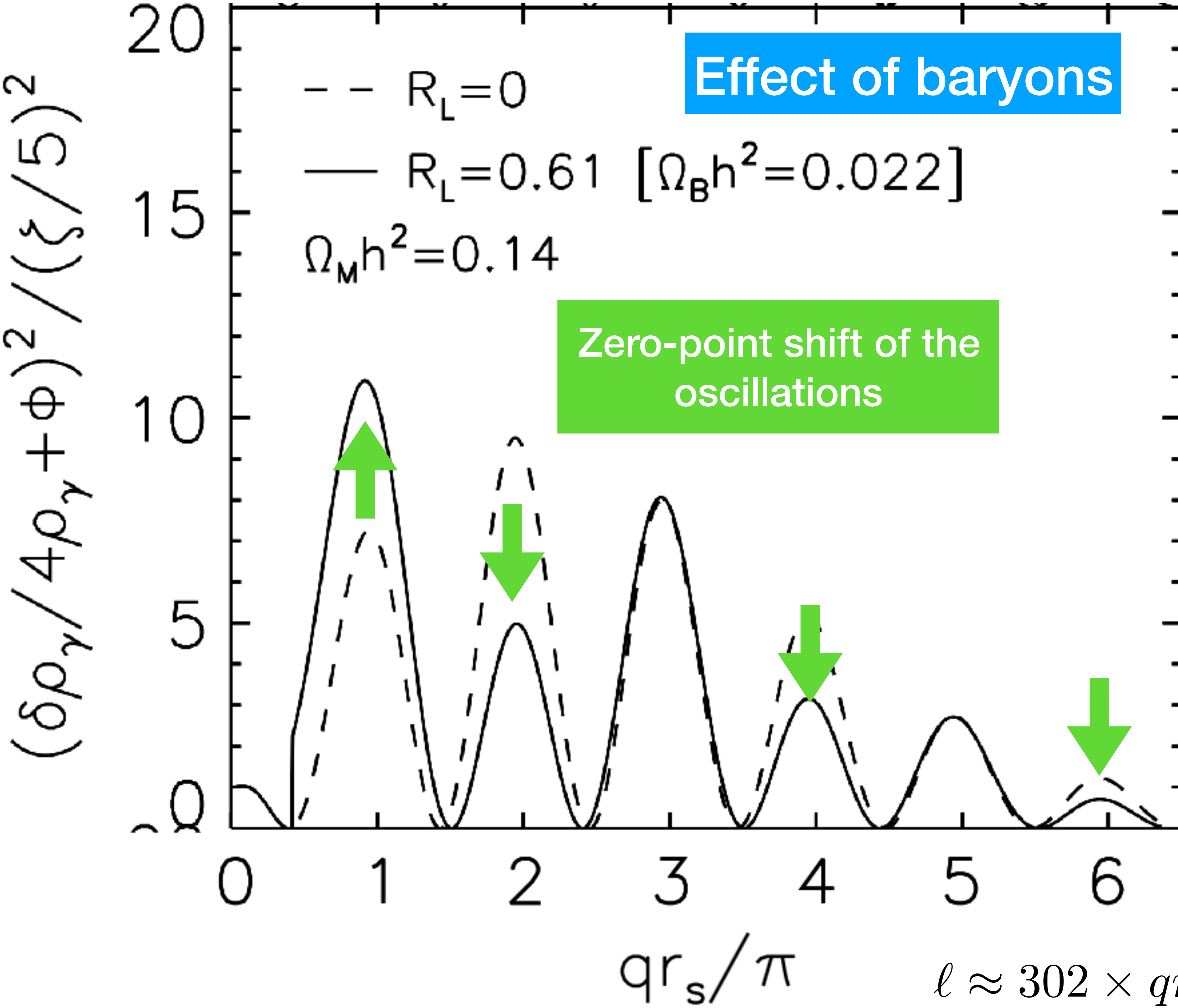
$\ell \approx 302 \times qr_s/\pi$



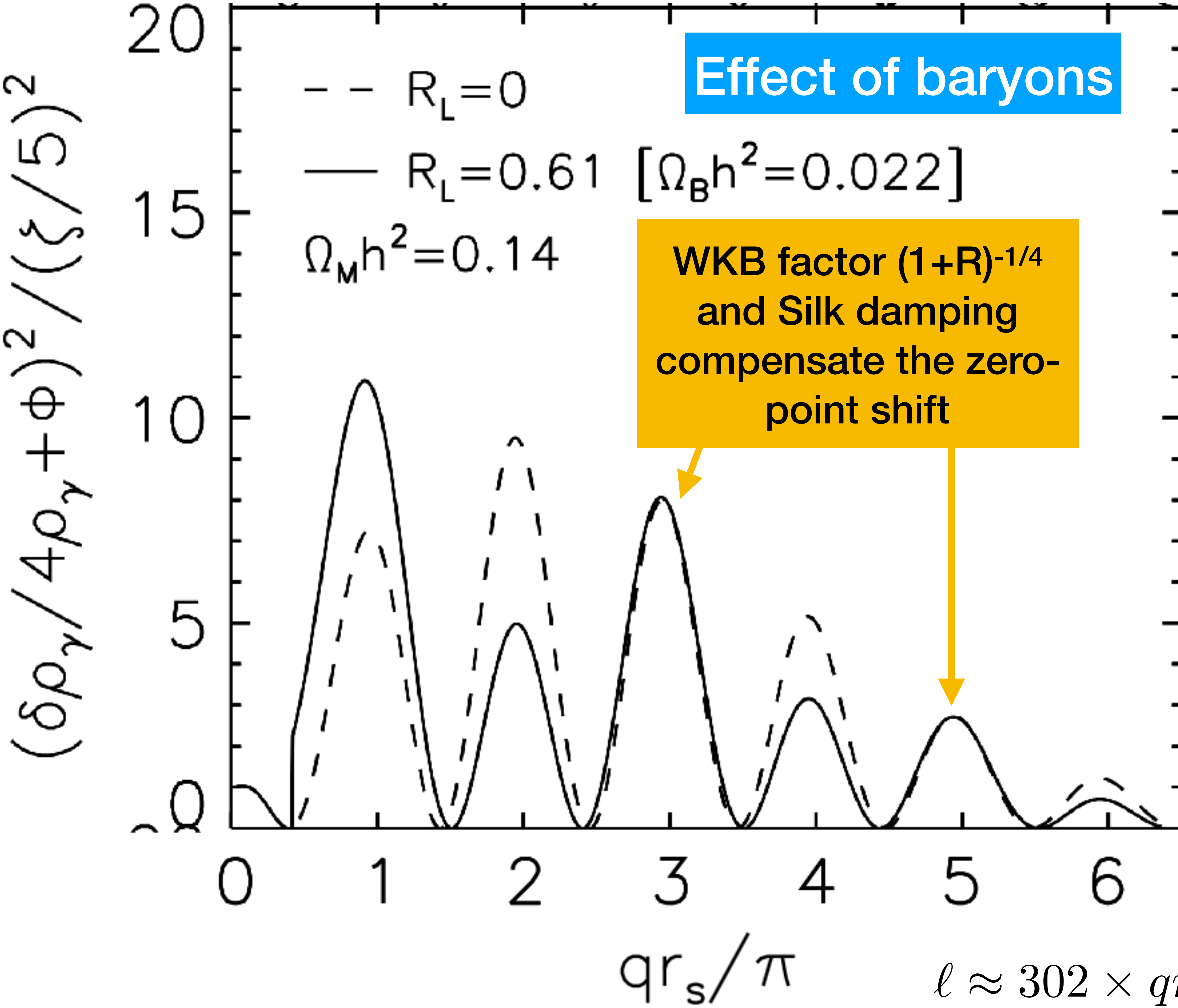
No damping
Damping
R_L=0
Silk damping
$(\delta\rho_\gamma/4\rho_\gamma+\Phi)^2/(\zeta/5)^2$
qr_s/π
0
5
10
15
20
0 1 2 3 4 5 6
No Baryon [R=0]
$\ell \approx 302 \times qr_s/\pi$



$\ell \approx 302 \times qr_s/\pi$



$\ell \approx 302 \times qr_s/\pi$



$\ell \approx 302 \times qr_s/\pi$

Effect of Total Matter

$$\frac{\delta\rho_\gamma}{4\bar{\rho}_\gamma} + \Phi = \frac{\zeta}{5}\left\{3R\mathcal{T}(q) - (1+R)^{-1/4}\mathcal{S}(q)\cos[qr_s + \theta(q)]\right\}$$

where *T*(q), *S*(q), *θ*(q) are “transfer functions” that smoothly interpolate two limits as

q << q_{EQ}: $\mathcal{S} \to 1,\ \mathcal{T} \to 1,\ \theta \to 0$

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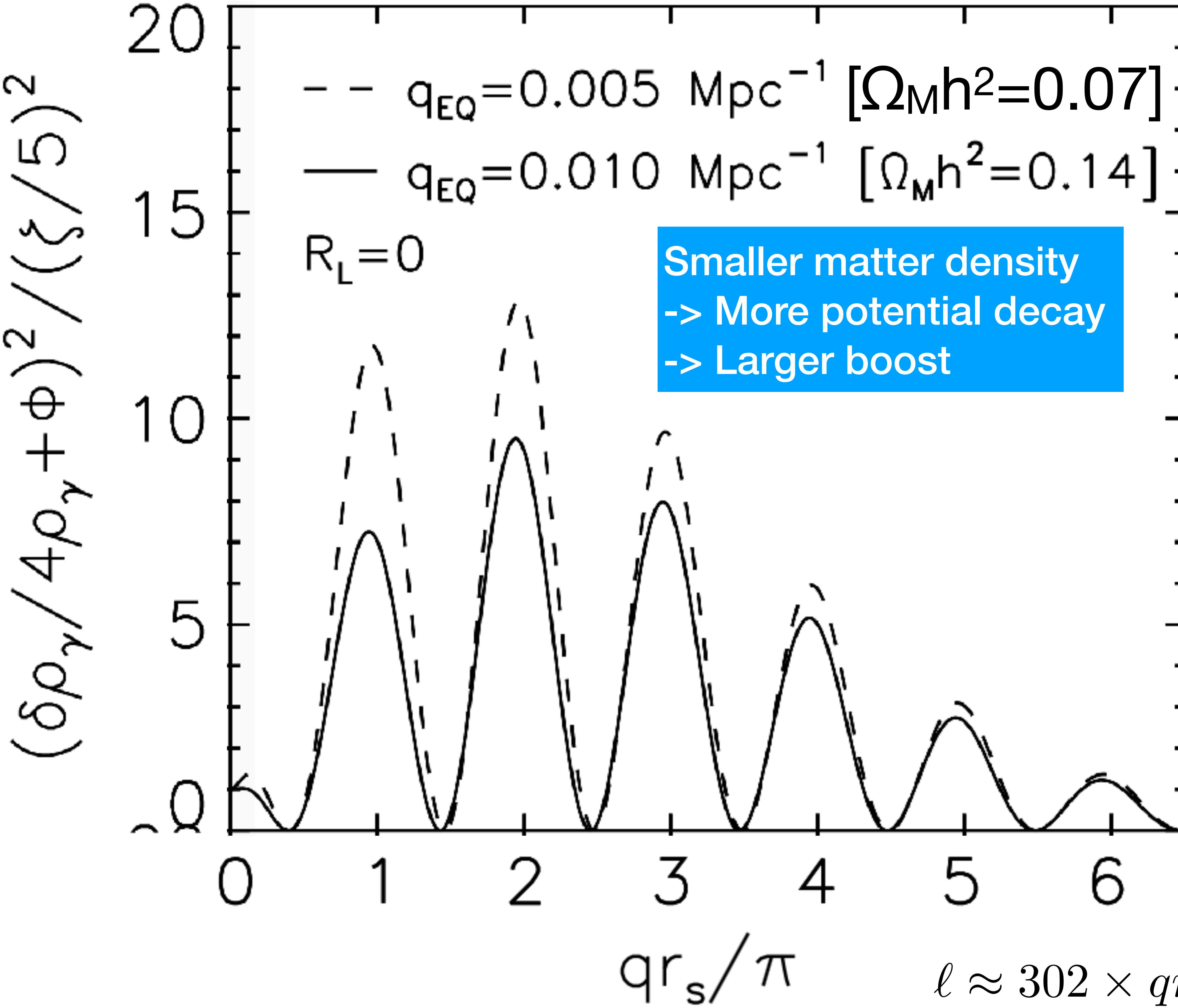
“EQ” for “matter-radiation Equality epoch”

with $q_{EQ} = a_{EQ}H_{EQ} \sim 0.01\ (\Omega_M h^2/0.14)\ \mathrm{Mpc}^{-1}$



$\ell \approx 302 \times qr_s/\pi$

Recap

- Decay of gravitational potentials boosts the temperature anisotropy dT/T at high multipoles by **a factor of 5 compared to the Sachs-Wolfe plateau**
  - Where this boost starts depends on the total matter density
- Baryon density shifts the zero-point of the oscillation, boosting the odd peaks relative to the even peaks
  - However, the WKB factor $(1+R)^{-1/4}$ and damping make the **boosting of the 3rd and 5th peaks not so obvious**

Not quite there yet…

- **The first peak is too low**
  - We need to include the “integrated Sachs-Wolfe effect”
- **How to fill zeros between the peaks?**
  - We need to include the Doppler shift of light

Doppler Shift of Light

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n}r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n}r_L) - \hat{n}\cdot\boldsymbol{v}_B(t_L, \hat{n}r_L)$$

V_B is the bulk velocity of a baryon fluid

- Using the velocity potential, we write $-\hat{n}\cdot\nabla\delta u_B/a$
- In tight coupling, $\delta u_B = \delta u_\gamma$
- Using energy conservation,

$$\delta u_\gamma = (3a^2/q^2)\partial(\delta\rho_\gamma/4\bar{\rho}_\gamma)/\partial t$$

Line-of-sight direction

$$\hat{n}^i = -\gamma^i$$

Coming distance (r)

$$x^i = \hat{n}^i r$$

$$r(t) = \int_t^{t_0} \frac{dt'}{a(t')}$$

Doppler Shift of Light

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta\rho_\gamma(t_L, \hat{n}r_L)}{4\bar{\rho}_\gamma(t_L)} + \Phi(t_L, \hat{n}r_L) - \hat{n}\cdot\boldsymbol{v}_B(t_L, \hat{n}r_L)$$

v_B is the bulk velocity of a baryon fluid

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- In tight coupling, $\delta u_B = \delta u_\gamma$
- Using energy conservation,

$$\delta u_\gamma = (3a^2/q^2)\partial(\delta\rho_\gamma/4\bar{\rho}_\gamma)/\partial t$$

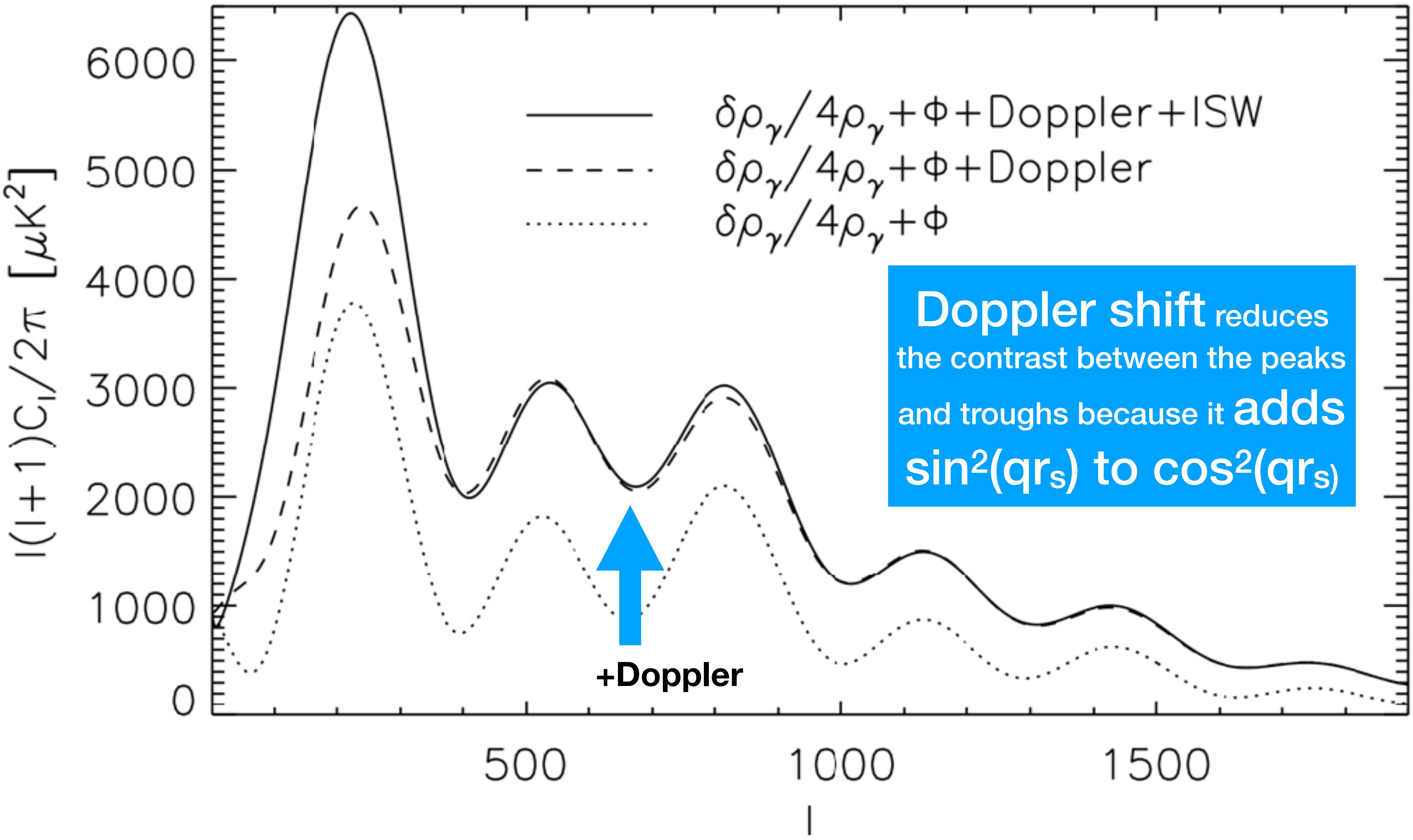
Velocity potential is a
time-derivative
of the energy density:
cos(qr_s) becomes sin(qr_s)!

Temperature Anisotropy from Doppler Shift

$$\frac{q}{a}\delta u_\gamma = \frac{\sqrt{3}\zeta}{5}(1+R)^{-3/4}\mathcal{S}(\kappa)\sin[qr_s + \theta(\kappa)]$$

- To this, we should multiply the damping factor

$$\exp(-q^2/q^2_{\text{Damp}})$$



6000
5000
4000
3000
2000
1000
0
500
1000
1500
l
l(l+1)C_l/2π [μK²]
δρ_γ/4ρ_γ+Φ+Doppler+ISW
δρ_γ/4ρ_γ+Φ+Doppler
δρ_γ/4ρ_γ+Φ
Doppler shift reduces the contrast between the peaks and troughs because it adds sin²(qr_s) to cos²(qr_s)
+Doppler

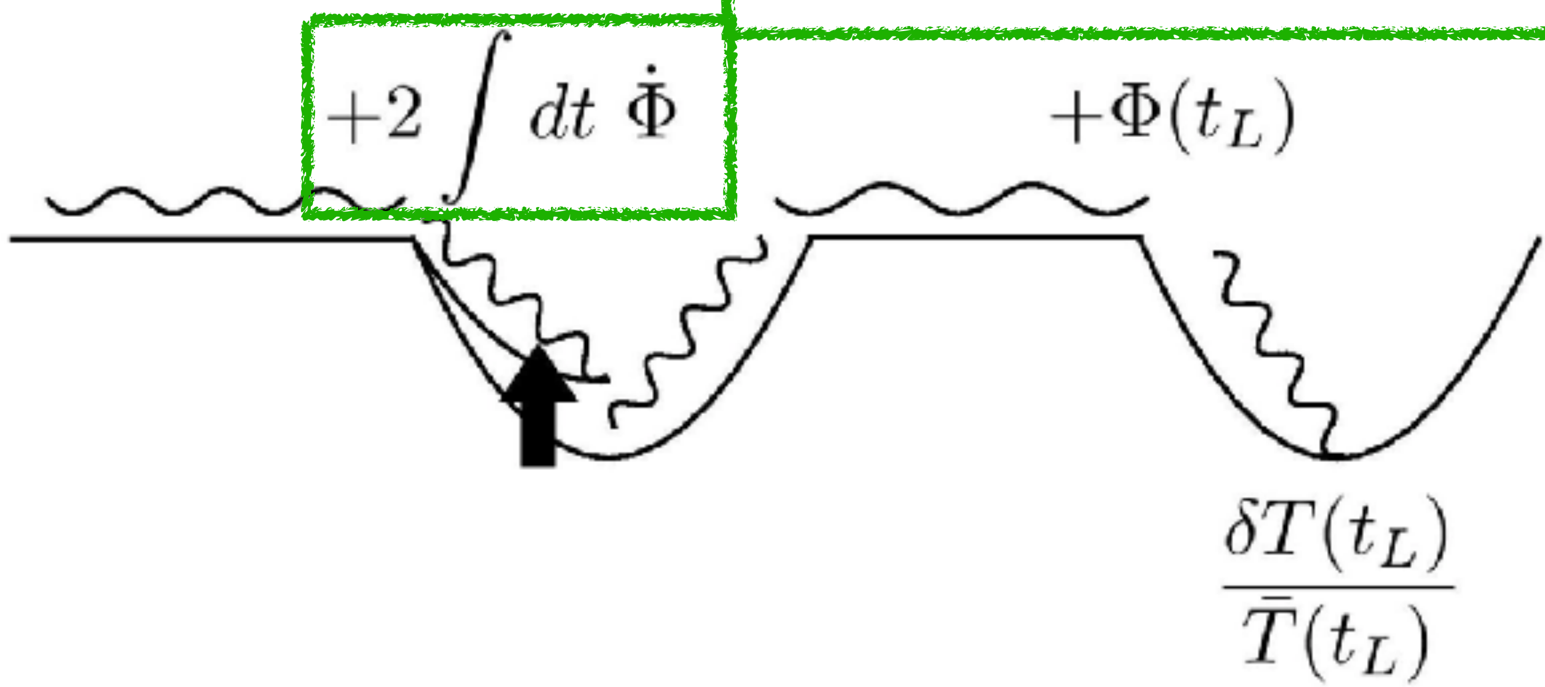
Hu & Sugiyama (1996)

(Early) ISW

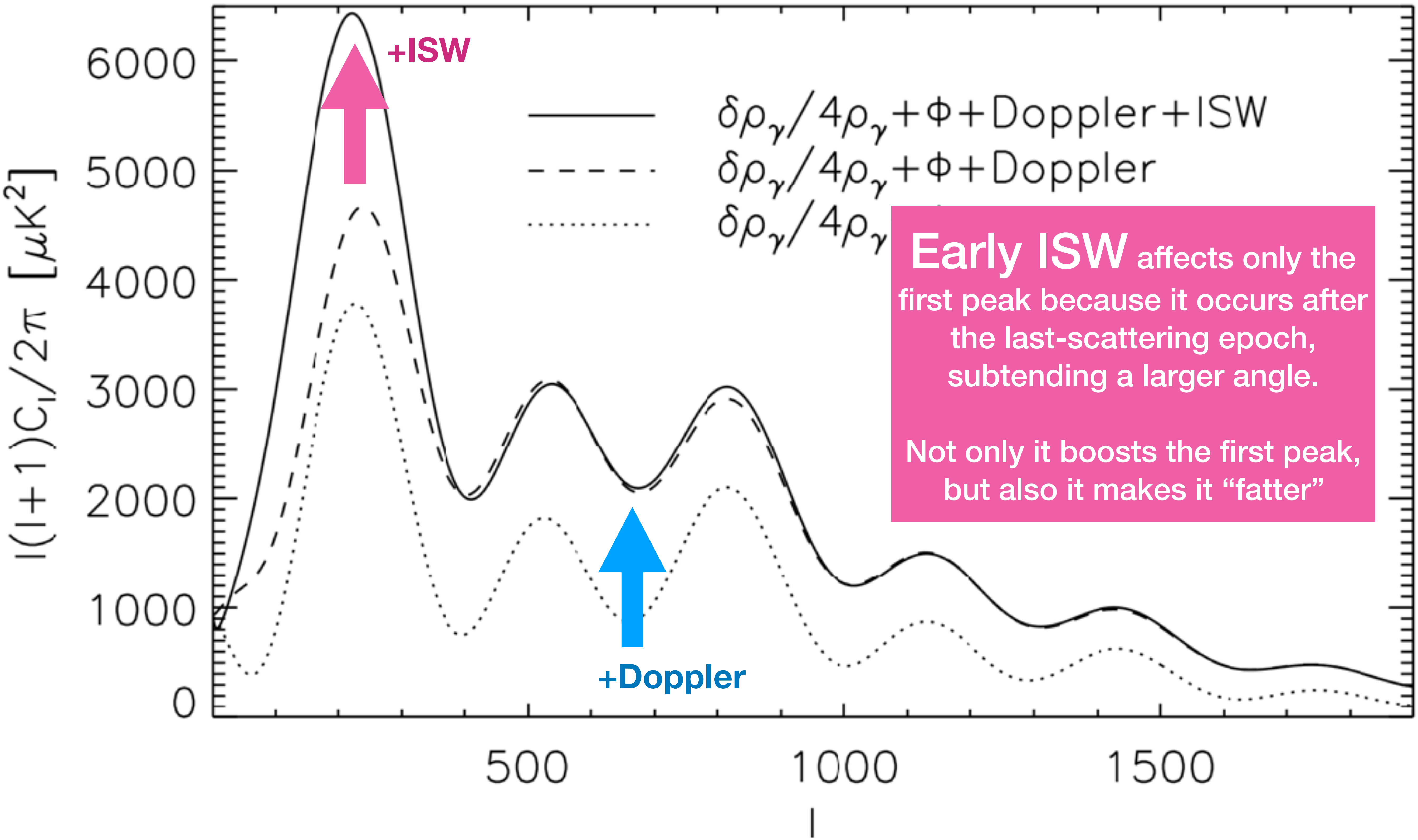
$$\frac{\Delta T(\hat{n})}{T_0} = \frac{\delta T(t_L, \hat{n} r_L)}{\bar{T}(t_L)} + \Phi(t_L, \hat{n} r_L) - \Phi(t_0, 0)$$

"integrated Sachs-Wolfe" (ISW) effect

$$+ \int_{t_L}^{t_0} dt\ (\dot{\Phi} + \dot{\Psi})(t, \hat{n} r)$$



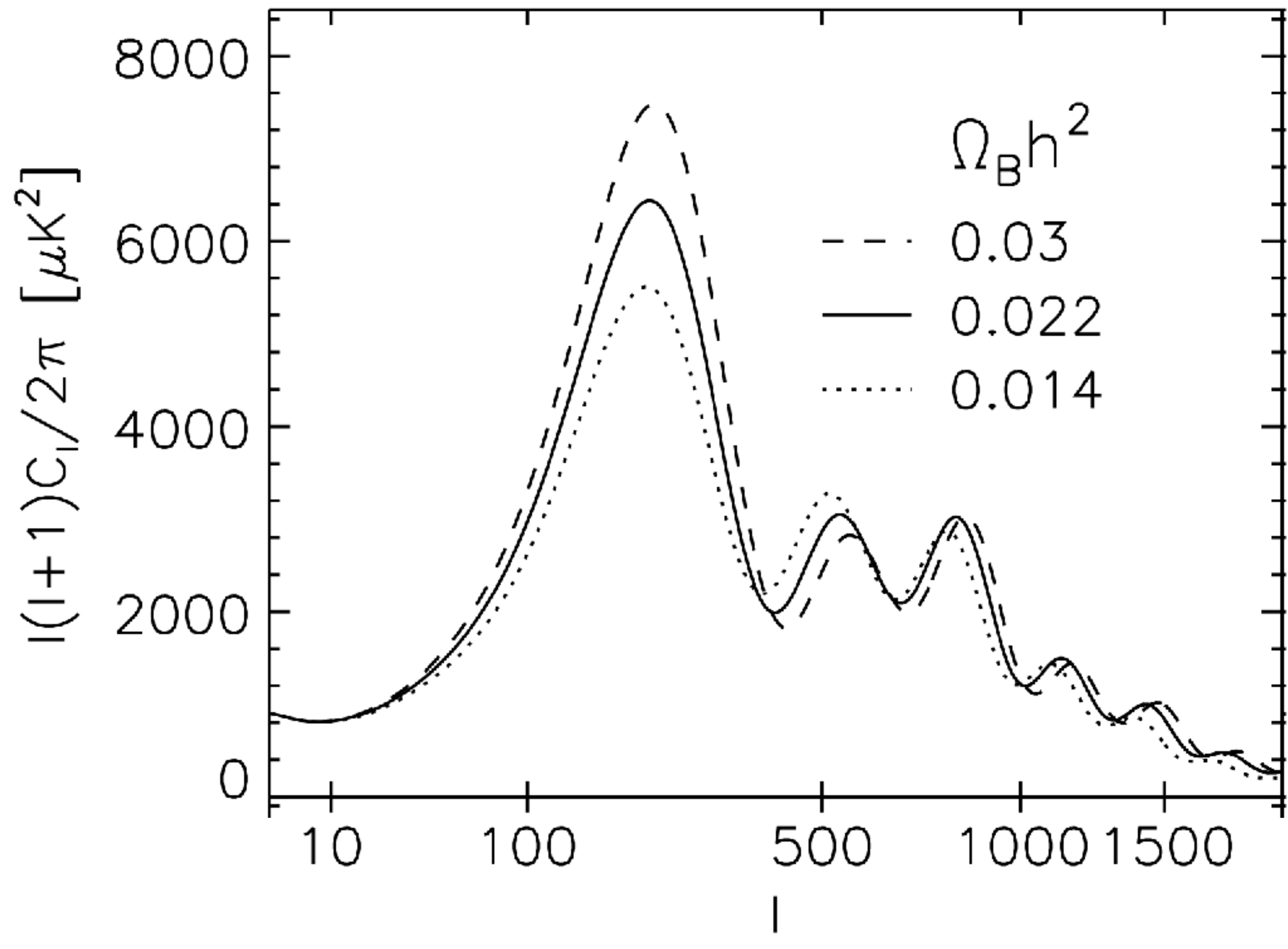
Gravitational potentials still decay after last-scattering because the Universe then was not completely matter-dominated yet



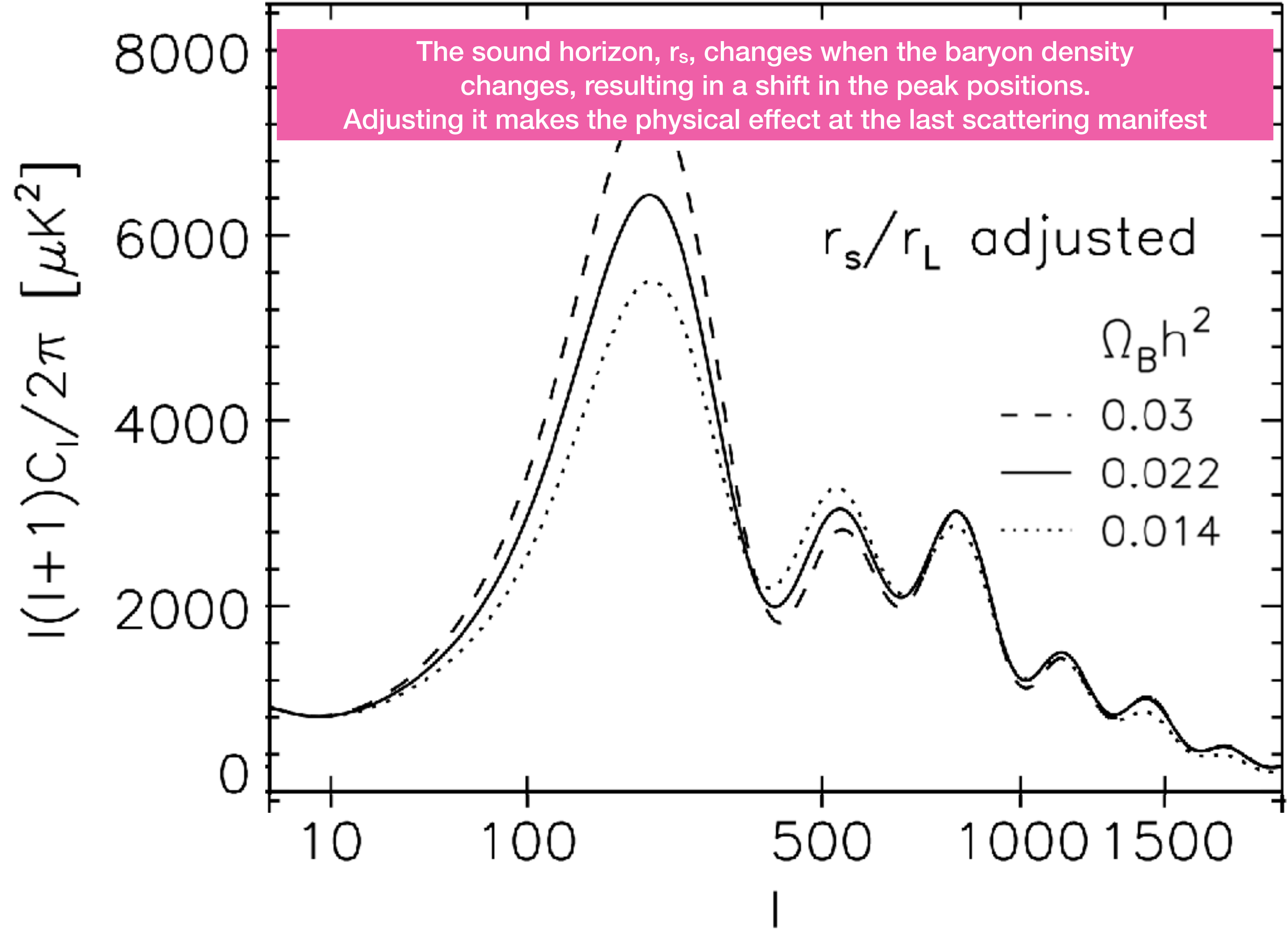
+ISW
$\delta\rho_\gamma/4\rho_\gamma + \Phi + \text{Doppler} + \text{ISW}$
$\delta\rho_\gamma/4\rho_\gamma + \Phi + \text{Doppler}$
$\delta\rho_\gamma/4\rho_\gamma$
Early ISW affects only the first peak because it occurs after the last-scattering epoch, subtending a larger angle.
Not only it boosts the first peak, but also it makes it "fatter"
+Doppler
$l(l+1)C_l/2\pi\ [\mu K^2]$
6000
5000
4000
3000
2000
1000
0
500
1000
1500
l

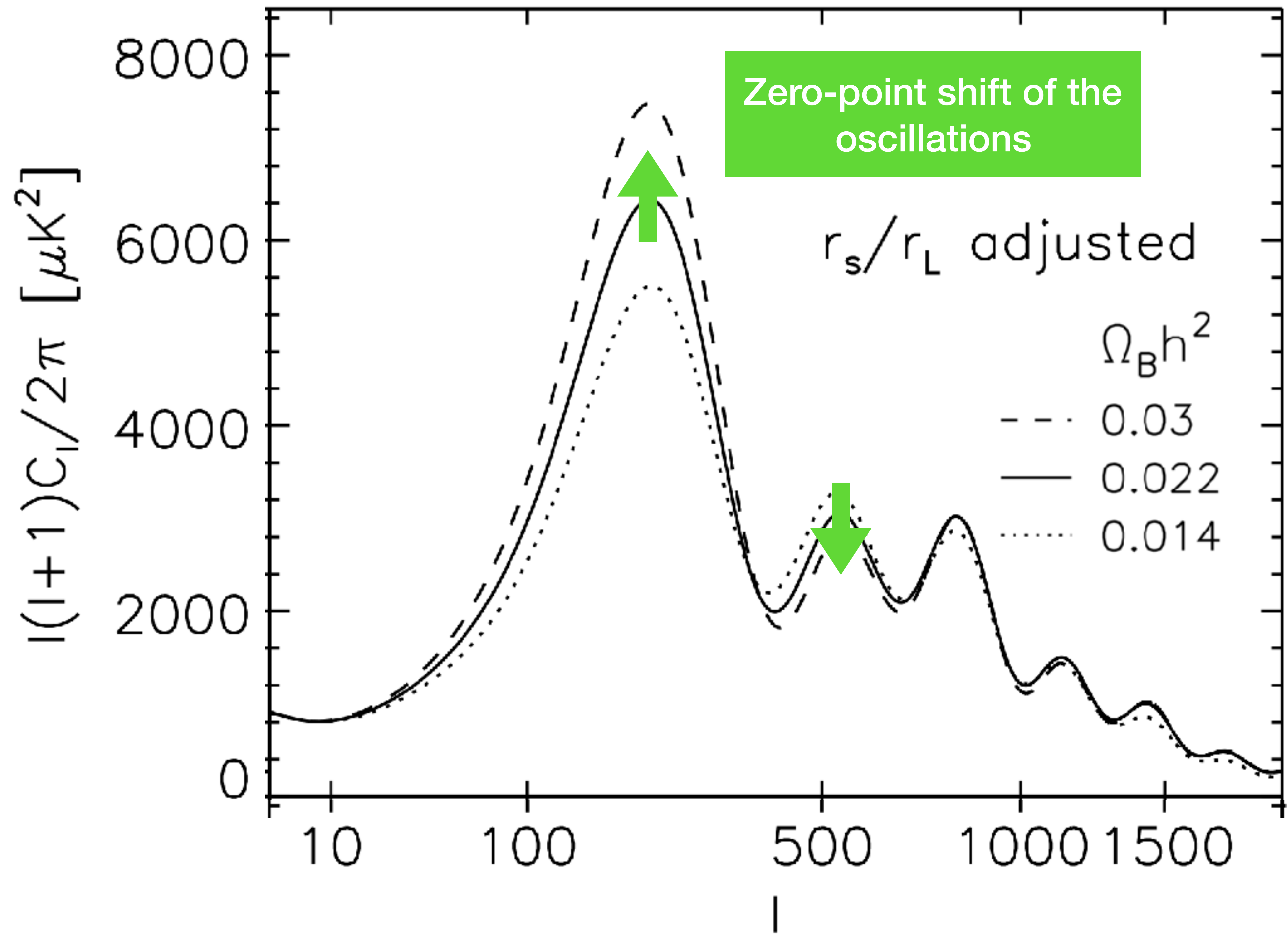
We are ready!

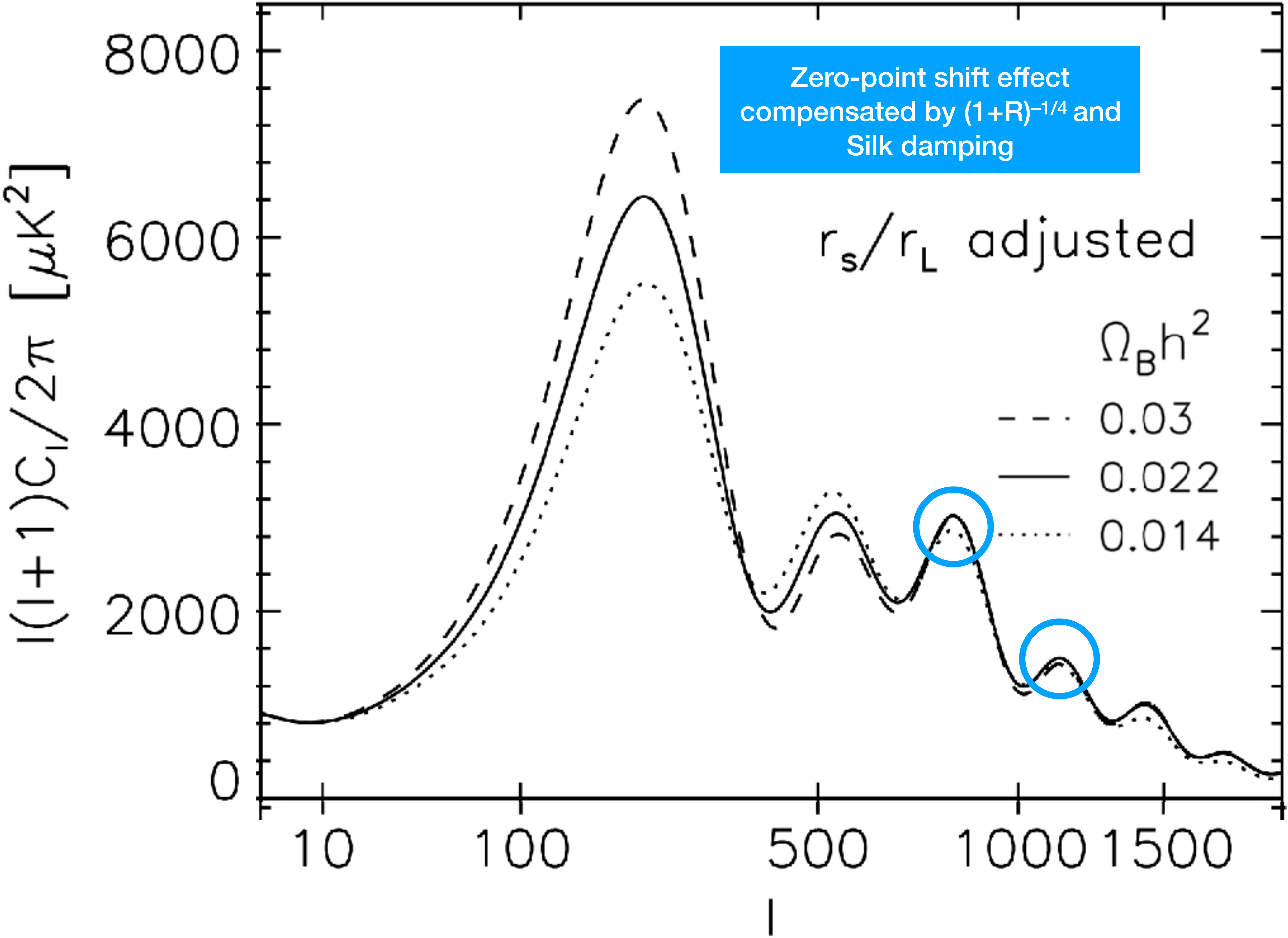
- We are ready to understand the effects of all the cosmological parameters.
- Let’s start with the baryon density



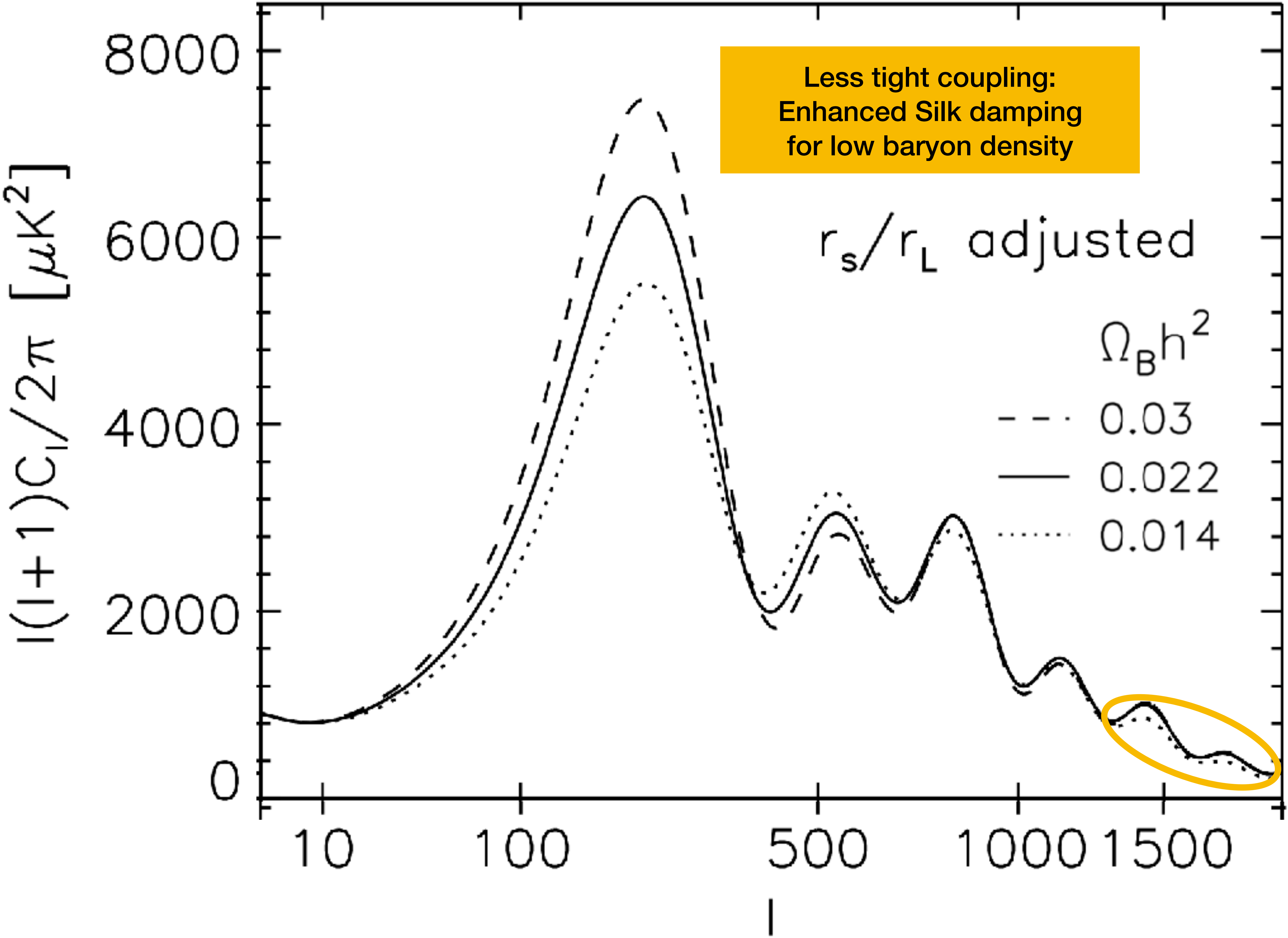
$l(l+1)C_l/2\pi$ [μK^2]
8000
6000
4000
2000
0
10
100
500
1000
1500
l
$\Omega_B h^2$
0.03
0.022
0.014


The sound horizon, r_s, changes when the baryon density
changes, resulting in a shift in the peak positions.
Adjusting it makes the physical effect at the last scattering manifest
r_s/r_L adjusted
$\Omega_B h^2$
0.03
0.022
0.014
$l(l+1)C_l/2\pi$ [μK^2]
8000
6000
4000
2000
0
10
100
500
1000
1500
l

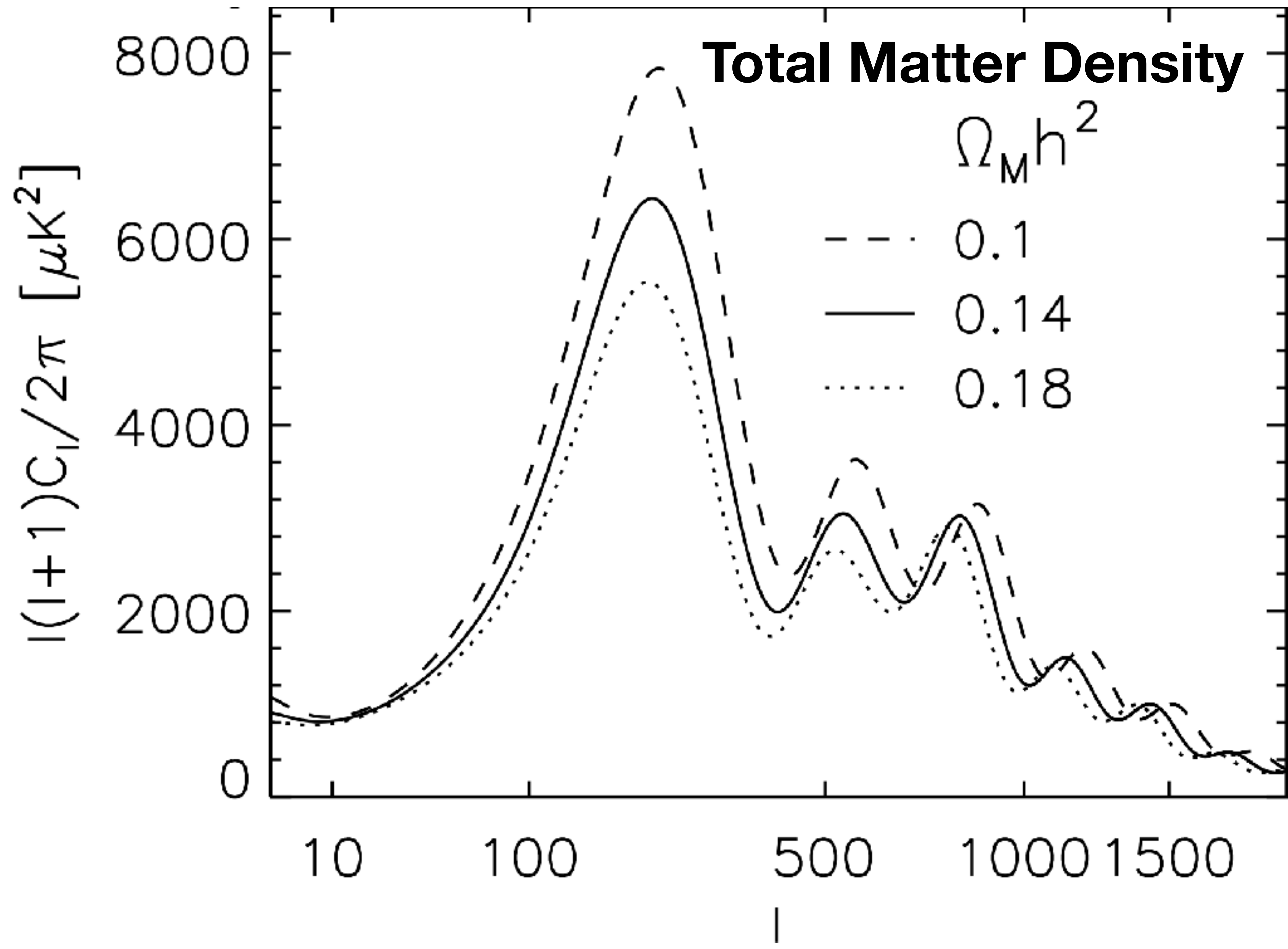

Zero-point shift of the oscillations
r_s/r_L adjusted
$\Omega_B h^2$
0.03
0.022
0.014
8000
6000
4000
2000
0
10
100
500
1000
1500
l
$l(l+1)C_l/2\pi$ [μK^2]



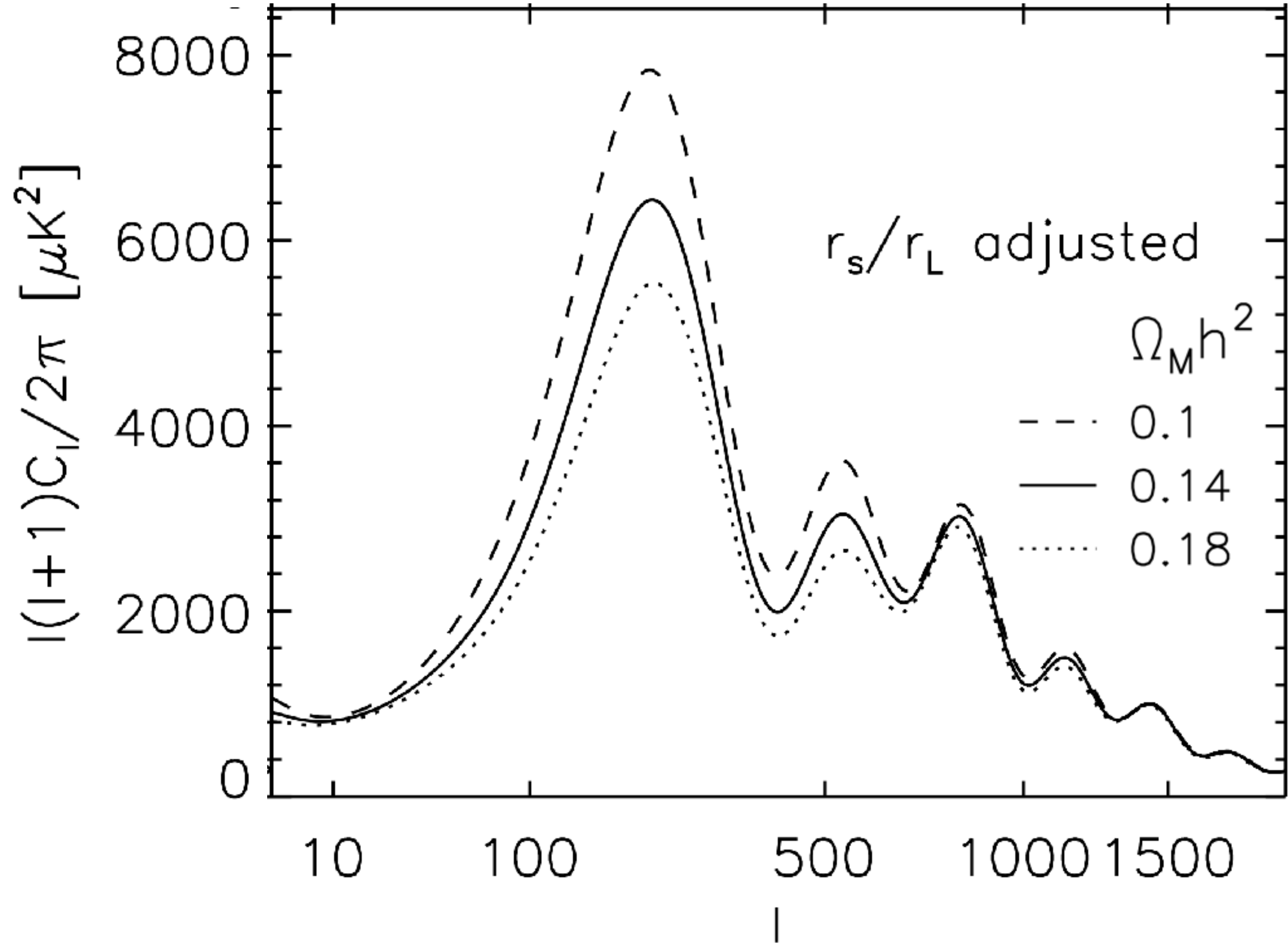
Zero-point shift effect compensated by (1+R)^-1/4 and Silk damping
r_s/r_L adjusted
Ω_B h^2
0.03
0.022
0.014
8000
6000
4000
2000
0
l(l+1)C_l/2π [μK^2]
10
100
500
1000
1500
l



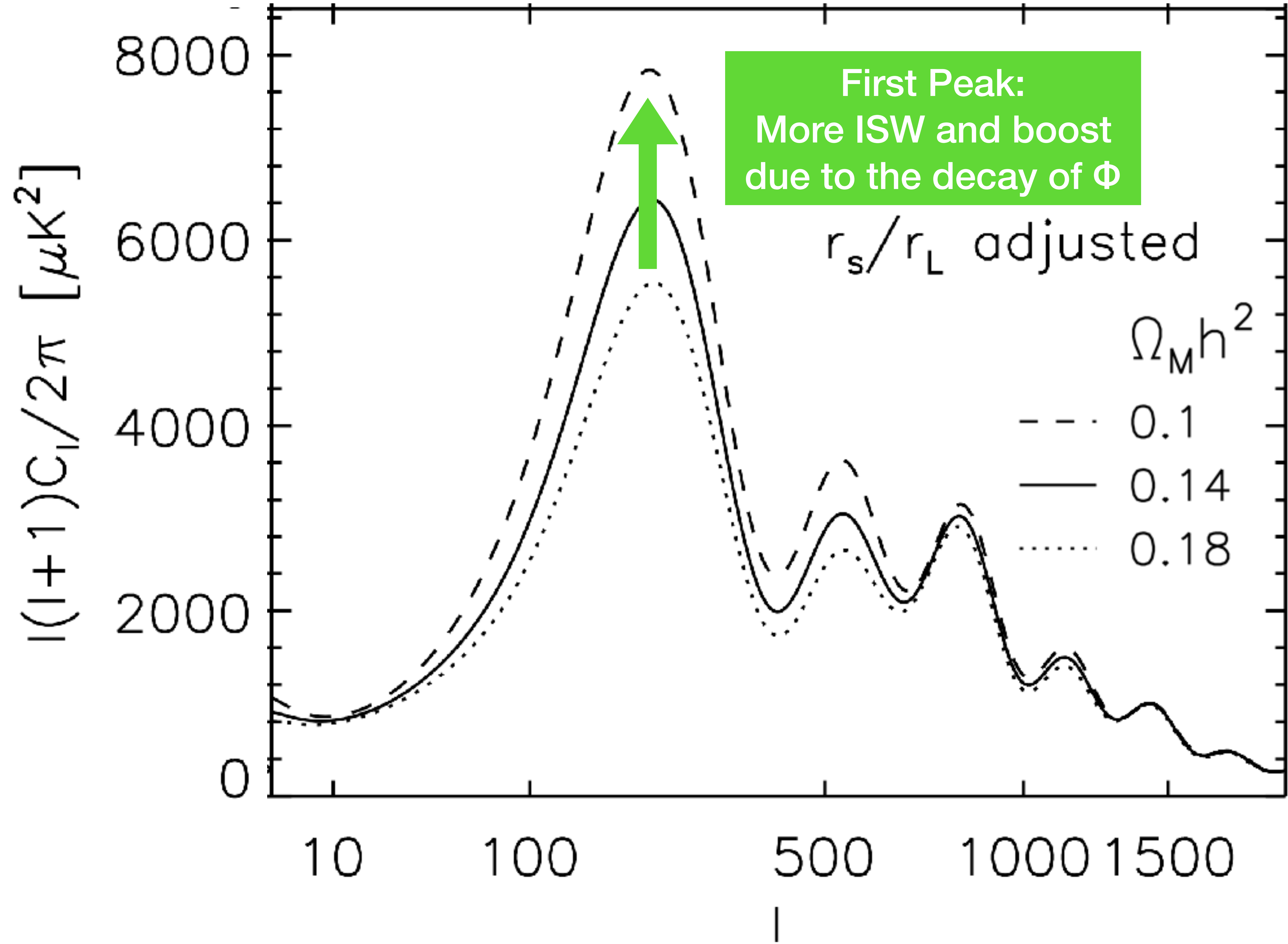
Less tight coupling:
Enhanced Silk damping
for low baryon density
r_s/r_L adjusted
$\Omega_B h^2$
0.03
0.022
0.014
8000
6000
4000
2000
0
$l(l+1)C_l/2\pi\ [\mu K^2]$
10
100
500
1000
1500
l



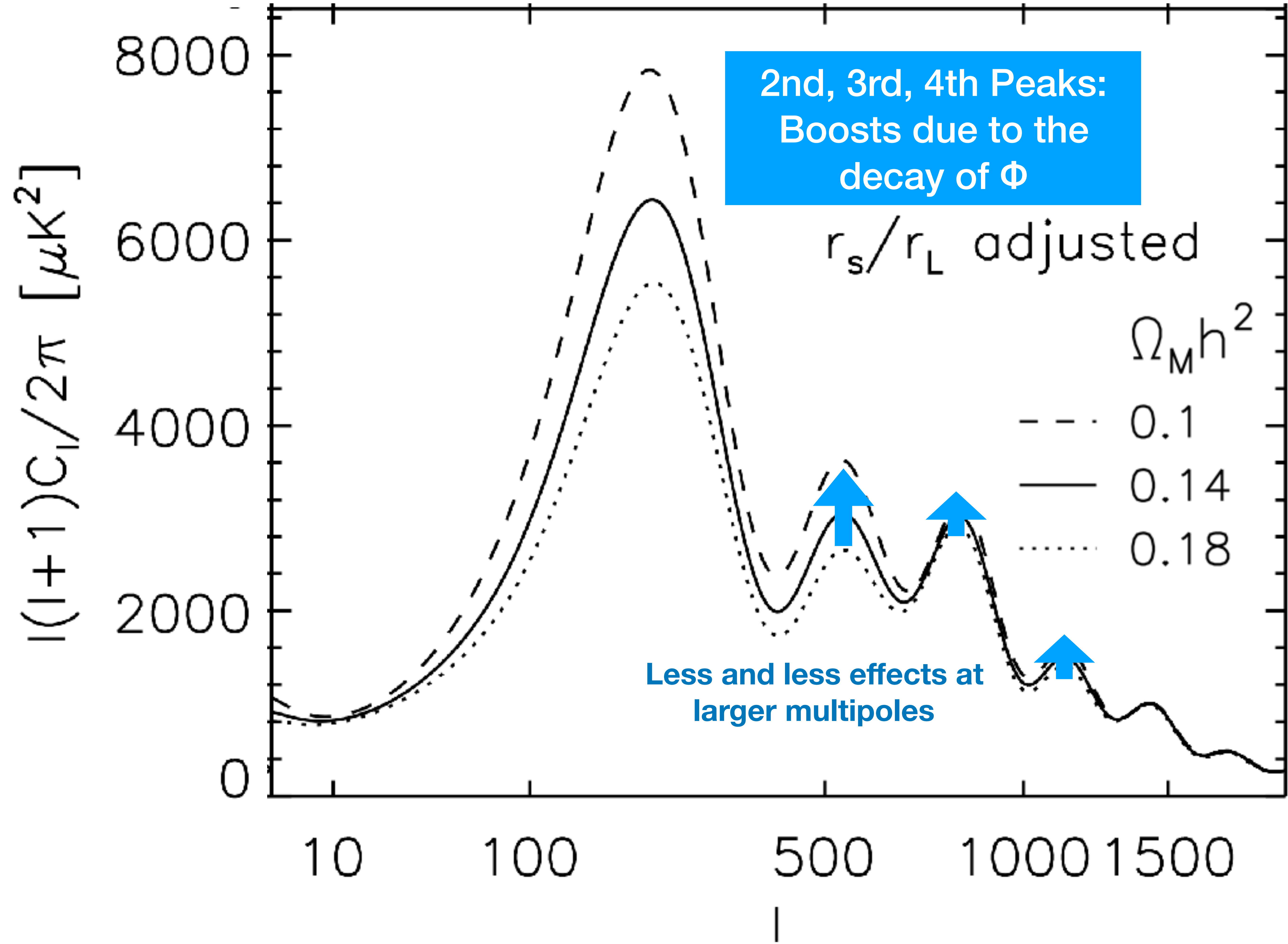
Total Matter Density
$\Omega_M h^2$
0.1
0.14
0.18
8000
6000
4000
2000
0
$l(l+1)C_l/2\pi$ [μK^2]
10
100
500
1000
1500
l



8000
6000
4000
2000
0
$l(l+1)C_l/2\pi$ [μK^2]
10
100
500
1000
1500
l
r_s/r_L adjusted
$\Omega_M h^2$
0.1
0.14
0.18



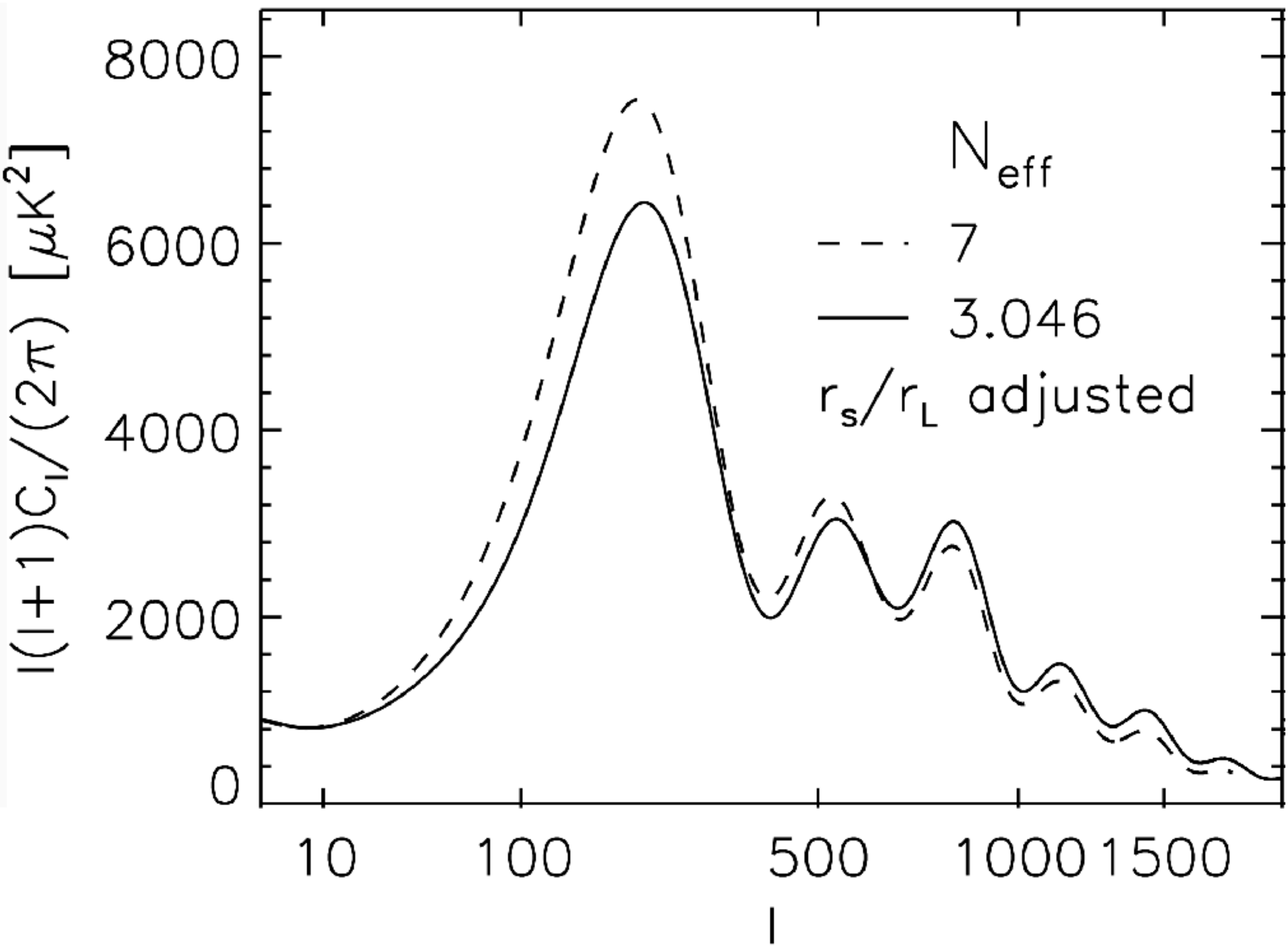
First Peak:
More ISW and boost due to the decay of Φ
r_s/r_L adjusted
$\Omega_M h^2$
0.1
0.14
0.18
8000
6000
4000
2000
0
10
100
500
1000
1500
l
$l(l+1)C_l/2\pi$ [μK^2]

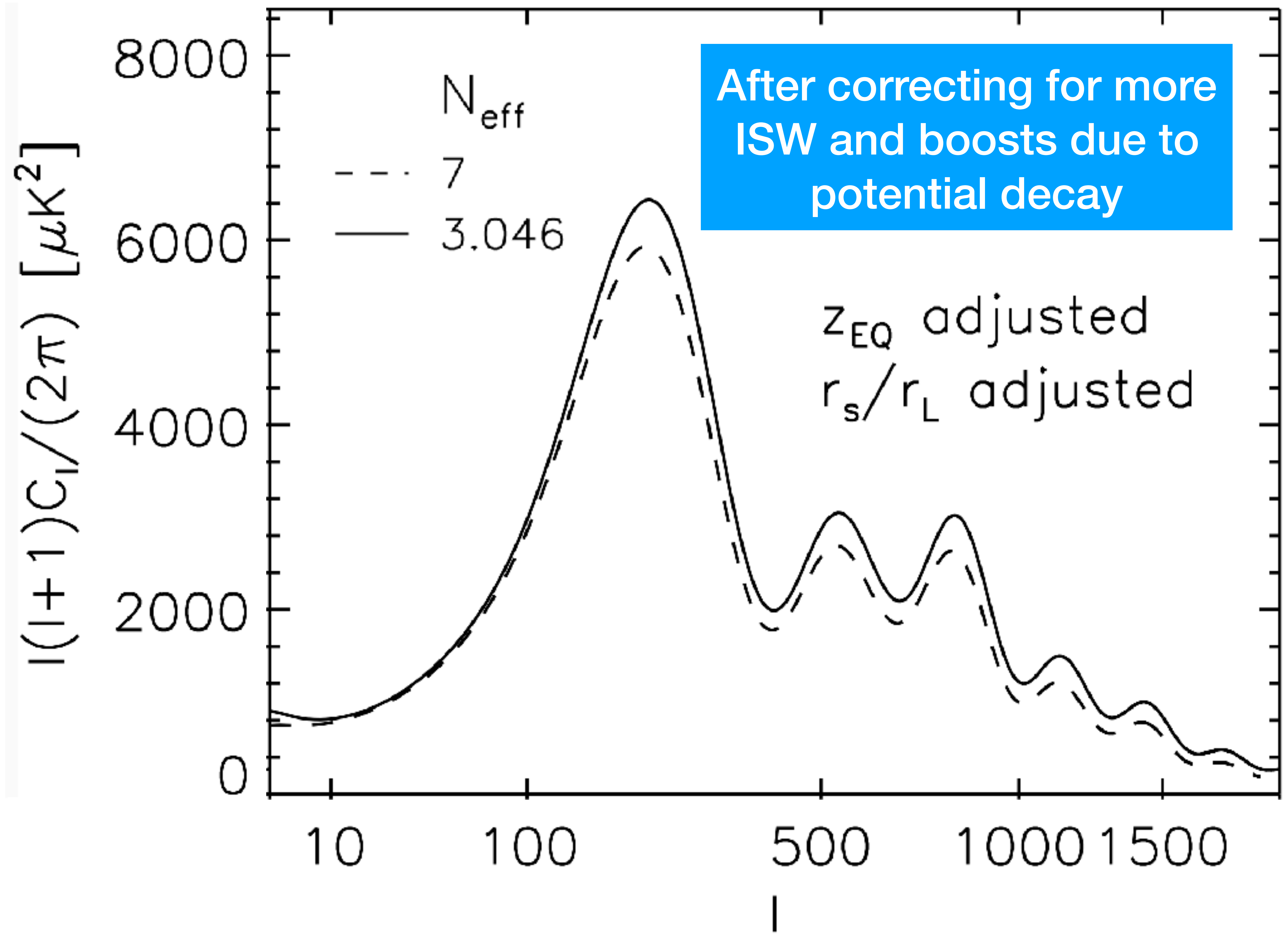


2nd, 3rd, 4th Peaks:
Boosts due to the
decay of Φ
r_s/r_L adjusted
Ω_M h^2
0.1
0.14
0.18
Less and less effects at
larger multipoles
l(l+1)C_l/2π [μK^2]
8000
6000
4000
2000
0
10
100
500
1000
1500
l

Effects of Relativistic Neutrinos

- To see the effects of relativistic neutrinos, we artificially increase the number of neutrino species from 3 to 7
  - Great energy density in neutrinos, i.e., greater energy density in radiation
  - (1) Longer radiation domination -> More ISW and boosts due to potential decay


l(l+1)C_l/(2π) [μK^2]
8000
6000
4000
2000
0
10
100
500
1000
1500
l
N_eff
7
3.046
r_s/r_L adjusted



After correcting for more ISW and boosts due to potential decay
N_{eff}
7
3.046
z_{EQ} adjusted
r_s/r_L adjusted
$l(l+1)C_l/(2\pi)$ [μK^2]
8000
6000
4000
2000
0
10
100
500
1000
1500
l

(2): Viscosity Effect on the Amplitude of Sound Waves

The solution is

$$X = -C\cos(\varphi + \theta)$$

$$R_\nu \equiv \bar{\rho}_\nu/(\bar{\rho}_\gamma + \bar{\rho}_\nu) \approx 0.409$$

where

$$C \equiv \sqrt{(-\zeta + \Delta A_\nu)^2 + \Delta B_\nu^2}$$

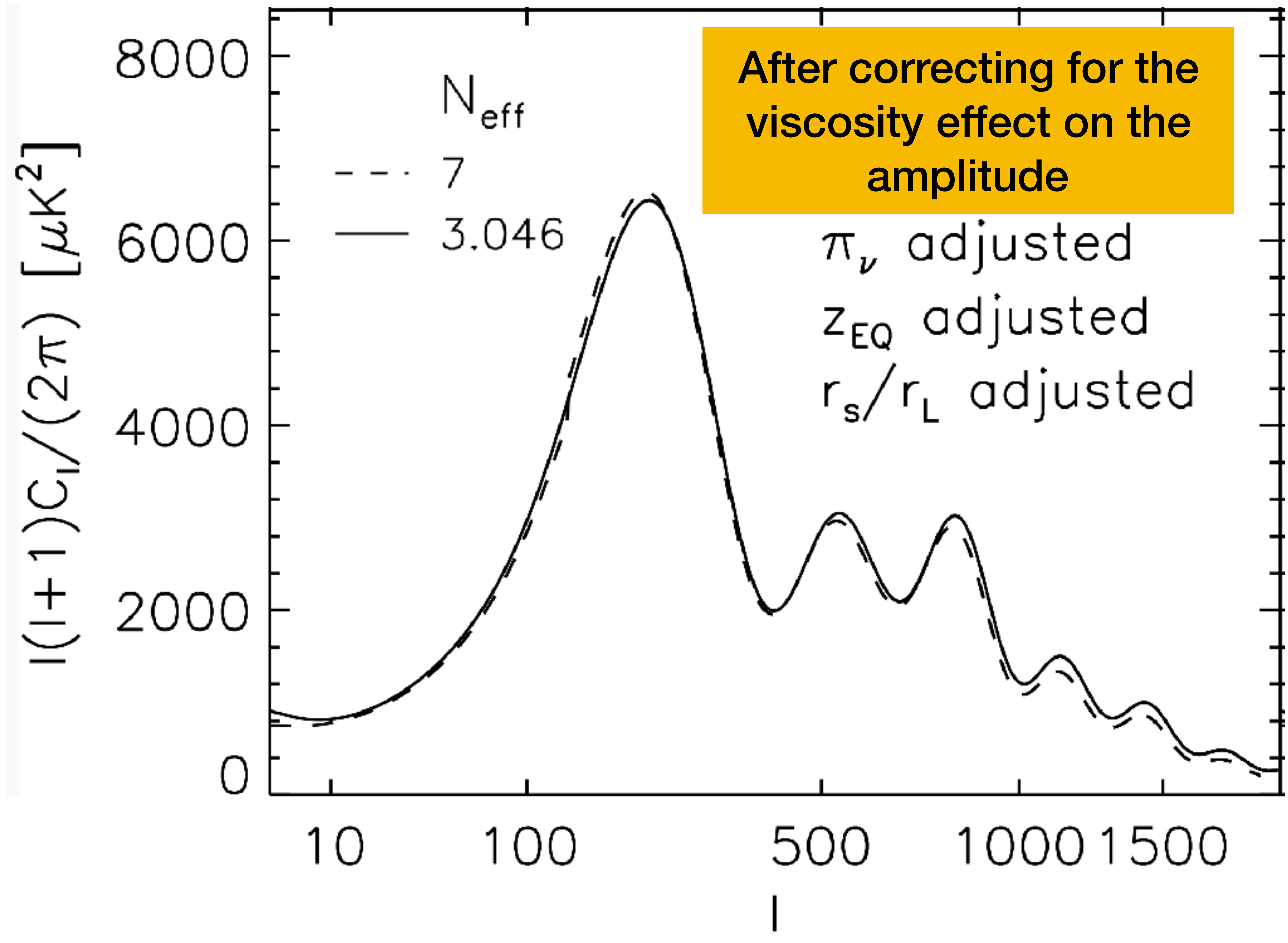
$$\approx \zeta(1 + 4R_\nu/15)^{-1}$$

Hu & Sugiyama (1996)

$$\tan\theta = -\frac{\Delta B_\nu}{-\zeta + \Delta A_\nu} \approx 0.063\pi$$

Phase shift!

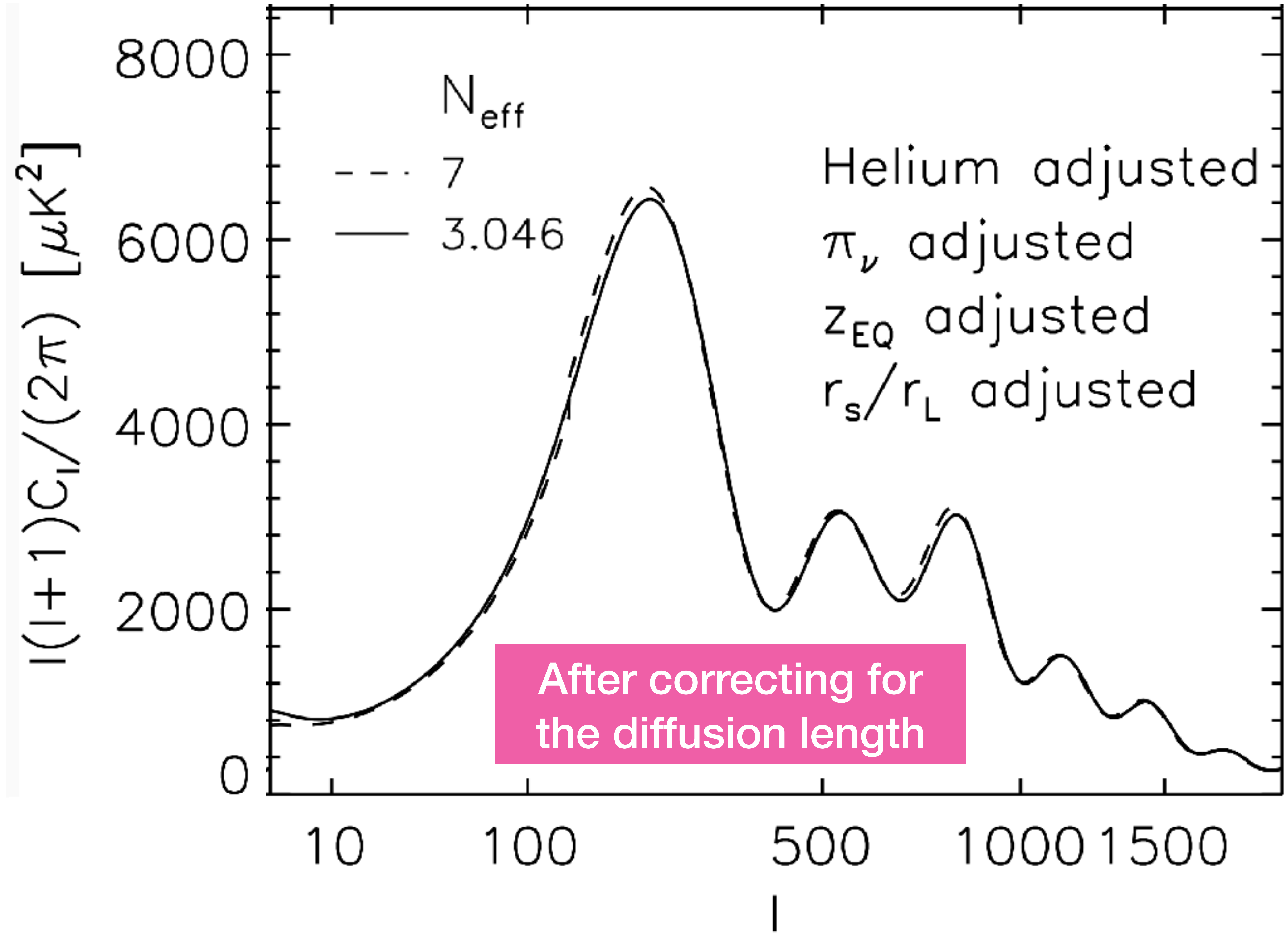
Bashinsky & Seljak (2004)


After correcting for the viscosity effect on the amplitude
N_eff
7
3.046
π_ν adjusted
z_EQ adjusted
r_s/r_L adjusted
8000
6000
4000
2000
0
l(l+1)C_l/(2π) [μK^2]
10
100
500
1000
1500
l

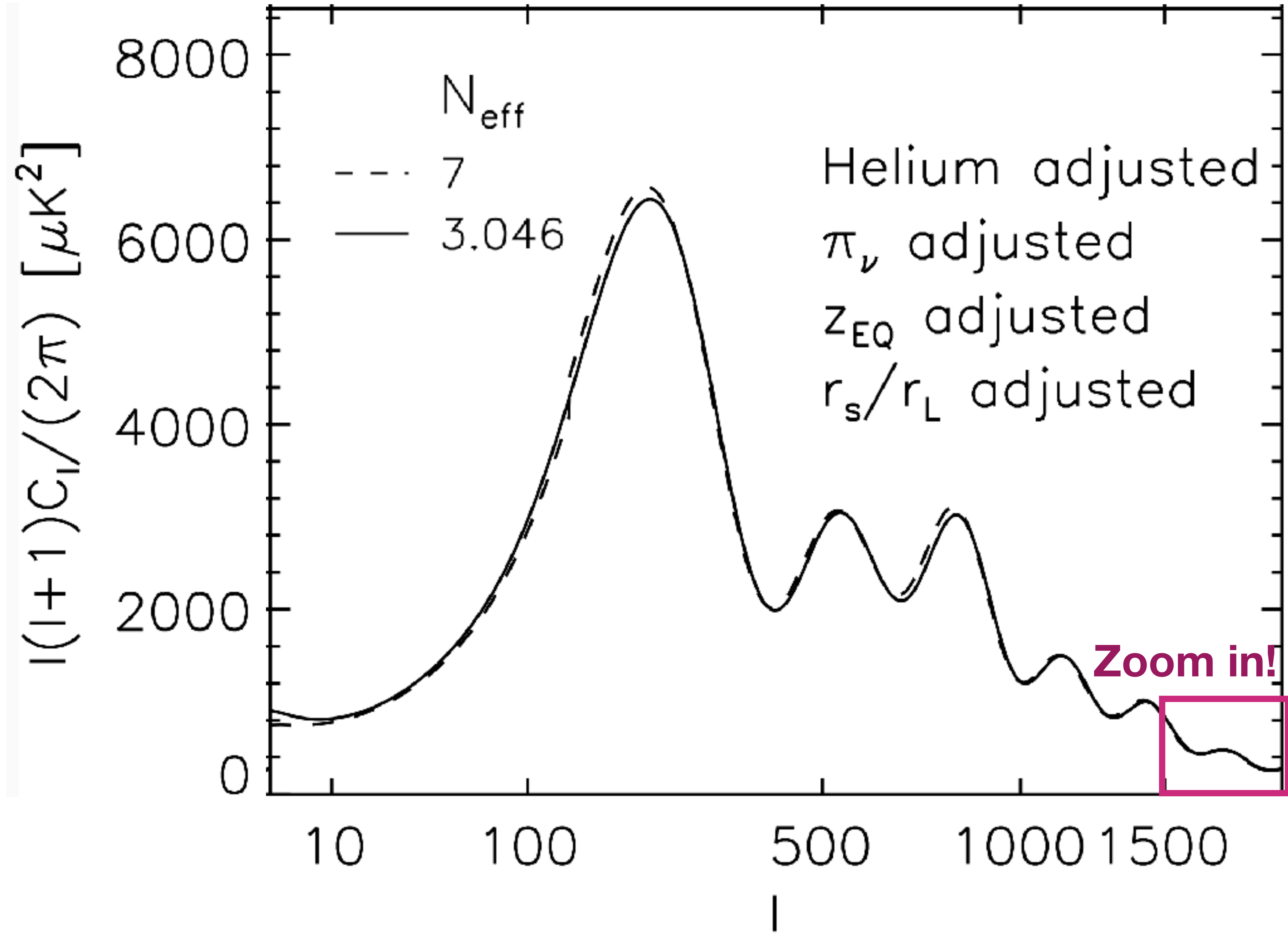
Bashinsky & Seljak (2004)

(3): Change in the Silk Damping

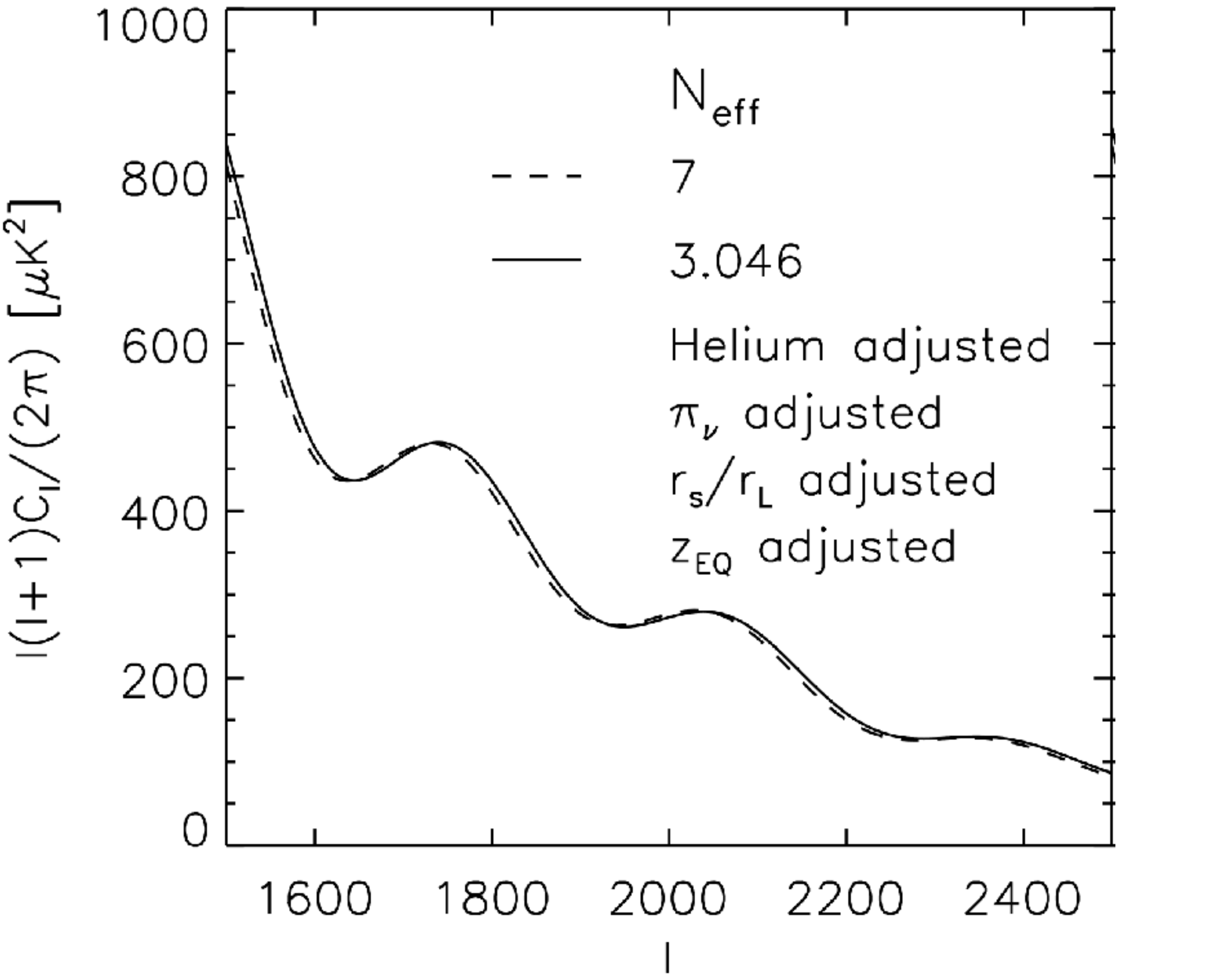
- Greater neutrino energy density implies greater Hubble expansion rate, $H^2=8\pi G\sum\rho_\alpha/3$
- This **reduces** the sound horizon in proportion to H^{-1}, as $r_s \sim c_s H^{-1}$
- This also reduces the diffusion length, but in proportional to $H^{-1/2}$, as $a/q_{silk} \sim (\sigma_T n_e H)^{-1/2}$ **Consequence of the random walk!**
- As a result, l_{silk} **decreases relative to the first peak position**, enhancing the Silk damping



N_eff
7
3.046
Helium adjusted
π_ν adjusted
z_EQ adjusted
r_s/r_L adjusted
After correcting for the diffusion length
l(l+1)C_l/(2π) [μK²]
8000
6000
4000
2000
0
10
100
500
1000
1500
l



$l(l+1)C_l/(2\pi)$ [μK^2]
8000
6000
4000
2000
0
N_{eff}
7
3.046
Helium adjusted
π_ν adjusted
z_{EQ} adjusted
r_s/r_L adjusted
Zoom in!
10
100
500
1000
1500
l



1000
800
600
400
200
0
1600
1800
2000
2200
2400
l
l(l+1)C_l/(2π) [μK^2]
N_eff
7
3.046
Helium adjusted
π_ν adjusted
r_s/r_L adjusted
z_EQ adjusted

(4): Viscosity Effect on the Phase of Sound Waves

$$R_\nu \equiv \bar{\rho}_\nu/(\bar{\rho}_\gamma + \bar{\rho}_\nu) \approx 0.409$$

The solution is

$$X = -C\cos(\varphi + \theta)$$

where

$$C \equiv \sqrt{(-\zeta + \Delta A_\nu)^2 + \Delta B_\nu^2}$$

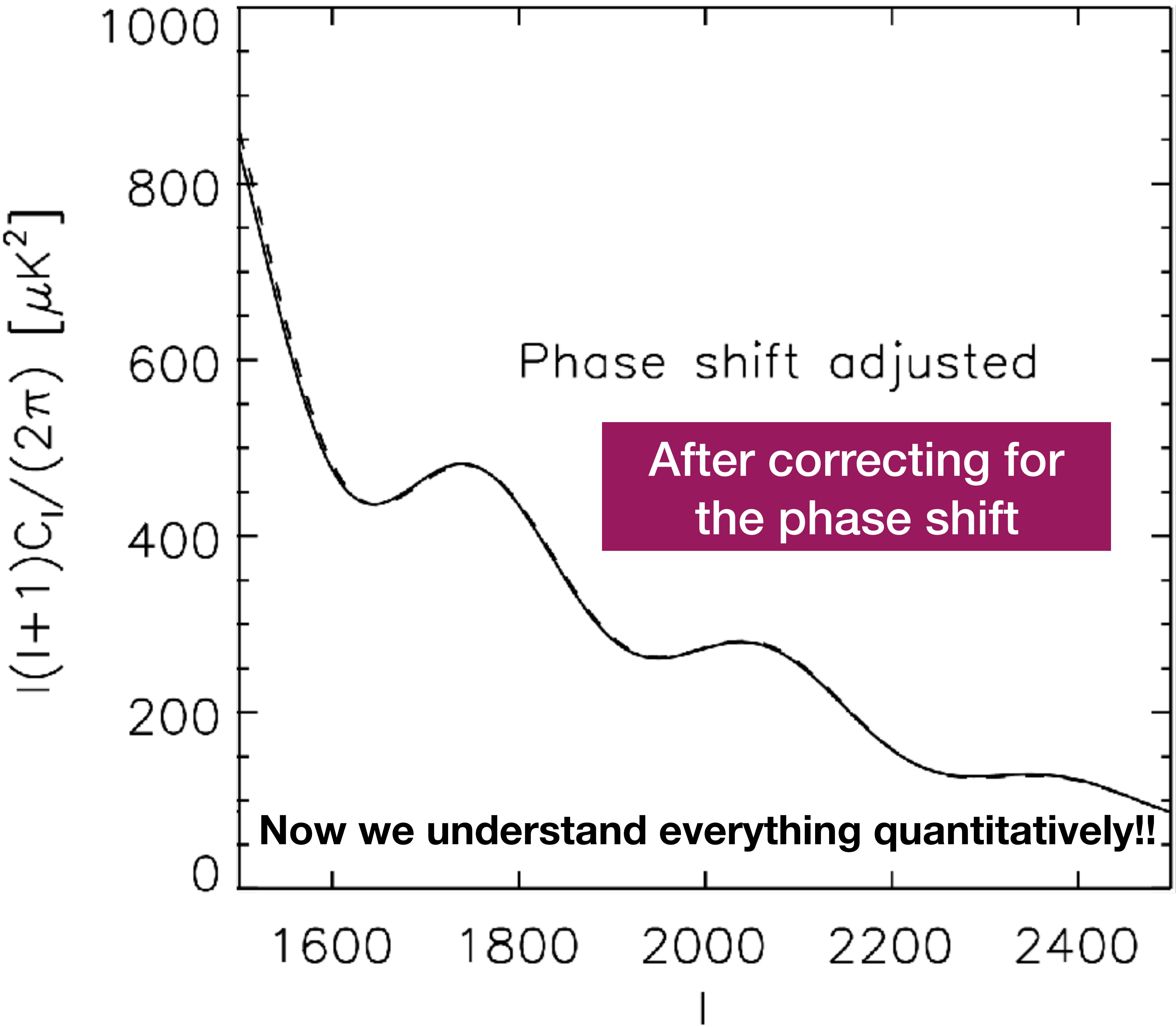
$$\approx \zeta(1 + 4R_\nu/15)^{-1}$$

Hu & Sugiyama (1996)

$$\tan\theta = -\frac{\Delta B_\nu}{\zeta + \Delta A_\nu} \approx 0.063\pi$$

Phase shift!

Bashinsky & Seljak (2004)



1000
800
600
400
200
0
1600
1800
2000
2200
2400
l
l(l+1)C_l/(2π) [μK^2]
Phase shift adjusted
After correcting for the phase shift
Now we understand everything quantitatively!!

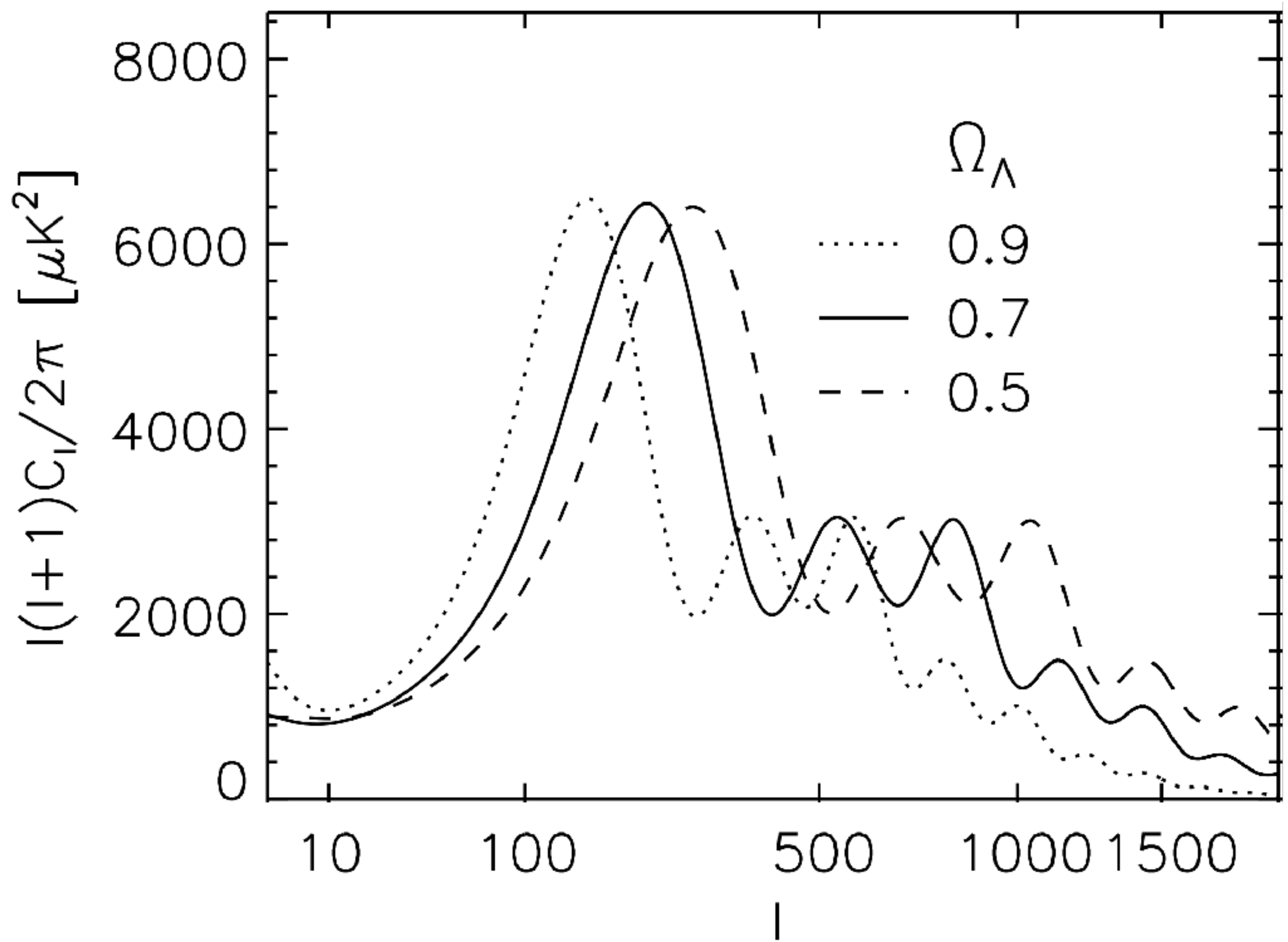
Two Other Effects

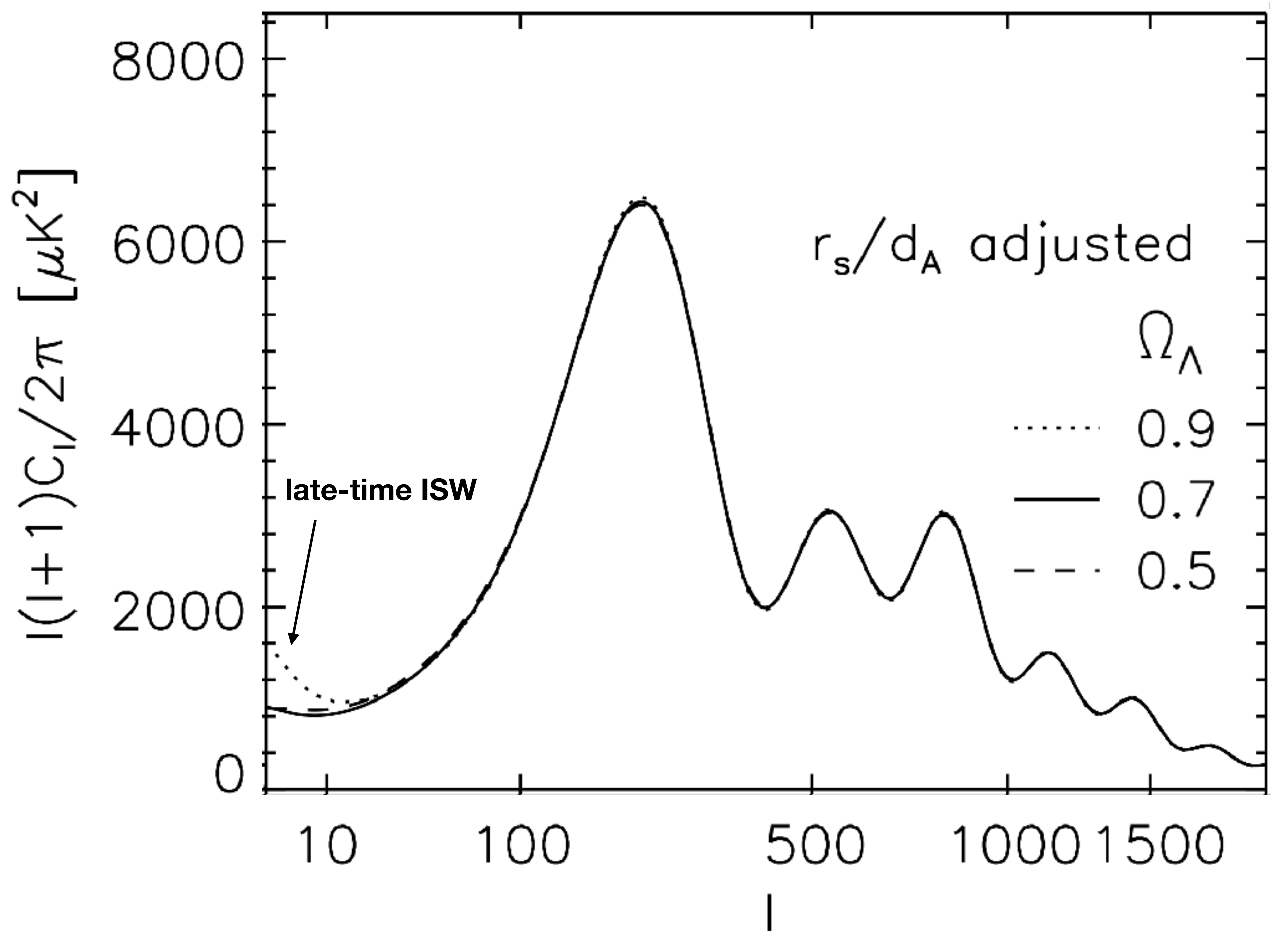
- **Spatial curvature**
  - We have been assuming spatially-flat Universe with zero curvature (i.e., Euclidean space). What if it is curved?
- **Optical depth to Thomson scattering in a low-redshift Universe**
  - We have been assuming that the Universe is transparent to photons since the last scattering at z=1090. What if there is an extra scattering in a low-redshift Universe?

Spatial Curvature

- It changes the angular diameter distance, d_A, to the last scattering surface; namely,
  - $r_L \to d_A = R\,\mathit{sin}(r_L/R) = r_L(1 - r_L^2/6R^2) + \ldots$ for positively-curved space
  - $r_L \to d_A = R\,\mathit{sinh}(r_L/R) = r_L(1 + r_L^2/6R^2) + \ldots$ for negatively-curved space

Smaller angles (larger multipoles) for a negatively curved Universe


8000
6000
4000
2000
0
$l(l+1)C_l/2\pi$ [μK^2]
Ω_Λ
0.9
0.7
0.5
10
100
500
1000
1500
l



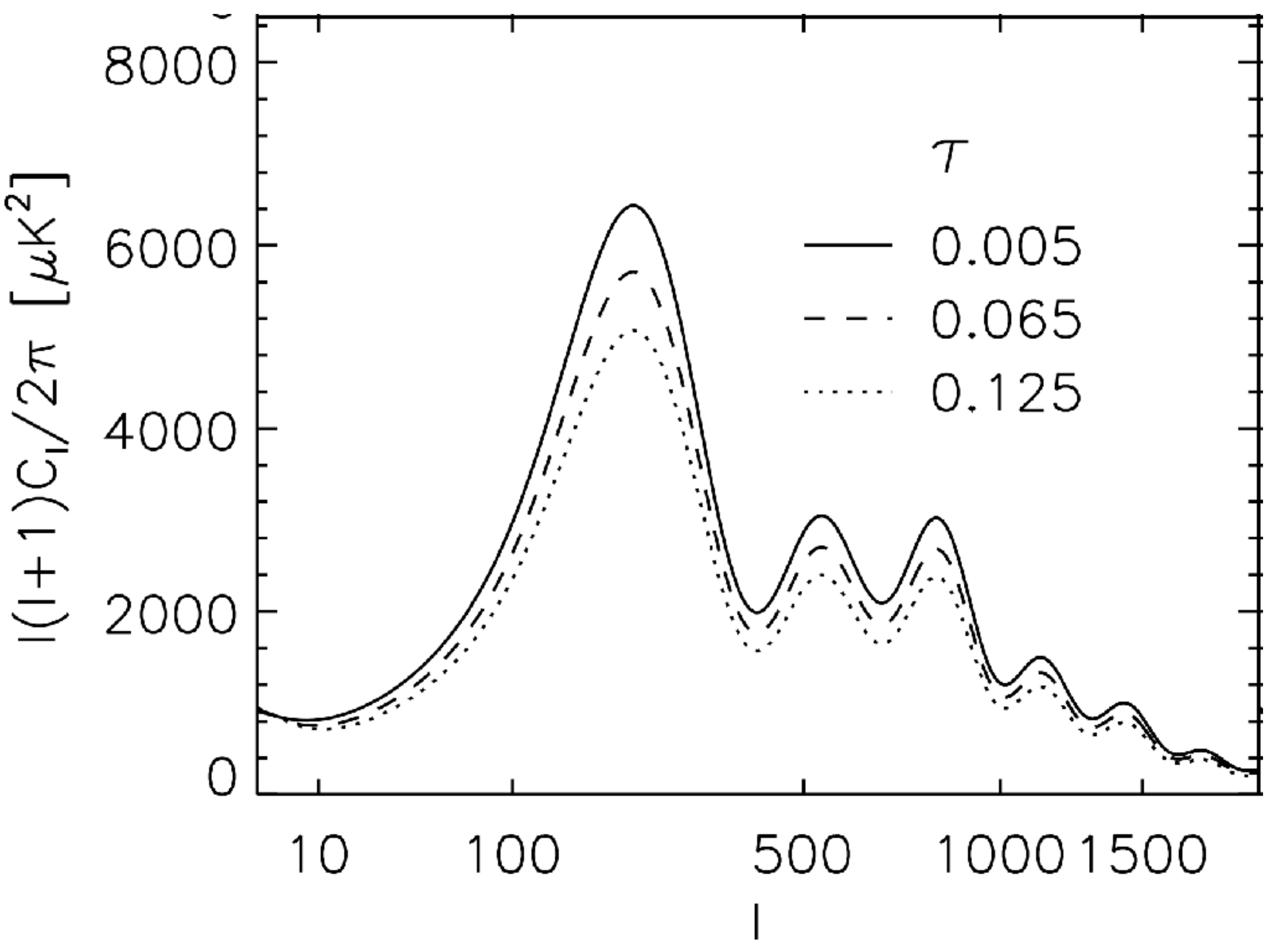
8000
6000
4000
2000
0
$l(l+1)C_l/2\pi$ [μK^2]
10
100
500
1000
1500
l
r_s/d_A adjusted
Ω_Λ
0.9
0.7
0.5
late-time ISW

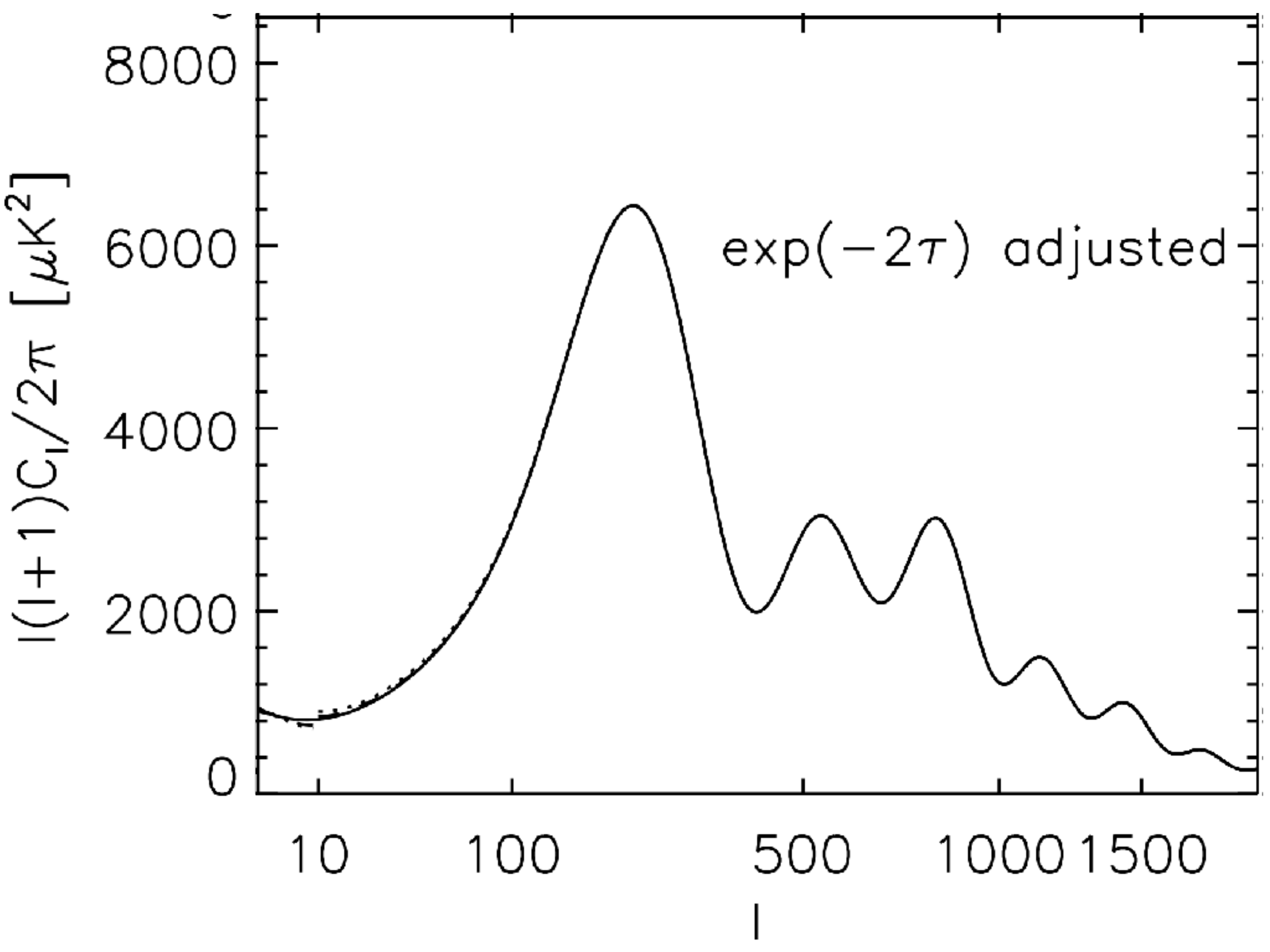
Optical Depth

- Extra scattering by electrons in a low-redshift Universe damps temperature anisotropy

- $C_l \rightarrow C_l \exp(-2\tau)$ at l >~ 10
  - where τ is the optical depth

$$\tau = c\sigma_T \int_{t_{\text{re-ionisation}}}^{t_0} dt\ \bar{n}_e$$


8000
6000
4000
2000
0
$l(l+1)C_l/2\pi$ [μK^2]
τ
0.005
0.065
0.125
10
100
500
1000
1500
l



8000
6000
4000
2000
0
l(l+1)C_l/2π [μK^2]
exp(−2τ) adjusted
10
100
500
1000
1500
l

Important consequence of the optical depth

- Since the power spectrum is uniformly suppressed by $\exp(-2\tau)$ at $l>\sim 10$, we cannot determine the amplitude of the power spectrum of the gravitational potential, $P_\Phi(q)$, independently of τ.
    - Namely, what we constrain is the combination:

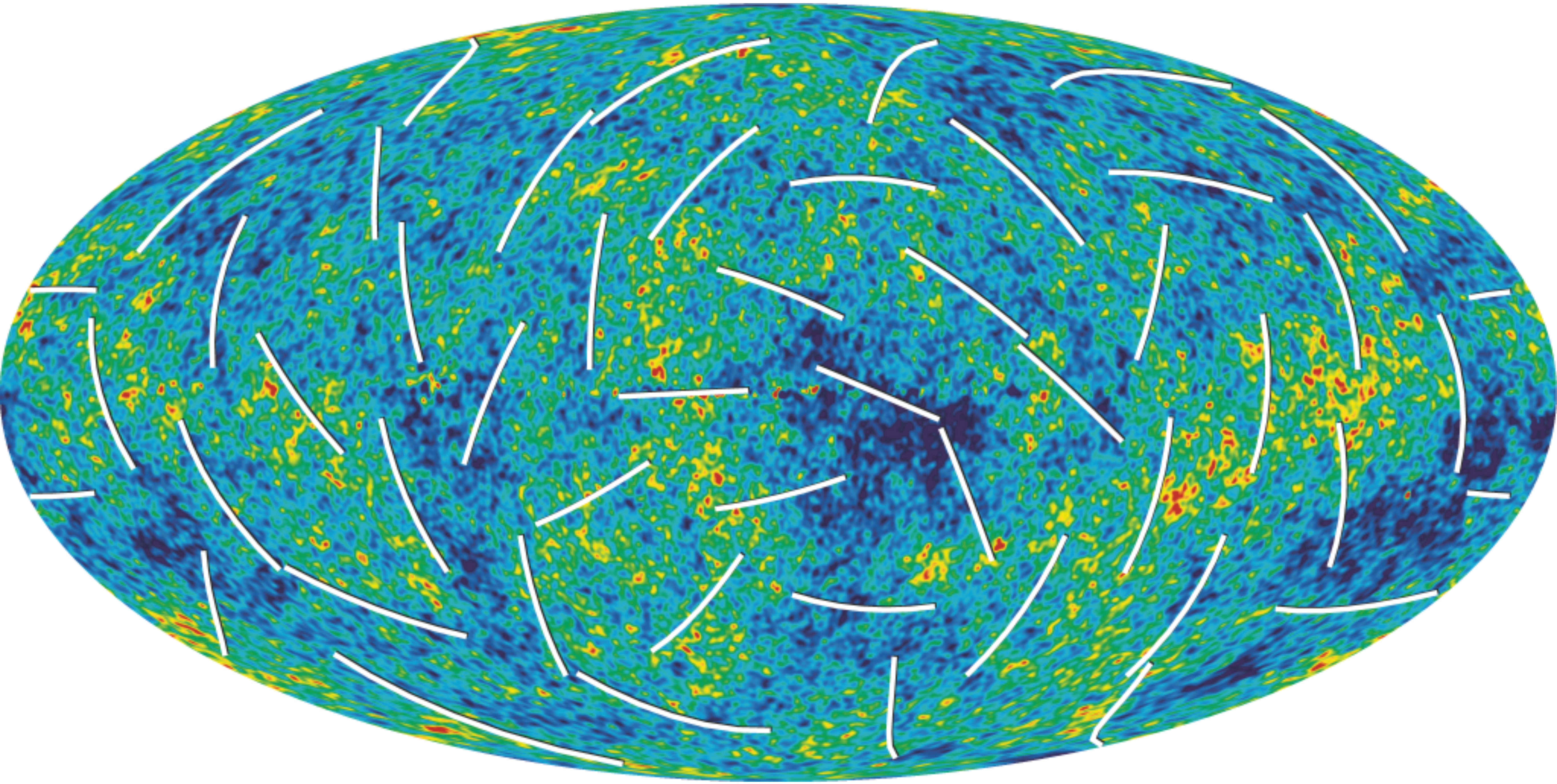
$$\exp(-2\tau)P_\Phi(q) \propto \exp(-2\tau)A_s$$

- Breaking this degeneracy requires an independent determination of the optical depth. This requires **POLARISATION** of the CMB.

Cosmological Parameters Derived from the Power Spectrum

	WMAP	Planck	+CMB Lensing
$100\Omega_B h^2$	2.264 ± 0.050	2.222 ± 0.023	2.226 ± 0.023
$\Omega_D h^2$	0.1138 ± 0.0045	0.1197 ± 0.0022	0.1186 ± 0.0020
Ω_Λ	0.721 ± 0.025	0.685 ± 0.013	0.692 ± 0.012
n	0.972 ± 0.013	0.9655 ± 0.0062	0.9677 ± 0.0060
$10^9 A_s$	2.203 ± 0.067	$2.198^{+0.076}_{-0.085}$	2.139 ± 0.063
τ	0.089 ± 0.014	0.078 ± 0.019	0.066 ± 0.016
t_0 [100 Myr]	137.4 ± 1.1	138.13 ± 0.38	137.99 ± 0.38
H_0	70.0 ± 2.2	67.31 ± 0.96	67.81 ± 0.92
$\Omega_M h^2$	0.1364 ± 0.0044	0.1426 ± 0.0020	0.1415 ± 0.0019
$10^9 A_s e^{-2\tau}$	1.844 ± 0.031	1.880 ± 0.014	1.874 ± 0.013
σ_8	0.821 ± 0.023	0.829 ± 0.014	0.8149 ± 0.0093

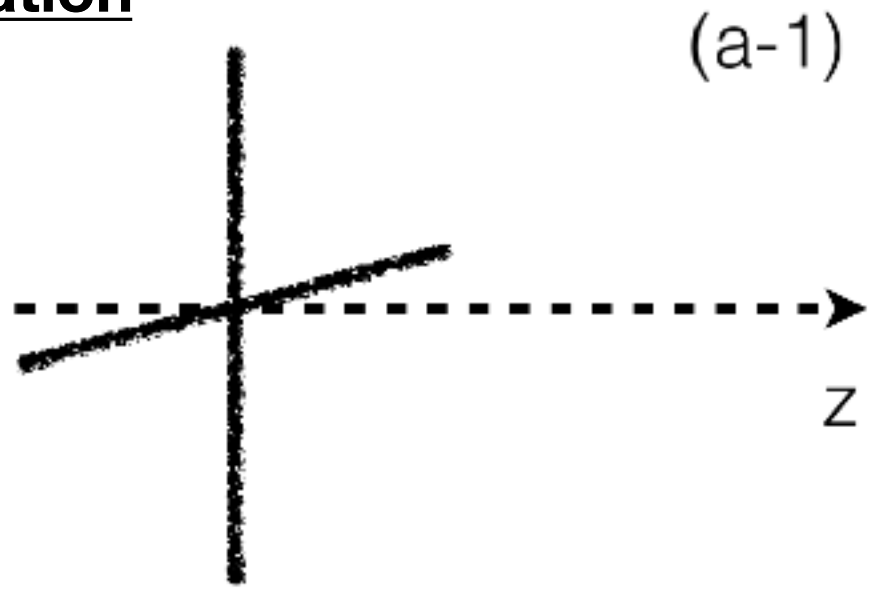
CMB Polarisation

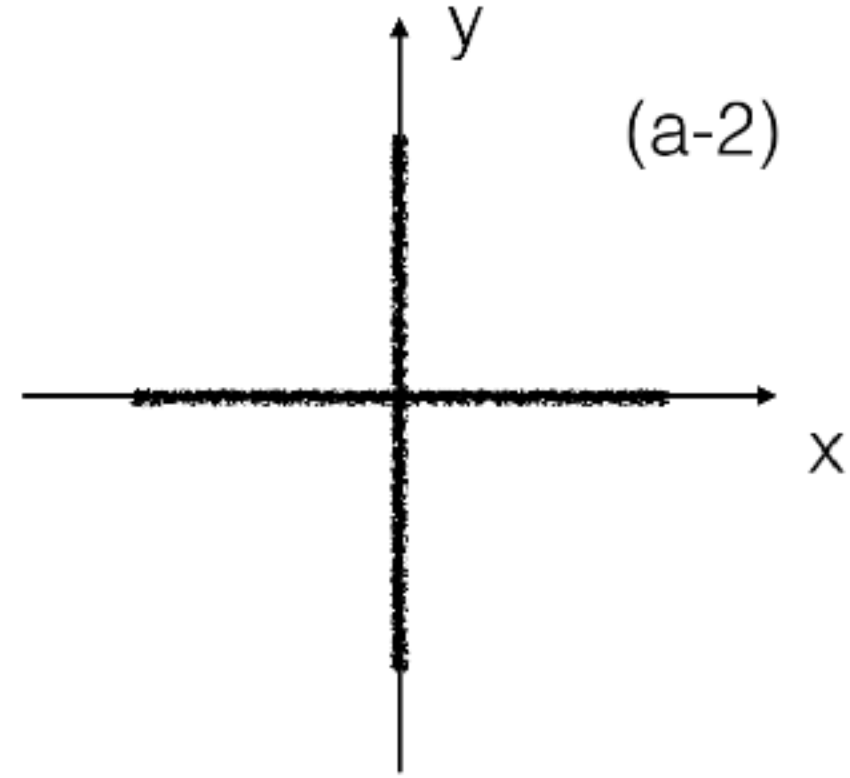


- CMB is weakly polarised!

Polarisation

No polarisation





(b-1)

Polarised in x-direction

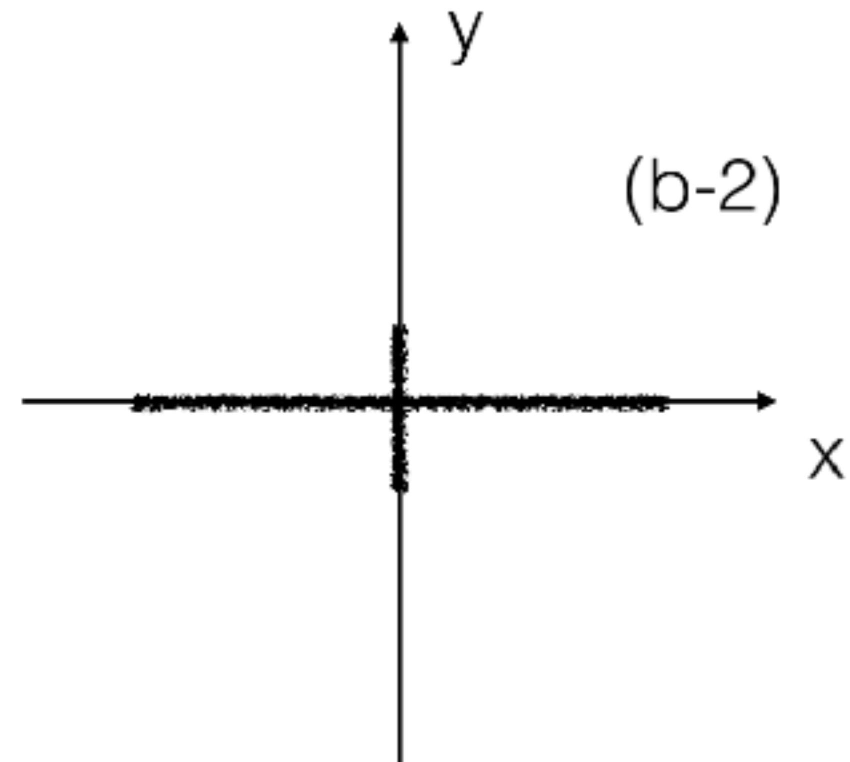


Photo Credit: TALEX


Photo Credit: TALEX
horizontally polarised

Photo Credit: TALEX

Necessary and sufficient conditions for generating polarisation

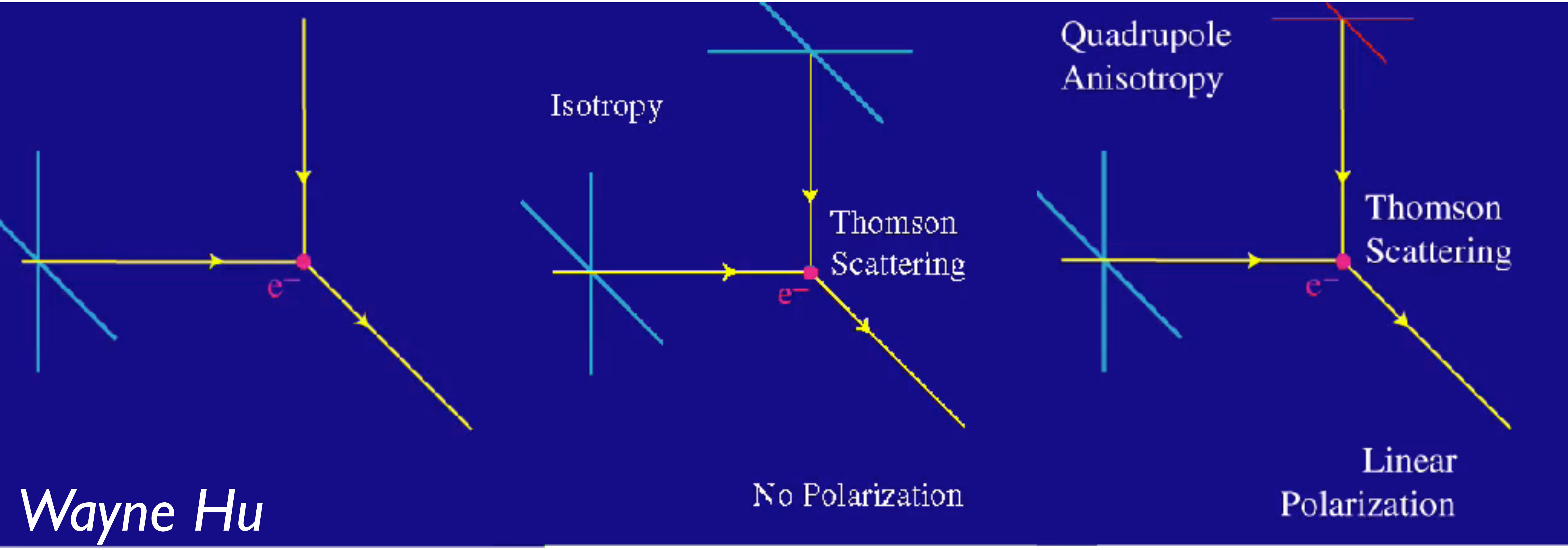
- You need to have two things to produce linear polarisation

  1. Scattering

  2. Anisotropic incident light

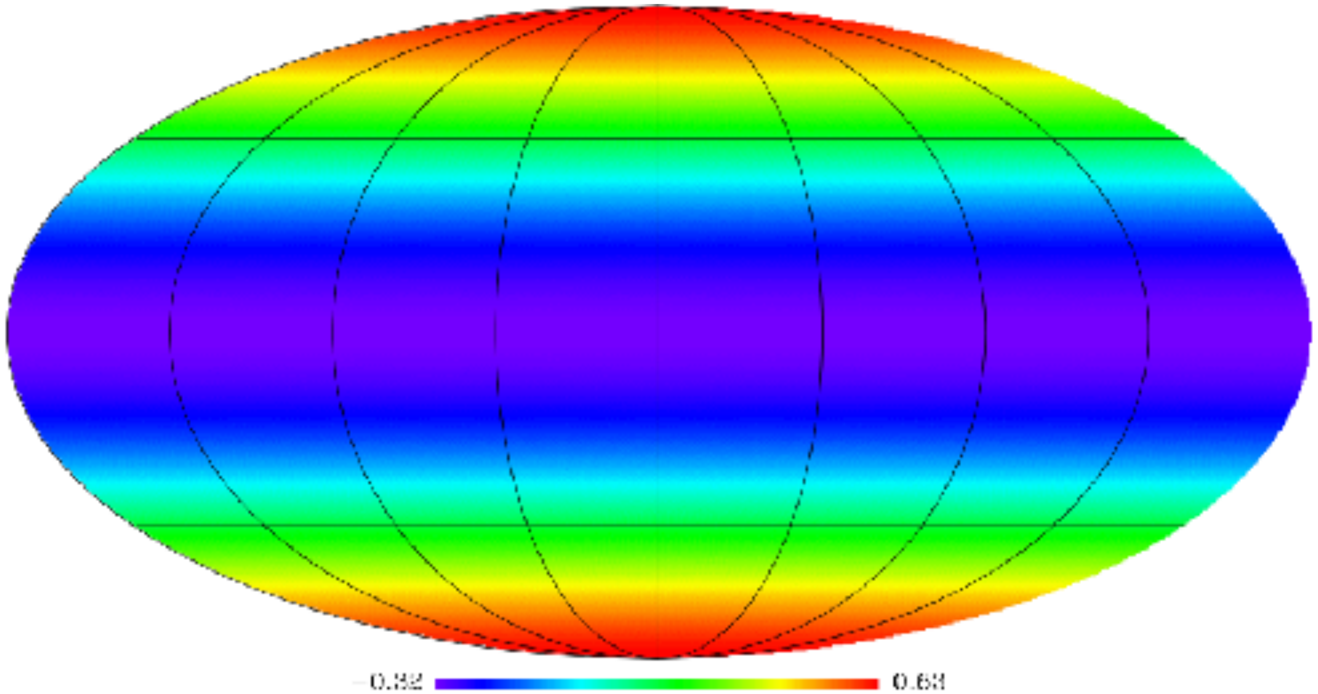
- However, the Universe does not have a preferred direction. How do we generate anisotropic incident light?

Need for a local quadrupole temperature anisotropy

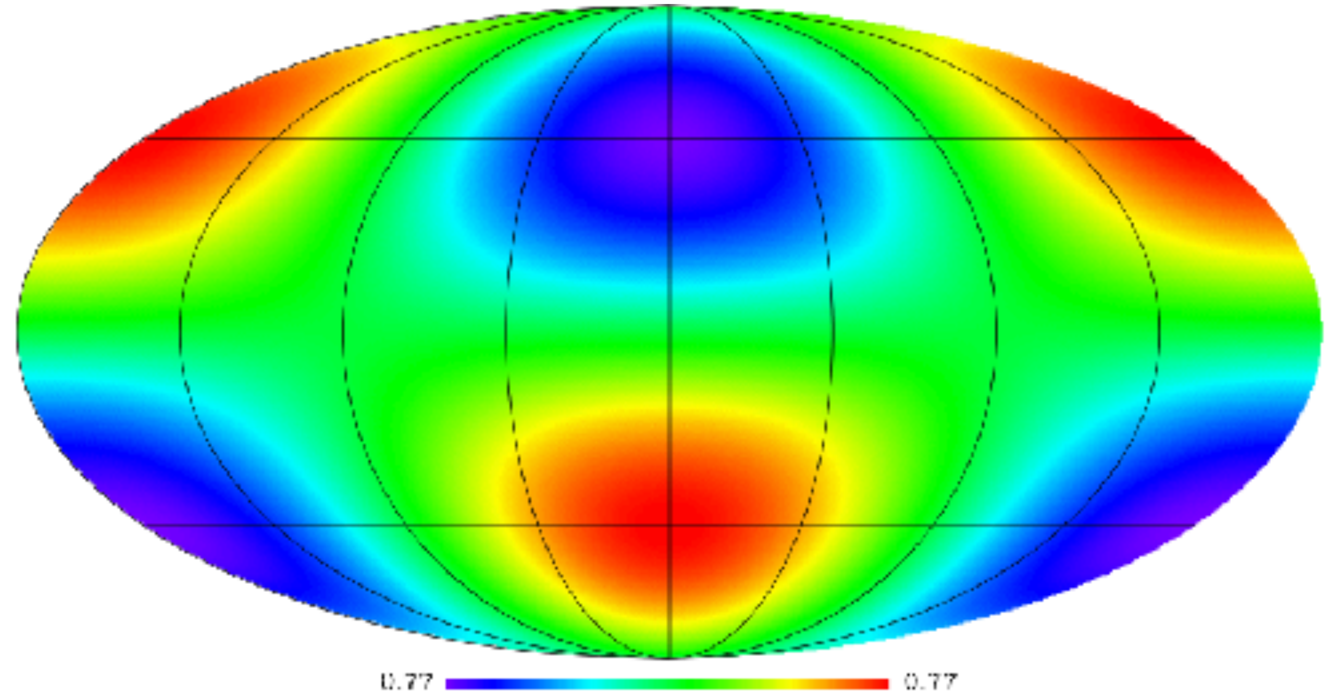


- How do we create a local temperature quadrupole?

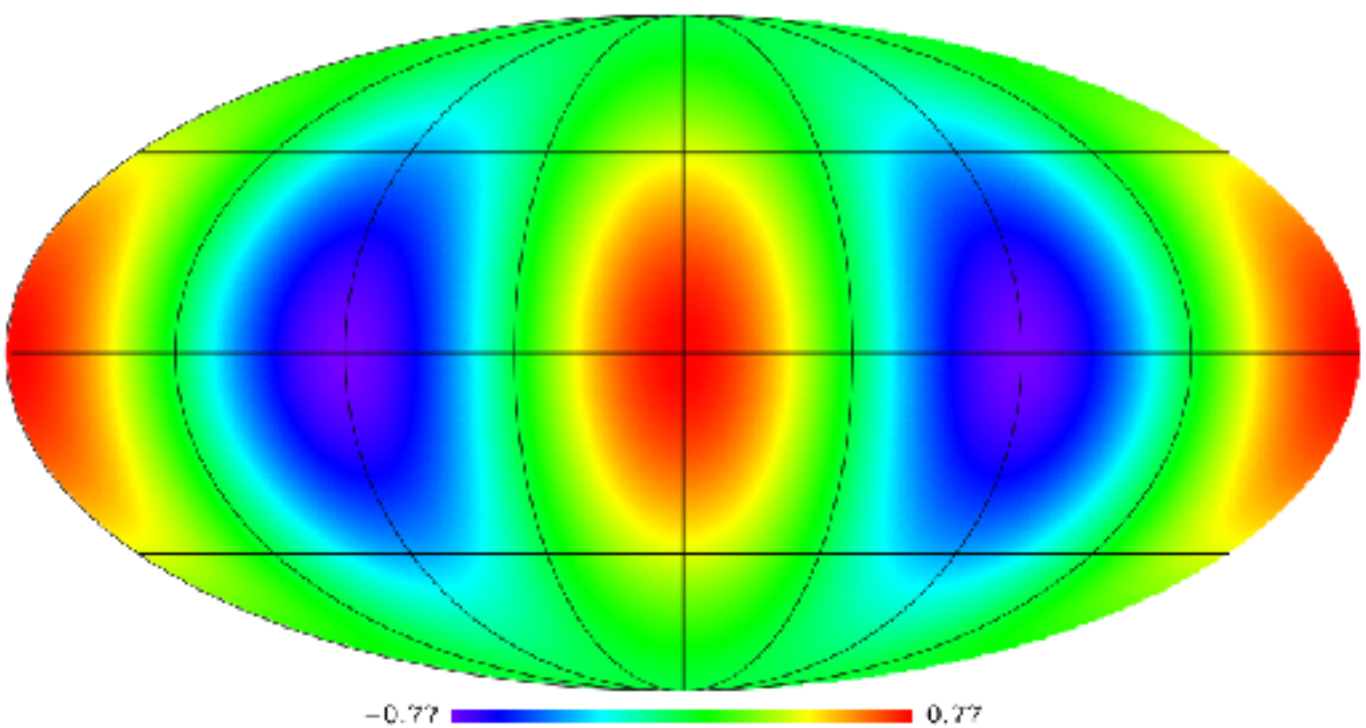
(l,m)=(2,0)



(l,m)=(2,1)



(l,m)=(2,2)



Quadrupole
temperature anisotropy
seen from an electron

Quadrupole Generation: A Punch Line

- When Thomson scattering is efficient (i.e., tight coupling between photons and baryons via electrons), the distribution of photons from the rest frame of baryons is isotropic

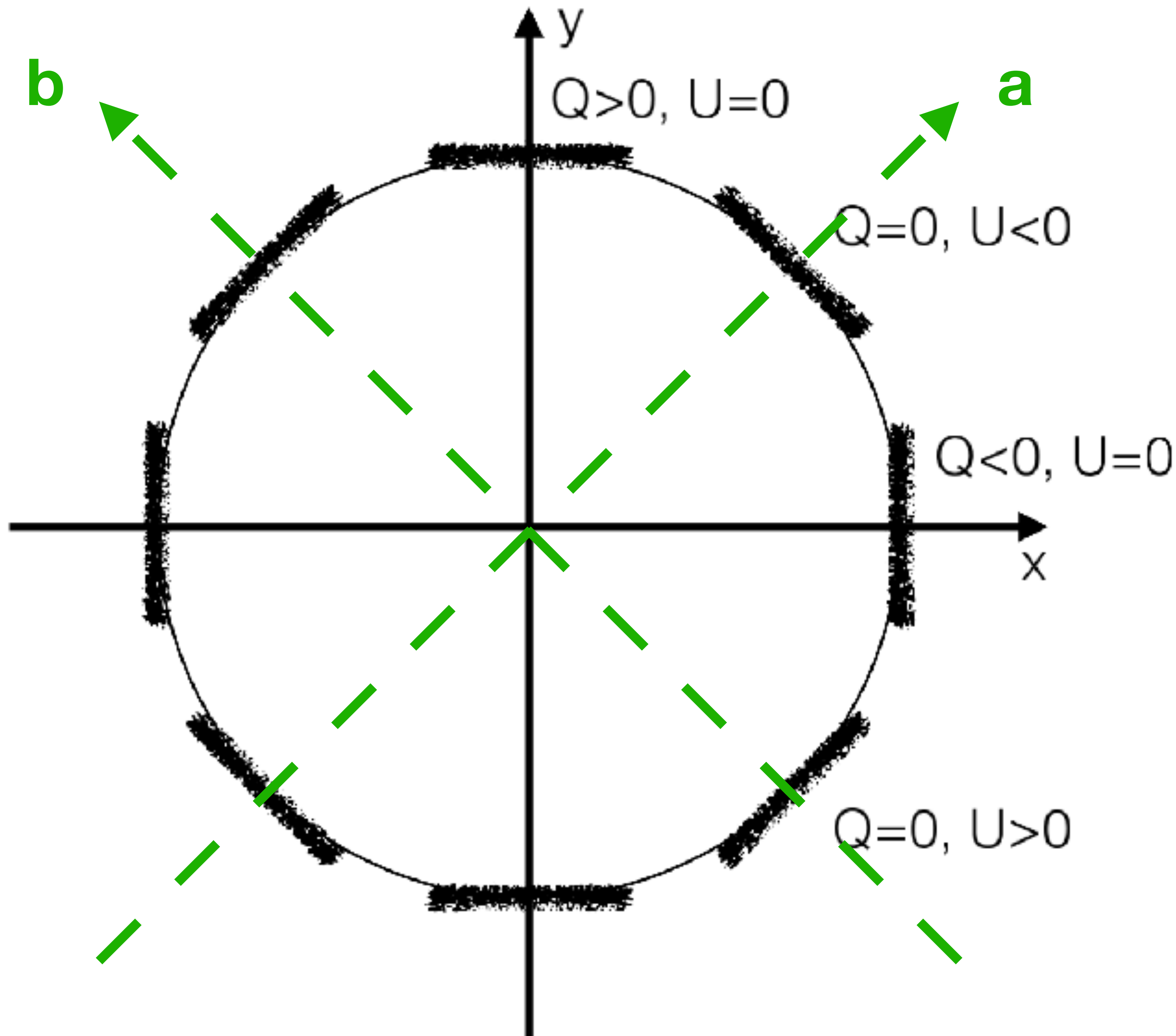
- **Only when tight coupling relaxes**, a local quadrupole temperature anisotropy in the rest frame of a photon-baryon fluid can be generated

- In fact, "a local *temperature anisotropy in the rest frame of a photon-baryon fluid*" is equal to **viscosity**

Stokes Parameters [Flat Sky, Cartesian coordinates]

$$Q \propto E_x^2 - E_y^2$$

$$U \propto E_a^2 - E_b^2$$

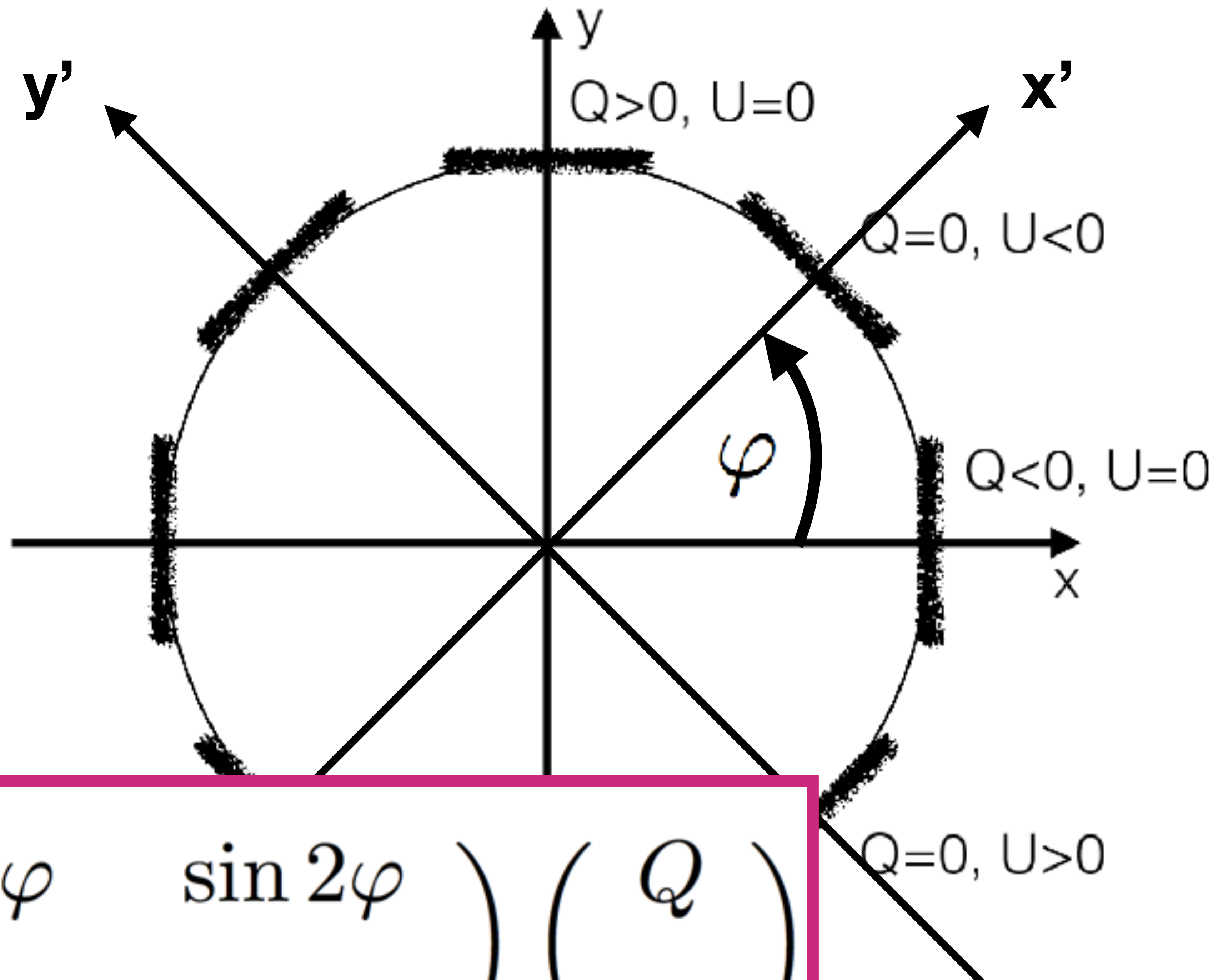


Stokes Parameters change under coordinate rotation

Under (x,y) -> (x',y'):

$$Q \longrightarrow \tilde{Q}$$

$$U \longrightarrow \tilde{U}$$



$$\begin{pmatrix} \tilde{Q} \\ \tilde{U} \end{pmatrix} = \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ -\sin 2\varphi & \cos 2\varphi \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix}$$

Compact Expression

- Using an imaginary number, write $Q + iU$

Then, under coordinate rotation we have

$$\tilde{Q} + i\tilde{U} = \exp(-2i\varphi)(Q + iU)$$

$$\tilde{Q} - i\tilde{U} = \exp(2i\varphi)(Q - iU)$$

Alternative Expression

- With the polarisation amplitude, P, and angle, α , defined by

$$P \equiv \sqrt{Q^2 + U^2}, \;\; U/Q \equiv \tan 2\alpha$$

We write

$$Q + iU \; = \; P \exp(2i\alpha)$$

Then, under coordinate rotation we have

$$\tilde{\alpha} = \alpha - \varphi$$

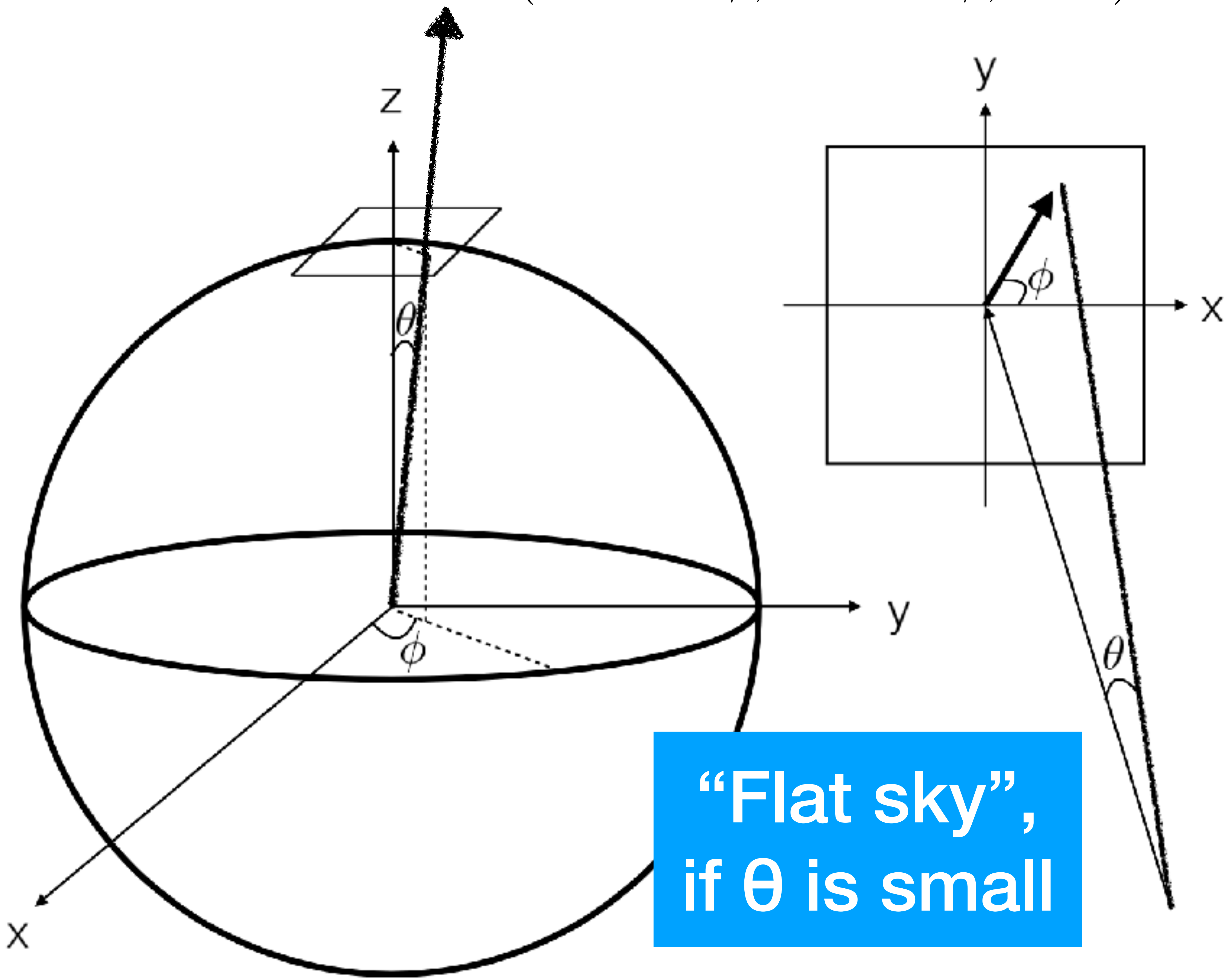
and P is invariant under rotation

E and B decomposition

- That Q and U depend on coordinates is not very convenient…
    - Someone said, “I measured Q!” but then someone else may say, “No, it’s U!”. They flight to death, only to realise that their coordinates are 45 degrees rotated from one another…
- The best way to avoid this unfortunate fight is to define a coordinate-independent quantity for the distribution of polarisation **patterns** in the sky

To achieve this, we need to go to Fourier space

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$



“Flat sky”,
if θ is small

Fourier-transforming Stokes Parameters?

$$Q(\boldsymbol{\theta}) + iU(\boldsymbol{\theta}) = \int \frac{d^2\ell}{(2\pi)^2}\; a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

where

$$\boldsymbol{\ell} = (\ell\cos\phi_\ell, \ell\sin\phi_\ell)$$

- As Q+iU changes under rotation, the Fourier coefficients $a_{\boldsymbol{\ell}}$ change as well
- So…

(*) Nevermind the overall minus sign. This is just for convention

Tweaking Fourier Transform

$$Q(\boldsymbol{\theta}) + iU(\boldsymbol{\theta}) = \int \frac{d^2\ell}{(2\pi)^2}\, a_{\boldsymbol{\ell}} \exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

where we write the coefficients as(*)

$$a_{\boldsymbol{\ell}} = -{}_2a_{\boldsymbol{\ell}} \exp(2i\phi_\ell)$$

- Under rotation, the azimuthal angle of a Fourier wavevector, ϕ_l, changes as $\phi_\ell \to \tilde{\phi}_\ell = \phi_\ell - \varphi$
- This cancels the factor in the left hand side:

$$\tilde{Q} + i\tilde{U} = \exp(-2i\varphi)(Q + iU)$$

Seljak (1997); Zaldarriaga & Seljak (1997); Kamionkowski, Kosowky, Stebbins (1997)

Tweaking Fourier Transform

- We thus write

$$Q(\boldsymbol{\theta}) \pm iU(\boldsymbol{\theta}) = -\int \frac{d^2\ell}{(2\pi)^2}\, {}_{\pm 2}a_{\boldsymbol{\ell}} \exp(\pm 2i\phi_\ell + i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

- And, defining ${}_{\pm 2}a_{\boldsymbol{\ell}} \equiv -(E_{\boldsymbol{\ell}} \pm iB_{\boldsymbol{\ell}})$

$$Q(\boldsymbol{\theta}) \pm iU(\boldsymbol{\theta}) = \int \frac{d^2\ell}{(2\pi)^2}\, (E_{\boldsymbol{\ell}} \pm iB_{\boldsymbol{\ell}}) \exp(\pm 2i\phi_\ell + i\boldsymbol{\ell}\cdot\boldsymbol{\theta})$$

By construction E_l and B_l do not pick up a factor of exp(2iϕ) under coordinate rotation. **That's great!** What kind of polarisation patterns do these quantities represent?

Pure E, B Modes

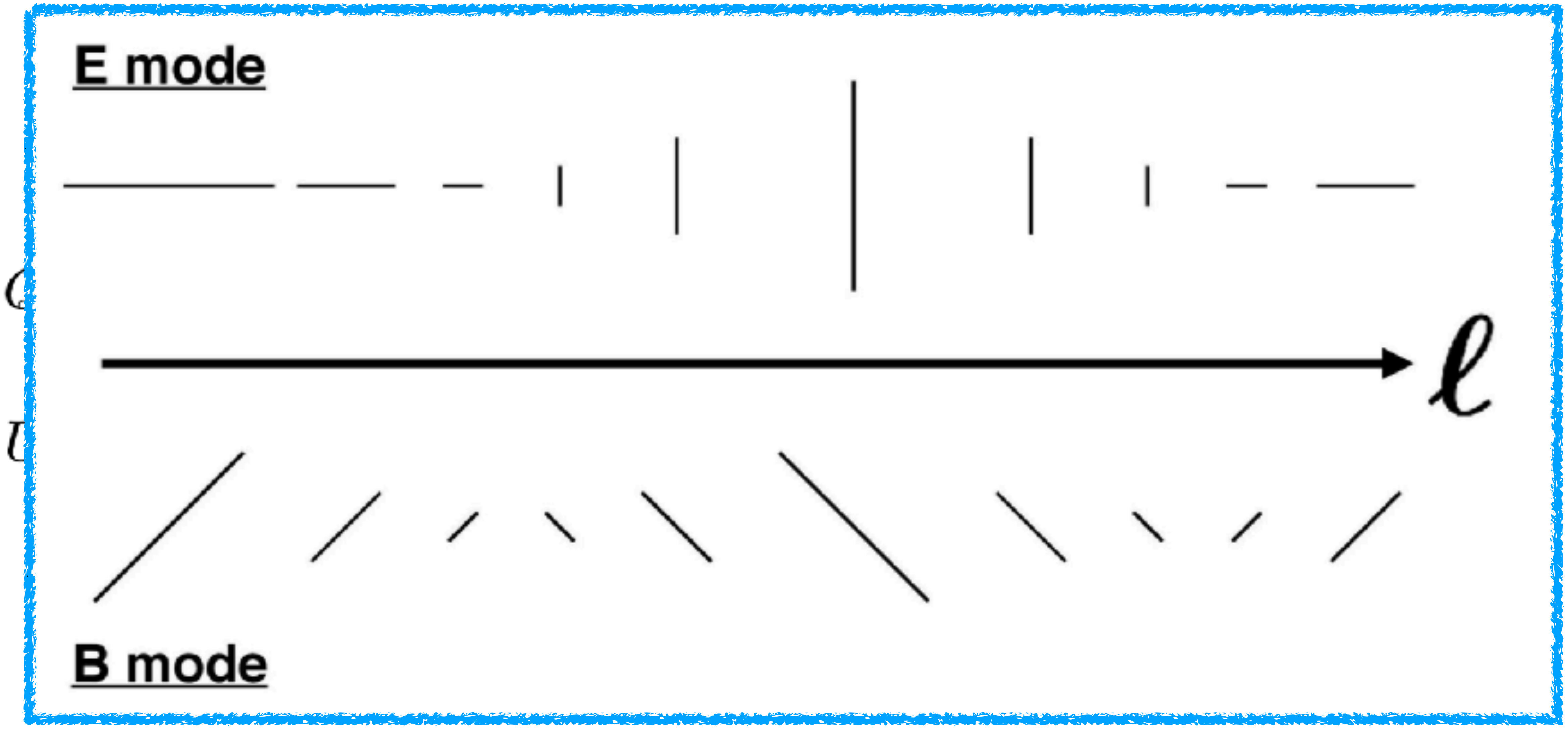
- Q and U produced by E and B modes are given by

$$
\begin{aligned}
Q(\boldsymbol{\theta}) &= \int \frac{d^2\ell}{(2\pi)^2}(E_{\boldsymbol{\ell}}\cos 2\phi_\ell - B_{\boldsymbol{\ell}}\sin 2\phi_\ell)\exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta}) \\
U(\boldsymbol{\theta}) &= \int \frac{d^2\ell}{(2\pi)^2}(E_{\boldsymbol{\ell}}\sin 2\phi_\ell + B_{\boldsymbol{\ell}}\cos 2\phi_\ell)\exp(i\boldsymbol{\ell}\cdot\boldsymbol{\theta})
\end{aligned}
$$

- Let's consider Q and U that are produced by a single Fourier mode
- Taking the x-axis to be the direction of a wavevector, we obtain

$$
\begin{aligned}
Q(\theta) &= E_\ell \exp(i\ell\theta) \\
U(\theta) &= B_\ell \exp(i\ell\theta)
\end{aligned}
$$

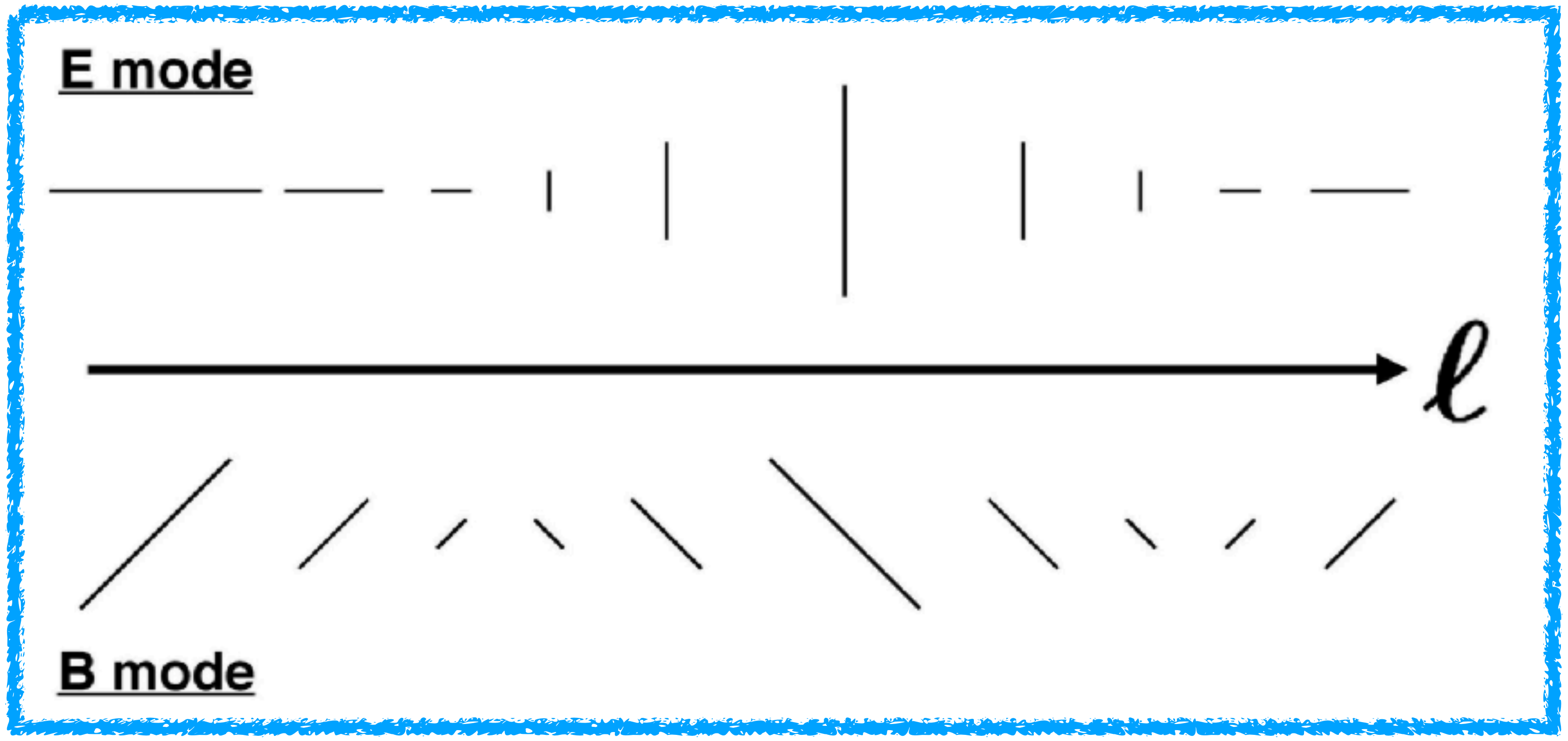
Pure E, B Modes



- Taking the x-axis to be the direction of a wavevector, we obtain

$$Q(\theta) = E_\ell \exp(i\ell\theta)$$
$$U(\theta) = B_\ell \exp(i\ell\theta)$$

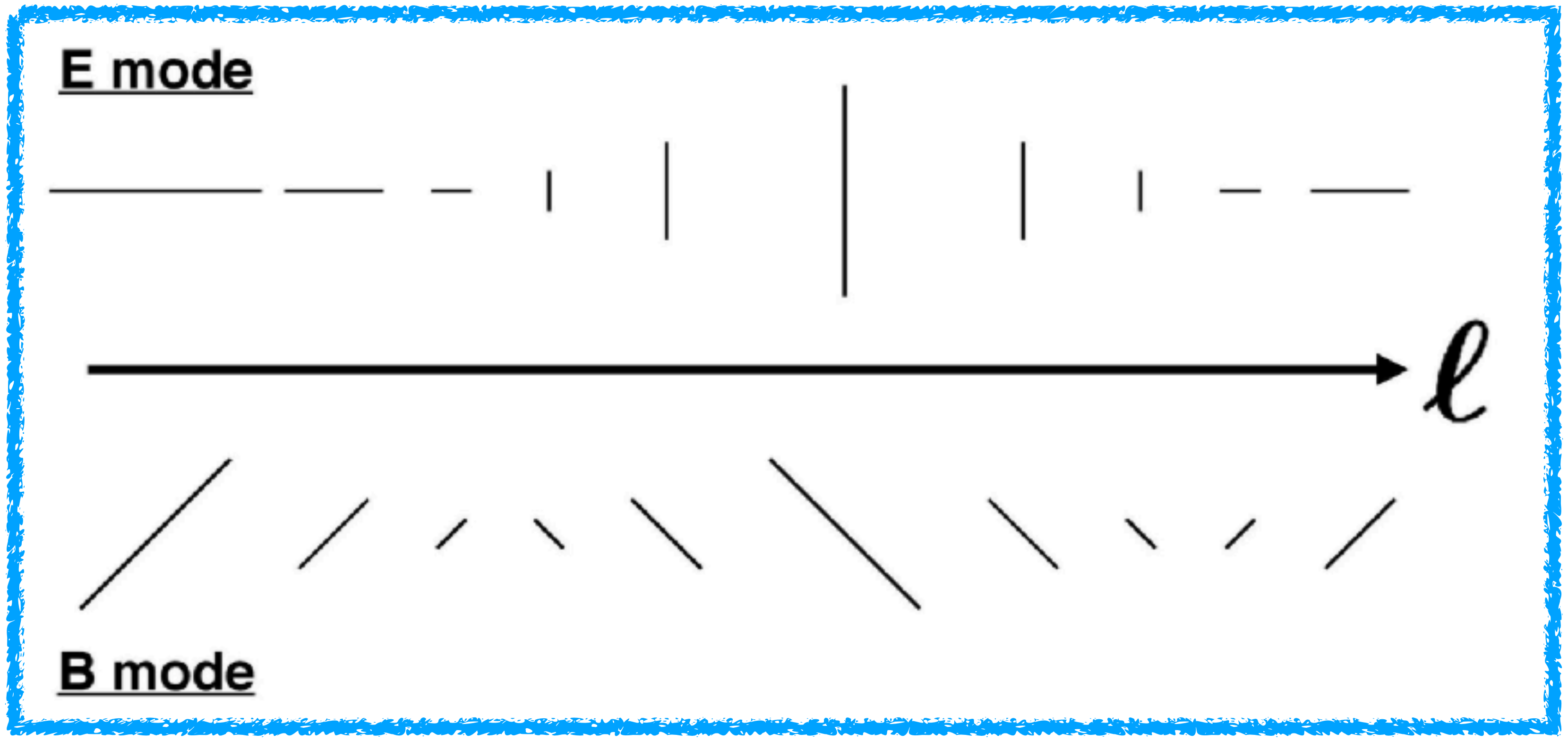
Geometric Meaning (1)

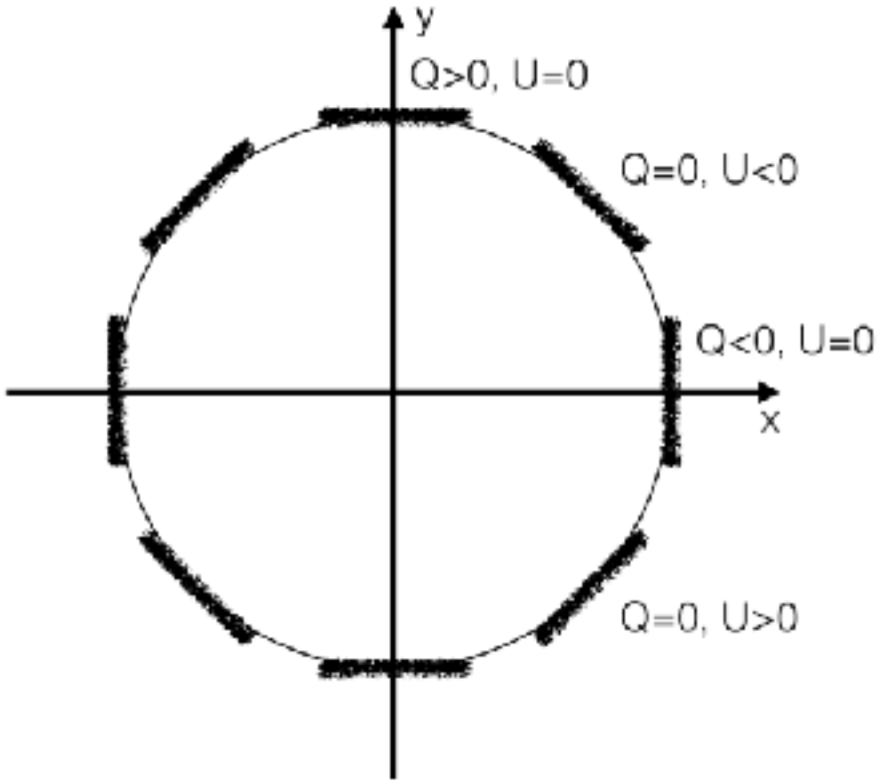


- **E mode**: Polarisation directions parallel or perpendicular to the wavevector

- **B mode**: Polarisation directions 45 degree tilted with respect to the wavevector

Geometric Meaning (2)

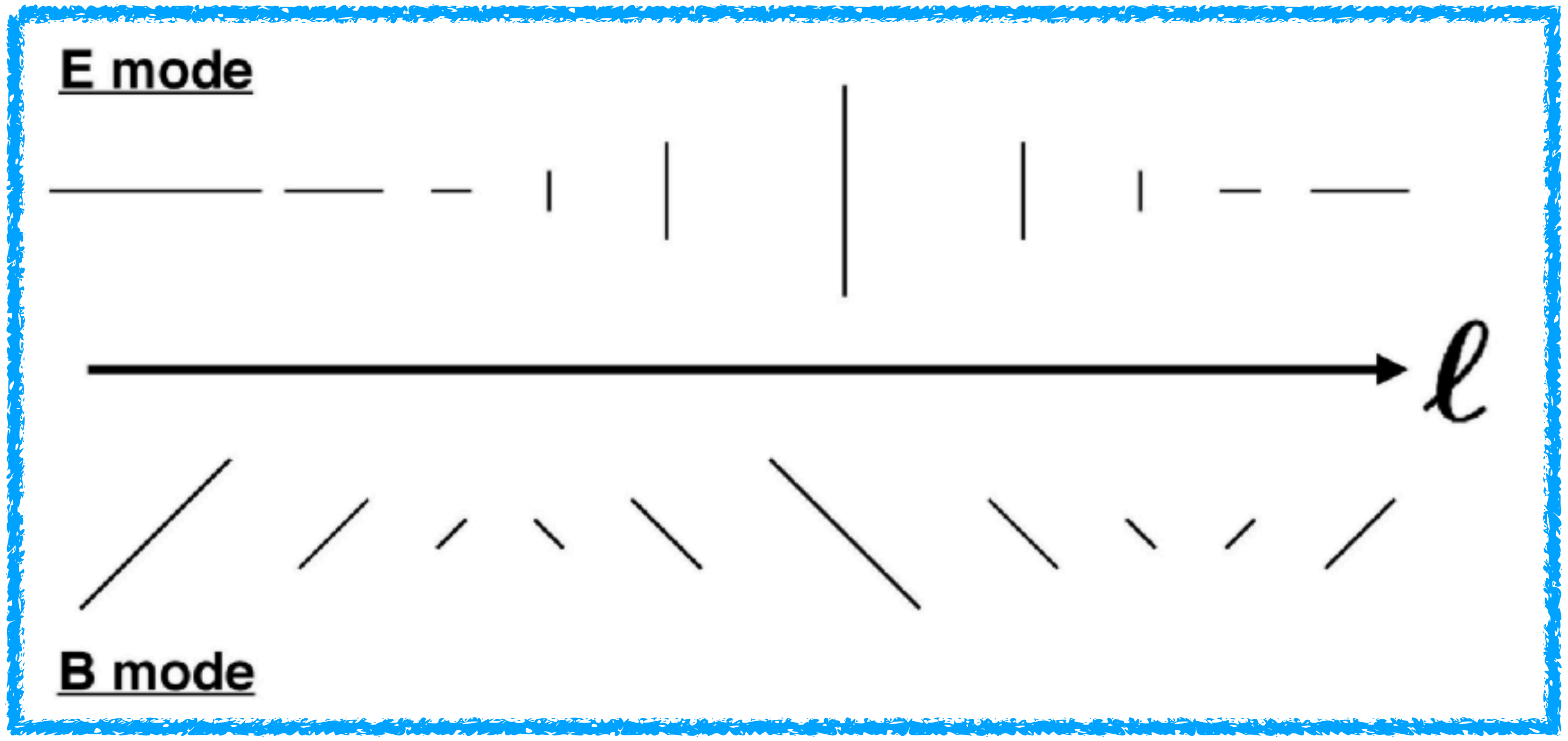




- **E mode**: Stokes Q, defined with respect to ℓ as the x-axis
- **B mode**: Stokes U, defined with respect to ℓ as the y-axis

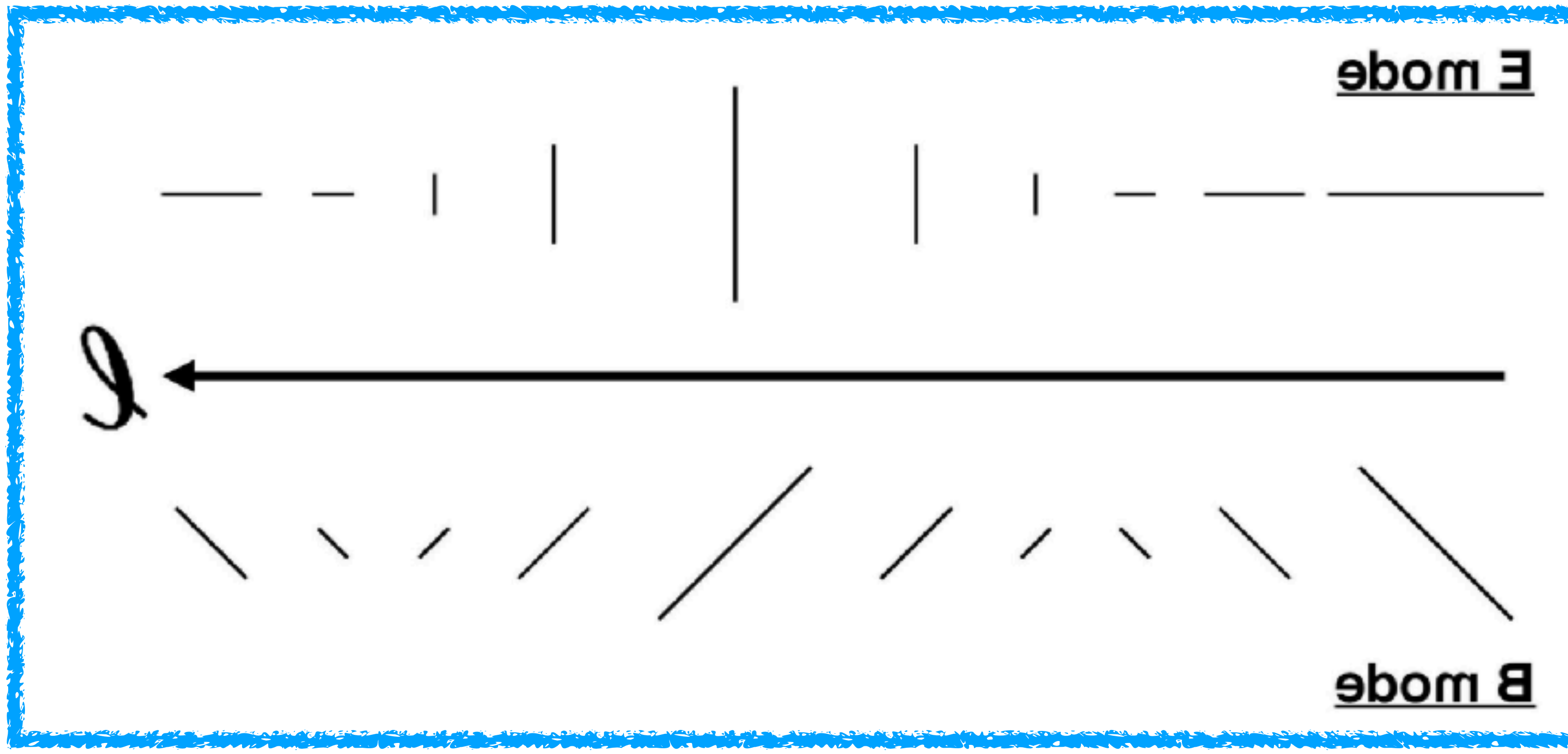
IMPORTANT: These are all coordinate-independent statements

Parity



- **<u>E mode</u>**: Parity even
- **<u>B mode</u>**: Parity odd

Parity



- **<u>E mode</u>**: Parity even
- **<u>B mode</u>**: Parity odd

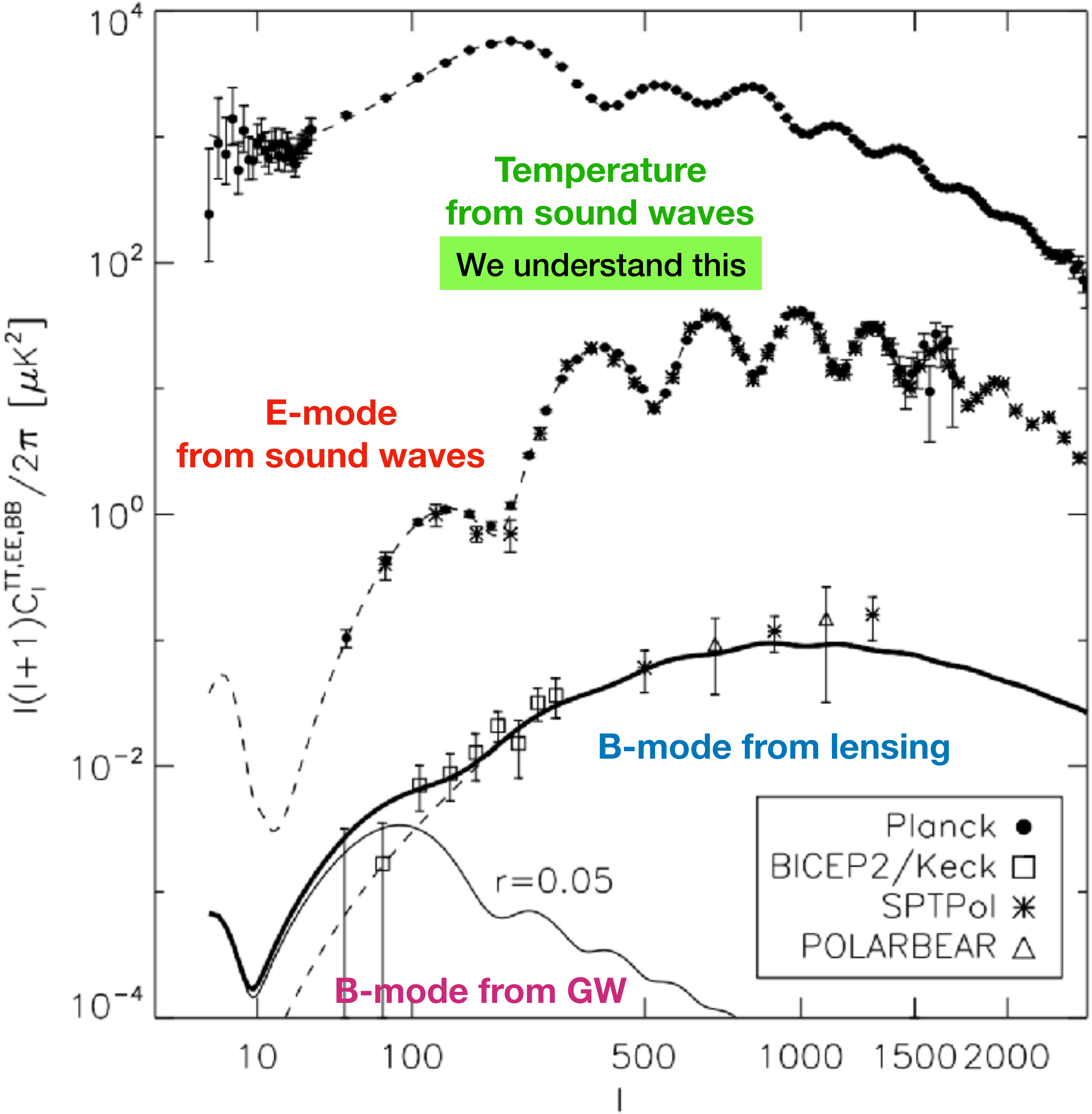
Power Spectra

$$\langle E_{\boldsymbol{\ell}} E^*_{\boldsymbol{\ell}'} \rangle = (2\pi)^2 \delta_D^{(2)}(\boldsymbol{\ell} - \boldsymbol{\ell}') C_\ell^{EE}$$

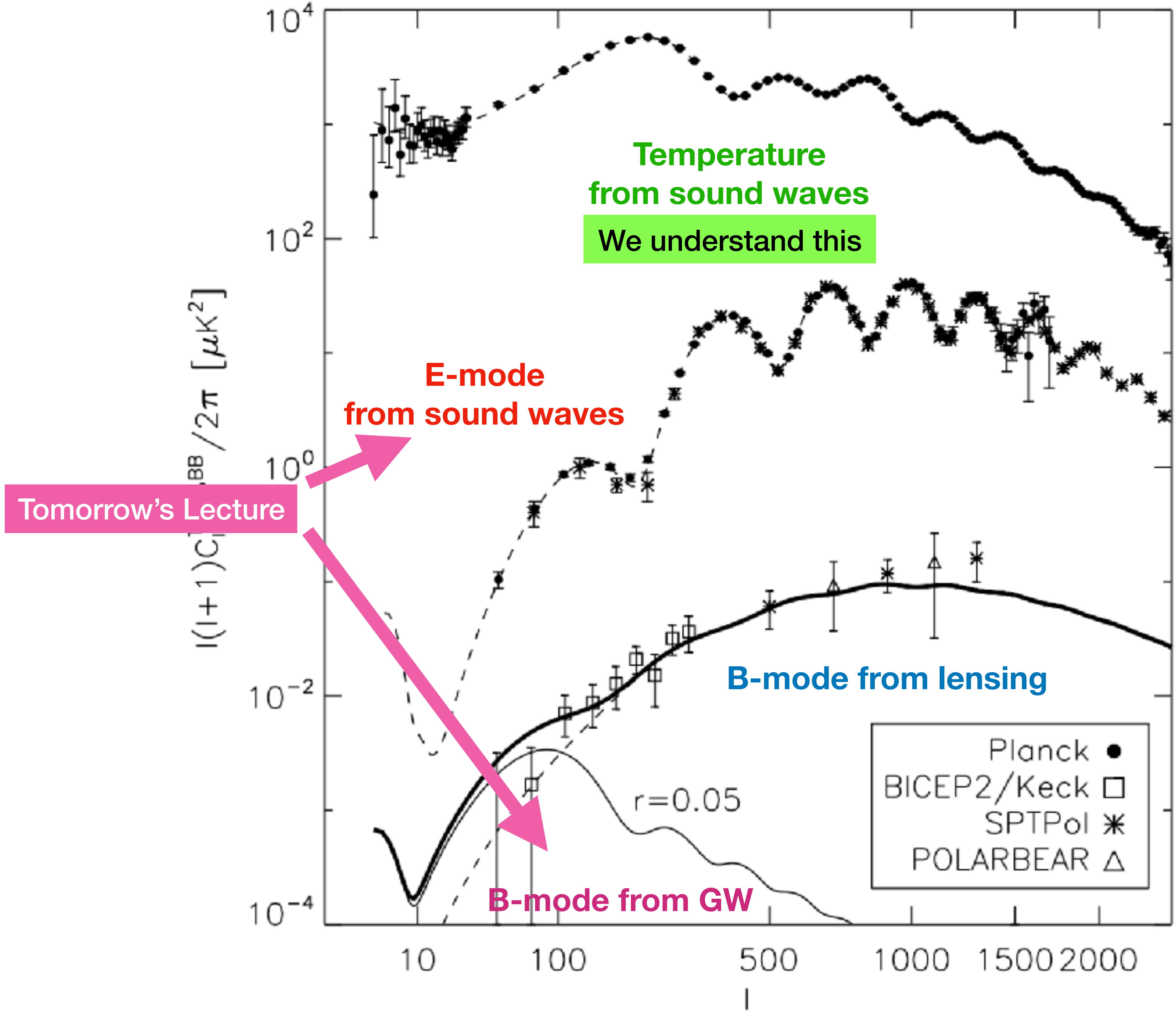
$$\langle B_{\boldsymbol{\ell}} B^*_{\boldsymbol{\ell}'} \rangle = (2\pi)^2 \delta_D^{(2)}(\boldsymbol{\ell} - \boldsymbol{\ell}') C_\ell^{BB}$$

$$\langle T_{\boldsymbol{\ell}} E^*_{\boldsymbol{\ell}'} \rangle = \langle T^*_{\boldsymbol{\ell}} E_{\boldsymbol{\ell}'} \rangle = (2\pi)^2 \delta_D^{(2)}(\boldsymbol{\ell} - \boldsymbol{\ell}') C_\ell^{TE}$$

- However, <EB> and <TB> vanish for parity-preserving fluctuations because <EB> and <TB> change sign under parity flip



Temperature
from sound waves
We understand this
E-mode
from sound waves
B-mode from lensing
B-mode from GW
r=0.05
Planck
BICEP2/Keck
SPTPol
POLARBEAR
$l(l+1)C_l^{TT,EE,BB}/2\pi$ [μK^2]
10^4
10^2
10^0
10^{-2}
10^{-4}
10
100
500
1000
1500
2000
l


Temperature
from sound waves
We understand this
E-mode
from sound waves
Tomorrow's Lecture
B-mode from lensing
B-mode from GW
r=0.05
Planck
BICEP2/Keck
SPTPol
POLARBEAR
10^4
10^2
10^0
10^{-2}
10^{-4}
10
100
500
1000
1500
2000
l