Fundamentals of Atmospheric Dynamics

focused on Rossby waves

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Content

1. Systematic approximation of Governing equations

Reading Materials Holton (Chapter 1, 2) for the exact equation set and primitive equation, and Geophysical Fluid Dynamics (Pedlosky, Chapter 6) for hydrostatic approximation

Before going into the lecture, the governing equation sets of the atmosphere with various complexity, from the most complete set of equations to the simplest possible equation, are introduced in this section. In particular, introduced is how the nondivergent barotropic vorticity equation, which is a simple equation and will be used for most of the topics in this lecture, is obtained from the exact equation set of fluid dynamics.

1) The exact equation set (compressible, inhomogeneous, inertial system)

The equation set of fluid dynamics includes (1) the Newton's second law of motion field, (2) the first law of thermodynamics, (3) the mass conservation, and (4) the ideal gas law.

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi \qquad (1)$$

$$c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = \dot{Q} \qquad (2)$$

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{v} \qquad (3)$$

$$p = \rho R T \qquad (4)$$

The above 4 equation set has four variables ($\vec{v}, p, \rho, T$), therefore it is a closed system. The exact equation set governs the motion field and thermodynamic property of all kinds of fluids. This equation set contains all kinds of waves, such as the sound waves, gravity waves, and large-scale Rossby waves when the governing equations of motion fields include the Coriolis force, particularly for large scale fluids whose time scale is about or longer than a day (earth rotational time scale). Note that the vorticity equation of the above system is

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{v} - \vec{\omega}(\nabla\cdot\vec{v}) - \nabla(\frac{1}{\rho})\times\nabla p. \tag{5}$$

2) The primitive equation set

Assumption: $L >> D$ $\Rightarrow$ Hydrostatic $\left(g = -\frac{1}{\rho}\frac{\partial p}{\partial z}\right)$ => pressure coordinate

=> pressure gradient $\frac{\delta p}{\rho} = g\delta z = \delta\phi$ (p & z are in opposite directions.)

$$\frac{Du}{Dt} = -\frac{d\phi}{dx} + fv + F_u \tag{6}$$

$$\frac{Dv}{Dt} = -\frac{d\phi}{dy} - fu + F_v \tag{7}$$

$$\frac{dp}{dz} = -\rho g, \quad \text{=>} \quad \frac{d\phi}{dp} = -\alpha \tag{8}$$

$$c_p \frac{dT}{dt} = -\alpha\frac{dp}{dt} + \dot{Q} \tag{9}$$

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{d\omega}{dp} = 0, \qquad \omega = \frac{dp}{dt} \tag{10}$$

$$p = \rho RT \tag{11}$$

The above primitive equation set is based on the hydrostatic approximation, where the vertical momentum equation becomes a diagnostic equation expressing the balance between the gravity force and the vertical pressure gradient. This system is applicable to the large-scale fluid motion whose horizontal scale is much larger than the vertical scale and therefore the Coriolis forcing terms should be included. This system provides a very good approximation of atmospheric state for a spatial scale of more than few tens of km. Therefore, the regional models as well as global general circulation models usually use this primitive equation set for the weather and climate simulations.

3) Shallow water system (hydrostatic & homogeneous $\rho = \rho_0$)

Assumption: $L >> D$**,** $\rho = \rho_0$**.** Hydrostatic equation $\frac{\partial p}{\partial z} = -\rho_0 g$.

Integrating the hydrostatic equation from the top of water surface to the height z,

$$\int_{p_0}^{p} \delta p = \int_{H}^{Z} -\rho_0 g \delta z \text{ , we get } p = p_0 + \rho_0 g(H - z).$$

Now the horizontal pressure gradient $\frac{\nabla_H p}{\rho_0} = g\nabla_H H$ is independent of z, which implies that $u \,\&\, v$ are independent of z, giving the barotropic (vertically uniform) condition. Also the constant density implies that the temperature is proportional to the pressure, which depends on the height H in this system,. Now the governing equations of the shallow water system are

$$\frac{du}{dt} = -g\frac{\partial H}{\partial x} + fv \quad (12)$$

$$\frac{dv}{dt} = -g\frac{\partial H}{\partial y} - fu \quad (13)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

From the above continuity equation, $\int_0^{W_H}\delta w=\int_0^H -\nabla_H\cdot\vec{v}_H\delta z$ which gives $w_{(z=H)}=-(\nabla_H\cdot\vec{v}_H)H$, and since $w_{(z=H)}=\frac{DH}{Dt}$,

$$\frac{DH}{Dt}=-(\nabla_H\cdot\vec{v}_H)H \qquad (14)$$

Equations (12), (13), and (14) consist of the equation set for the shallow water system. Applying the curl to Eqs. (13) and (14),

$$\frac{d}{dt}(\zeta+f)=-(\zeta+f)(\nabla_H\cdot\vec{v}_H) \qquad (15)$$

Combining Eqs. (14) and (15), we can obtain the following potential vorticity equation.

$$\frac{D}{Dt}\left(\frac{\varsigma+f}{H}\right)=0 \qquad (16)$$

Now, for H=constant, the rigid top condition is equivalent to the (horizontal) divergence to be zero from Eq. (14), $\nabla_H\cdot\vec{v}_H=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. Then,

$$\frac{D}{Dt}(\varsigma+f)=0 \qquad (17)$$

The above equation is called the "nondivergent barotropic vorticity equation," which will be a base equation of this lecture. Comparing of (17) with (5), we see the simplicity of (17), which contains only the large-scale Rossby waves. The nondivergence condition introduces the streamfunction (ψ) as below

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad => \quad u=-\frac{\partial\psi}{\partial y}\ and\ v=\frac{\partial\psi}{\partial x} \quad => \quad \zeta=-\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\nabla^2\psi$$

Eq. (17) $\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + v\frac{\partial f}{\partial y} = 0$ can be linearized with respect to the basic state of $\bar{u}$ and $\beta = df/dy$, which are assumed to be constant in space and time, and $\bar{v} = 0$. After neglecting the small nonlinear perturbation terms,

$$\frac{\partial \varsigma'}{\partial t} + \bar{u}\frac{\partial \varsigma'}{\partial x} + \beta v' = 0$$

The above equation is a simplest possible equation of describing large-scale atmospheric and ocean circulations. It expresses that the vorticity change can be due to the mean advection of relative vorticity and the meridional advection of planetary vorticity. Note that for the planetary-scale waves, the planetary vorticity advection (the third term) is much larger than the relative vorticity advection (the second term). -

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It is also noted that Eq. (15) can be written in a quasi-geostrophic approximation, where $f >> \zeta$ and the absolute magnitude of ζ is much larger than that of divergence D, Then, the vorticity equation in a quasi-geostrophic approximation can be written as

$$\frac{\partial \zeta}{\partial t} + \vec{v}_{\psi} \cdot \nabla(\zeta + f) = -fD \qquad (18)$$

where $\vec{v}_{\psi}$ is the streamfunction (non-divergent) component of wind and $\psi = gH/f_o$. It is noted that the wind associated with the advection term in (18) is non-divergent, and therefore the Eq. (18) is same as Eq. (16) except the right-hand side term fD which is often considered as a forcing term of circulation (streamfunction).

2. Rossby waves

1) Rossby wave dispersion in a β-plane and wave selection

The linearized nondivergent barotropic vorticity equation, driven in the previous section, will a base function in this section.

$$\frac{\partial \varsigma'}{\partial t} + \bar{u}\frac{\partial \varsigma'}{\partial x} + \beta v' = 0 \tag{1}$$

Applying the following plane wave solution $\psi' = \hat{\Psi}\exp[i(kx + ly - \omega t)]$ to Eq. (1) after substituting $\varsigma' = \nabla^2\psi' = (\partial^2/\partial x^2 + \partial^2/\partial y^2)\psi'$ and $v' = \partial\psi'/\partial x$, the following dispersion relationship can be obtained.

$$\omega = \bar{u}k - \frac{\beta k}{k^2 + l^2} \tag{2}$$

In the above, the phase speed $c = \frac{\omega}{k} = \bar{u} - \frac{\beta}{k^2 + l^2}$ is a function of wave number, and therefore the Rossby wave is dispersive. If the basic zonal wind is zero, the phase speed of Rossby wave is always negative (westward propagating).

The group velocity of the Rossby wave can be written as

$$G_x = \frac{\partial \omega}{\partial k} = \bar{u} + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2} \tag{3}$$

$$G_y = \frac{\partial \omega}{\partial l} = \frac{2\beta kl}{(k^2 + l^2)^2} \tag{4}$$

Substituting the equation of phase speed $c = \bar{u} - \frac{\beta}{k^2 + l^2}$ into (3), one obtains $G_x - c = \frac{2\beta k^2}{(k^2 + l^2)^2}$. Thus, in a frame of reference which moves with the phase speed c, the group velocity can be expressed as

$$\vec{G} = \frac{2\beta k\vec{k}}{(k^2 + l^2)^2} \quad \text{and} \quad \frac{G_y}{G_x} = \frac{l}{k} \tag{5}$$

For stationary waves, $\omega = 0$ and $c = 0$, Eq. (2) gives

$$k^2 + l^2 = \frac{\beta}{\overline{u}} \tag{7}$$

Combination of Eqs. (3) and (4) and using Eq. (7), we obtain the equation below

$$(G_x - \overline{u})^2 + G_y{}^2 = \frac{\beta^2}{(k^2 + l^2)^2} = \overline{u}^2 \tag{8}$$

Eq. (7) shows the constraint of Rossby wave number in a steady state. The Rossby wavenumbers exist only in the circle of the k and l space shown in the figure below. From the (0, 0) source point, the particles after one second by the group velocities, Gx and Gy, should be distributed as below (RHS figure).

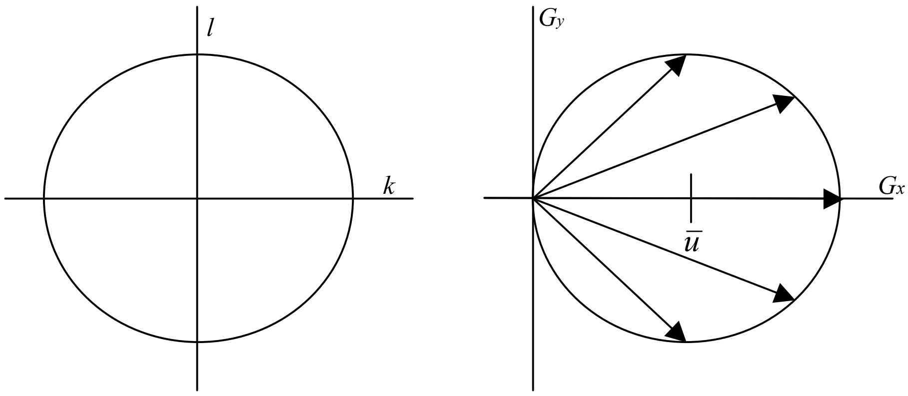


Fig. The wavenumber selection of k and l and wave front displacement (group velocity) for the stationary waves.

2) Pseudo-momentum equation

For the waves superimposed on the basic flow $\bar{u}(y)$, the linearized vorticity equation becomes

$$\frac{\partial \varsigma'}{\partial t} = -\bar{u}\frac{\partial \varsigma'}{\partial x} - \gamma v' \quad , \qquad \gamma \equiv \beta + \frac{\partial \bar{\varsigma}}{\partial y} \tag{9}$$

Multiplying ς' and taking zonal mean, the above equation becomes

$$\frac{\partial}{\partial t}[\frac{\overline{\varsigma'^2}}{2\gamma}] = -\overline{v'\varsigma'} \tag{10}$$

For non-divergent eddies, the meridional vorticity advection is the same as the meridional convergence of zonal momentum.

$$\overline{v'\varsigma'} = -\frac{\partial}{\partial y}\overline{u'v'} \tag{11}$$

Non-divergence $\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$ and using $\overline{\frac{\partial}{\partial x}(\ )} = 0$, $\overline{u'\frac{\partial u'}{\partial x}} = \overline{\frac{\partial}{\partial x}\left(\frac{1}{2}u'^2\right)}$,

$$\overline{v'\varsigma'} = \overline{v'\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)} = \overline{\frac{\partial}{\partial x}\left(\frac{1}{2}v'^2\right)} - \overline{v'\frac{\partial u'}{\partial y}} = -\overline{\frac{\partial (u'v')}{\partial y}} + \overline{u'\frac{\partial v'}{\partial y}} + \overline{u'\frac{\partial u'}{\partial x}} = -\overline{\frac{\partial (u'v')}{\partial y}}.$$

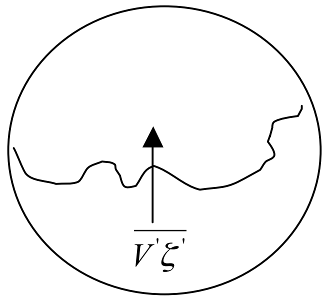
From the equation $\frac{\overline{du}}{dt} = f\bar{v} - \frac{d\bar{\varphi}}{dx}$, the R.H.S terms are all zero since $\bar{v} = \frac{\partial \bar{\psi}}{\partial x}$, therefore $\frac{\overline{du}}{dt} = \frac{\overline{\partial u}}{\partial t} + \overline{u\frac{\partial u}{\partial x}} + \overline{v\frac{\partial u}{\partial y}} = 0$, where the second term of RHS should be zero and the third term is $\overline{v'\frac{\partial u'}{\partial y}} = \frac{\partial}{\partial y}\overline{u'v'} - \overline{u'\frac{\partial v'}{\partial y}} = \frac{\partial}{\partial y}\overline{u'v'}$, since $\overline{u'\frac{\partial v'}{\partial y}} = -\overline{u'\frac{\partial u'}{\partial x}} = 0$ ($\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$)

Therefore $\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial y}\overline{u'v'}$.

The above zonal-mean zonal wind equation combined with (11) gives

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial y}\overline{u'v'} = \overline{v'\zeta'} \qquad (12)$$

The above equation indicates that the rate of change the circulation along a latitude is the change of vorticity in the cap north of the latitude, which is equivalent to the vorticity flux along the latitude. This states the so called "**circulation theorem**".



Using Eq. (10), Eq. (12) becomes

$$\frac{\partial}{\partial t}[\bar{u} + \frac{1}{2\gamma}\overline{\zeta'^2}] = 0 \qquad (13)$$

Now let $M = \frac{\overline{\zeta'^2}}{2\gamma}$. M has a dimension of wind and it is so called to as "***pseudo-momentum***" or a measure of "***wave activity***". Eq. (13) indicates that if the wave activity (M) increases with time, the zonal mean flow decreases with time: ***Eddy-mean flow interaction.*** Also, it tells that since $\bar{u}$ has a finite value in the domain, $\overline{\zeta'^2}$ can have a very big value in the domain only if γ changes the sign in some of the domain. This is called "***Rayleigh-Kuo barotropic instability." The flow can be unstable only if*** $\gamma < 0$ ***somewhere in the domain.***

Now we develop the conservation of wave pseudo-momentum. Eqs. (12) and (13) give the following equation.

$$\frac{\partial}{\partial t}[\frac{\overline{\zeta'^2}}{2\gamma}] + \frac{\partial}{\partial y}(-\overline{u'v'}) = 0 \qquad (14)$$

or $\frac{\partial M}{\partial t} + \frac{\partial F}{\partial y} = 0, \qquad F = -\overline{u'v'}$ (15)

If $F = G_y M$, where G_y is the group velocity of the wave expressed by Eq. (5), M is conserved in the frame of reference moving with the group velocity. The conservation of pseudo momentum in the frame of moving with the group velocity is expressed as

$$\frac{D_G M}{Dt} = 0 \tag{16}$$

Now we will prove $F = G_y M$ using a plane wave solution, $\Psi' = \mathrm{Re}[\tilde{\Psi} e^{i(kx+ly)}]$.

$$u' = -\frac{\partial \Psi'}{\partial y} = -\mathrm{Re}[il\tilde{\Psi} e^{i(kx+ly)}] = -\frac{1}{2}[il\tilde{\Psi} e^{i(kx+ly)} - il\tilde{\Psi}^* e^{-i(kx+ly)}]$$

$$v' = \frac{\partial \Psi'}{\partial x} = \mathrm{Re}[ik\tilde{\Psi} e^{i(kx+ly)}] = \frac{1}{2}[ik\tilde{\Psi} e^{i(kx+ly)} - ik\tilde{\Psi}^* e^{-i(kx+ly)}]$$

Using the above equations, we can express the meridional momentum flux in terms of wave number and wave amplitude as below.

$$\overline{u'v'} = -\frac{1}{2} kl \left|\tilde{\Psi}\right|^2 \tag{17}$$

The above equation indicates that the northward zonal momentum transport is accompanied by the southward wave propagation and vice versa. Now we solve $G_y M$.

Since $\varsigma' = \mathrm{Re}[-(k^2 + l^2)\tilde{\Psi} e^{i(kx+ly)}]$, $\overline{\varsigma'^2} = \frac{1}{2}(k^2 + l^2)^2 \left|\tilde{\Psi}\right|^2$ (18)

Using Eqs. (5) and (18), $G_y M = G_y \frac{\overline{\varsigma'^2}}{2\gamma} = \frac{1}{2} kl \left|\tilde{\Psi}\right|^2 = -\overline{u'v'}$. Therefore $G_y M = F$. For stationary waves, $\frac{\partial}{\partial y}(-\overline{u'v'}) = 0$ from (17), indicating that $\overline{u'v'}$ is independent of y, and then $\bar{u}$ is constant in time, which is so called "non-acceleration theorem" in this two dimensional flow.

3. Meridional dispersion of Rossby wave and teleconnection dynamics

Reading Materials

Hoskins, B. J. and D. J. Karoly, 1981: The Steady Linear Response of a Spherical Atmosphere to Thermal and Orographic Forcing. *J. Atmos. Sci.*, 38, 1179-1196.

1) Two-dimensional Rossby waves

In the previous section, the Rossby wave was treated in a constant basic zonal flow. However, observation shows that the zonal flow has a large-scale meridional structure. Also, the planetary-scale waves usually cover several ten degrees of latitude, and for such waves the assumption of constant basic state is hard to be justified. In this section, we will study the two-dimensional Rossby wave based on the "**Ray Theory**" developed by Hoskins and Karoly (1981).

Observations indicate:

(1) Away from a tropical wave forcing region, the planetary-scale wavetrain has a barotropic structure in the extratropics.

(2) Planetary-scale waves (i.e., with small zonal wavenumber) propagate poleward as well as eastward, with a wavetrain path similar to a great circle path.

(3) Shorter waves propagate eastward and are trapped along the jet-stream region (~40°).

They developed the wave theory propagating in a slowly varying medium. In the Mercator projection of the sphere, the linearized nondivergent vorticity equation can be written as

$$[\frac{\partial}{\partial t} + \bar{u}_M \frac{\partial}{\partial x}][\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}] + \beta_M \frac{\partial \Psi}{\partial x} = 0 \quad (1)$$

where, $x = a\lambda$ and $y = a \ln\left[\frac{(1+\sin\Phi)}{\cos\Phi}\right]$.

$$\bar{u}_M = \frac{\bar{u}}{\cos\Phi}$$

$$\beta_M = \frac{2\Omega}{a}\cos^2\Phi - \frac{d}{dy}\frac{1}{\cos^2\Phi}\frac{d}{dy}(\cos^2\Phi \bar{u}_M)$$

In a local analysis with an assumption that the basic state does not much change locally, we can introduce a plane wave solution $\Psi_o \exp[i(kx + ly - \omega t)]$, in which k, l and ω are determined by the property of local basic state. Then the dispersion relationship can be obtained as before,

$$\omega = \bar{u}_M k - \frac{\beta_M k}{k^2 + l^2} \quad (2)$$

Based on the above dispersion relation, we can derive the group velocities for x and y directions. For a stationary wave ($\omega = 0$), the ray path can be obtained as

$$\frac{dy}{dx} = \frac{v_g}{u_g} = \frac{l}{k} \quad (3)$$

Eq. (3) indicates that the slope of wave ray path is simply expressed by the ratio of the meridional wave number to the zonal wave number. Relatively small-scale waves (large k) such as the synoptic waves propagate along the zonal direction and planetary-scale waves (small k) propagate along the meridional direction.

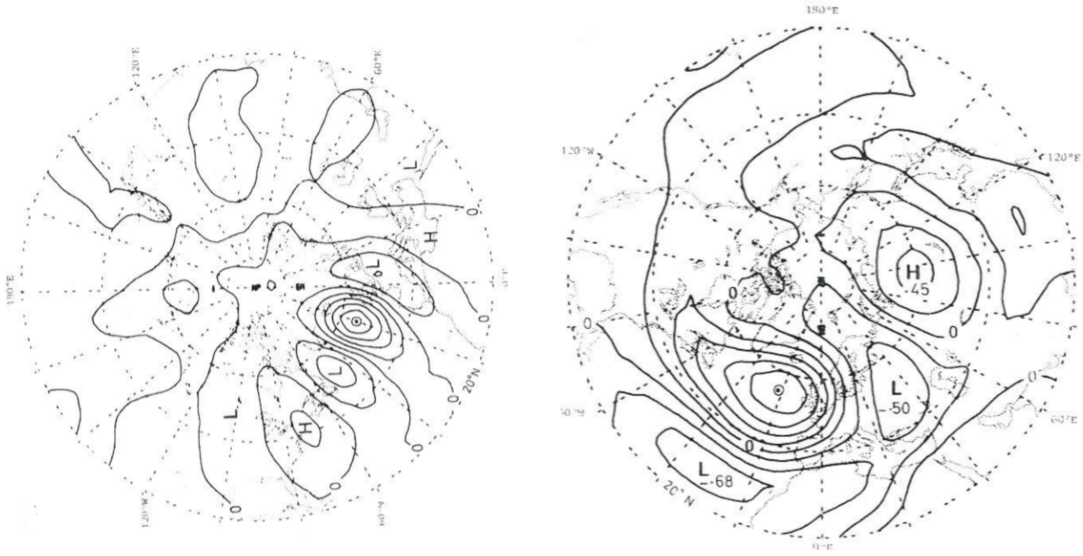


Fig. Teleconnection patterns obtained by simultaneous temporal correlation between 500 mb geopotential heights at 55N, 20W and other Northern Hemisphere locations during boreal winter, using LHS: band pass filtered data of 2.5-6 days and RHS: monthly mean data.

For a stationary wave with a zonal wave number k, the meridional wave number l is defined as

$$l^2 = {K_s}^2 - k^2 \quad ; \quad {K_s}^2 = \frac{\beta_M}{\bar{u}_M} \qquad (4)$$

where K_s is called "**stationary wave number**", which is determined by the basic state. It is also noted that the meridional wave propagation is possible only if the zonal wave number is smaller than the stationary wave number, that is, the meridional propagation is more favorable for planetary-scale waves. Observed characteristics of meridional wave propagation for different zonal wavenumbers can be discussed with the figures below.

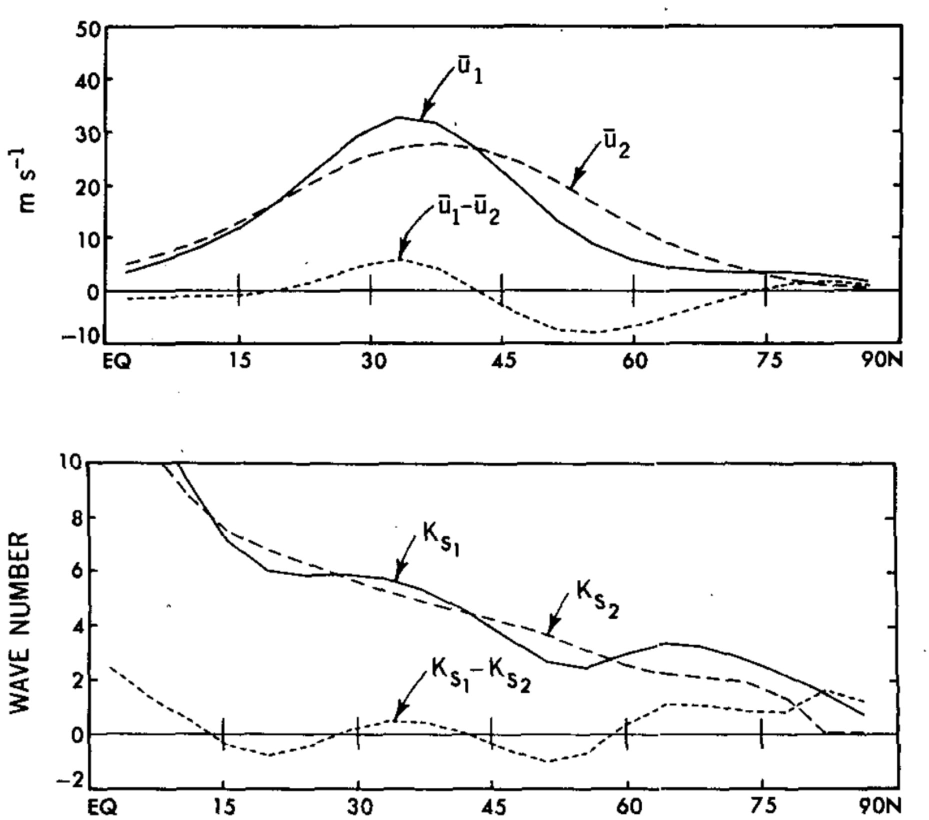


Fig. Two states of zonal-mean zonal wind obtained from the EOF analysis (upper figure) and the associated stationary wave numbers (lower figure).

Now we derive the solution of Eq. (1) more rigorously for $\bar{u} = f(y)$. The general solution has a form,

$$\Psi = P(y)\exp i(kx - \omega t) \tag{7}$$

Substituting Eq. (7) into Eq. (1),

$$i(\bar{u}_M k - \omega)(-k^2 P + \frac{\partial^2 P}{\partial y^2}) + i\beta_M kP = 0$$

With a manipulation, the above equation can be expressed as

$$\frac{\partial^2 P}{\partial y^2} + l(y)^2 P = 0 \tag{8}$$

where, $l^2 = \frac{\beta_M k}{\bar{u}_M k - \omega} - k^2 = \frac{K_s^{\ 2}}{1 - \frac{\omega}{k\bar{u}_M}} - k^2$

The solution of Eq. (8) depends on the structure of $l(y)$ and is not easy to solve analytically. Now we examine the solution using the WKB method, which is a kind of local analysis by assuming that the length scale of l^{-1} is a slowly varying function of y. Then, let $Y = \varepsilon y$ and $\varepsilon << 1$, then Eq. (8) becomes

$$\frac{d^2 P}{dY^2} + \frac{l^2}{\varepsilon^2} P = 0 \tag{9}$$

Look for the solution of the above equation with a form below, which is a first order of approximation of the WKB series of solution.

$$P = \alpha(Y)\exp[-i\frac{l_0}{\varepsilon}\beta(Y)] \tag{10}$$

Substituting the solution (eq. (10)) into eq. (9),

$$[\alpha^{''} - \alpha\frac{l_0^{\ 2}}{\varepsilon^2}\beta^{'2} + \alpha\frac{l^2}{\varepsilon^2}] - i\frac{l_0}{\varepsilon}(2\alpha^{'}\beta^{'} + \alpha\beta^{''}] = 0 \tag{11}$$

In the above equation, the real and imaginary parts should be zero separately. The zero condition of *real part* gives $\alpha^{''} - \alpha\frac{l_0^{\ 2}}{\varepsilon^2}\beta^{'2} + \alpha\frac{l^2}{\varepsilon^2} = 0$. Now, since $\varepsilon \to 0$, the first term can be neglected, and therefore

$$\beta^{'} = \pm(\frac{l}{l_0}) \tag{13}$$

The *imaginary part* of Eq. (11) gives after dividing it by $2\alpha\beta^{'}$.

$$\frac{\alpha^{'}}{\alpha} + \frac{\beta^{''}}{2\beta^{'}} = 0 \quad \rightarrow \quad \frac{d}{dy}(\ln\alpha + \ln(\beta^{'})^{\frac{1}{2}}) = 0 \quad \rightarrow \quad \alpha(\beta^{'})^{\frac{1}{2}} = constant.$$

Substitution of Eq. (13) into the above gives

$$\alpha = const(\frac{l}{l_0})^{-\frac{1}{2}} = const(l^{-\frac{1}{2}}) \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (10), one can obtain

$$P = Cl^{-\frac{1}{2}} \exp[\pm i \frac{1}{\varepsilon} \int_0^Y l dY] \quad \text{or} \quad P = Cl^{-\frac{1}{2}} \exp[\pm i \int_0^y l dy]$$

The above equation describes the meridional structure of wave, and the full solution can be obtained by substituting the above equation into Eq. (7).

$$\Psi = C \frac{1}{\sqrt{l}} \exp[\pm i(kx + \int_0^y l dy - \omega t)] \tag{15}$$

Along the ray of a particular zonal wave number $k = constant$, the wave amplitude is a function of meridional wave number ~ $1/\sqrt{l} = 1/\sqrt{K_S^2 - k^2}$. Usually K_s is a decreasing function of latitude as shown above, and therefore l decreases with latitude, resulting that ***the wave amplitude increases with latitude until the turning latitude where*** $l = 0$ ***with largest wave amplitude. Therefore, the largest amplitude of the waves forced from the tropics, particularly during ENSO, is appeared not near the forcing region but away from the forcing region, particularly in the high latitude where the turning latitudes locates.***

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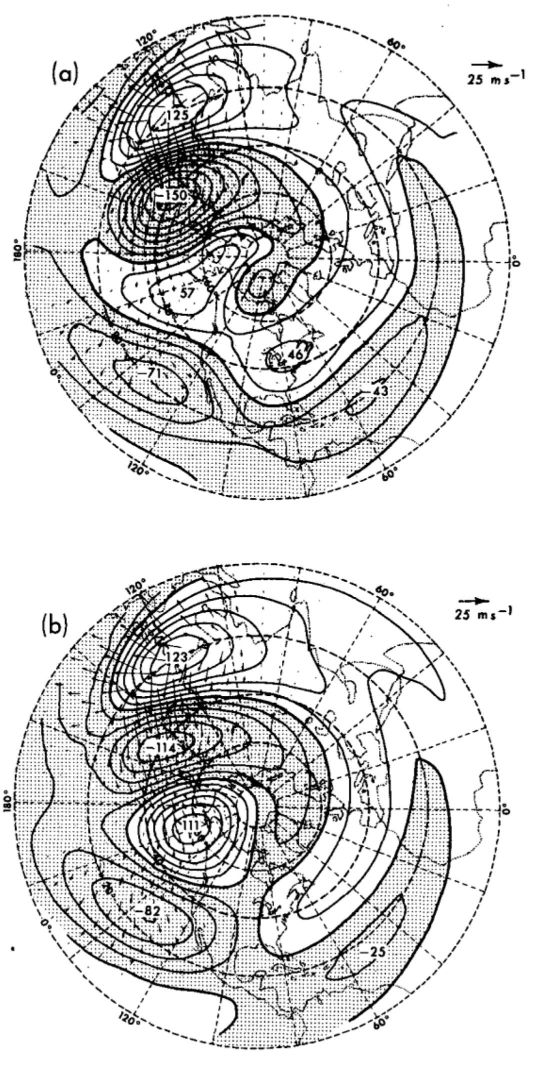


b. The model responses to idealized vorticity sources

We now examine the linear barotropic model responses to idealized vorticity sources located at various latitudes. The following elliptical formula is used for the forcing function.

$$F(\lambda, \phi) = \begin{cases} A\left[\sin\dfrac{\pi(\phi - \phi_1)}{(\phi_2 - \phi_1)} \sin\dfrac{\pi(\lambda - \lambda_1)}{(\lambda_2 - \lambda_1)}\right], \\ \quad \text{for } \lambda_1 < \lambda < \lambda_2 \text{ and } \phi_1 < \phi < \phi_2 \\ 0, \quad \text{otherwise,} \end{cases}$$

where $A = -1 \times 10^{-10}\ \text{s}^{-1}$, $\lambda_1 = 160°\text{E}$, $\lambda_2 = 100°\text{E}$, and $\phi_1 = \phi_0 - 10°$, $\phi_2 = \phi_0 + 10°$, where ϕ_0 is the latitude of the maximum forcing.

For the idealized vorticity source centered at 20°N, the streamfunction responses of the model with basic states $\bar{u}_1$ and $\bar{u}_2$, respectively, are shown in Figs. 8a and 8b. Both figures are characterized by a wave train with a great circle path emanating from the source region. The wave trains are more clearly represented by the two-dimensional wave flux vector $\mathbf{F}_S$ (arrows in the figures) formulated by Plumb (1985). This wave flux vector is defined as

$$\mathbf{F}_S \approx \begin{pmatrix} \dfrac{1}{2a^2\cos^2\phi}\left[\left(\dfrac{\partial\Psi^*}{\partial\lambda}\right)^2 - \Psi^*\dfrac{\partial^2\Psi^*}{\partial\lambda^2}\right] \\ \dfrac{1}{2a^2\cos\phi}\left[\dfrac{\partial\Psi^*}{\partial\lambda}\dfrac{\partial\Psi^*}{\partial\phi} - \Psi^*\dfrac{\partial^2\Psi^*}{\partial\lambda\partial\phi}\right] \end{pmatrix}$$

The vector $\mathbf{F}_S$ is parallel to the group velocity of stationary waves. The divergence and convergence of the flux vector are, respectively, related to the generation and dissipation of wave activity.

In the figures the forced responses near the source are similar to each other, because of negligible difference between $\bar{u}_1$ and $\bar{u}_2$ at 20°N. The local responses can be explained in terms of steady-state vorticity bal-

Fig. Streamfunction respnse of a linearized barotropic model to the idealized vorticity forcing center at 140E and 20N and the Plumb's wave flux vector (arrow). a) is for the zonal-mean basic state of the stronger jet case $\bar{u}_1$, b) for the broader jet case $\bar{u}_2$.

On the other hand, ***the critical latitude*** is defined as the latitude where $\bar{u}$ = c ($\bar{u}$ = 0 for stationary waves). As seen above, the meridional structure of Rossby wave is described by the following equation.

$$\frac{d^2\tilde{\Psi}}{dy^2} + l^2\tilde{\Psi} = 0 \text{ and } l^2 = \frac{\gamma}{\bar{u} - c} - k^2$$

Therefore, in the critical latitude, l has an infinite value when $\bar{u}$ = c. In fact, the equation breaks down at the critical latitude. The meridional group velocity near the latitudes becomes zero, since $G_y = \frac{2\gamma kl}{(k^2 + l^2)^2} \sim \frac{1}{l^3}$ --> 0, and the waves never reaches the critical latitude, meaning that the linear waves never passes the C.L. The critical latitude usually locates at the subtropical latitude (around 10-20N) and therefore the waves generated in one Hemisphere can not propagate to the other Hemisphere. It is noted that the Rossby waves generated in the extratropics are trapped mostly there between the subtropics (the critical latitude) and the high latitude (the turning latitude).

4. Forced Rossby waves by topography

A primary energy source (forcing) of upper-level planetary waves is known to be orography. The flow over topographic obstacles generates planetary scale waves in several ways. .Firstly, the vortex shrinking and stretching caused by compression and expansion of air column, respectively, are balanced by vorticity advection of planetary waves in the presence of mean zonal flow. Secondly, adiabatic heating and cooling induced by rising and sinking motions as air parcel flows uneven topography can be balanced by the temperature advection provided by planetary waves in the presence of mean zonal flow. There are other secondary effects such as orography precipitation and momentum transport by gravity waves, etc. Here we consider only the first mechanism. Note that for a given the observed zonal mean flow and realistic topography, the linear barotropic model can simulate the observed upper-tropospheric geopotential height (streamfunction) reasonably well, as shown by *Grose and Hoskins (JAS, 1979)* and Held (1983).

Orography forcing in a potential vorticity equation

As introduced before, the potential vorticity conservation in a shallow water system can be written as

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)\left(\frac{\zeta+f}{H}\right)=0 \qquad (1)$$

where H is the thickness, $H = h_0 + \eta - h_T$. h_0 is the mean height of the fluid, η is the deviation of free surface from the mean height, and h_T is the height of the rigid lower boundary or topography. In the quasi-geostrophic approximation on a beta-plane where $h_o >> \eta$ & h_T, Eq. (1) becomes to:

$$\left(\frac{\partial}{\partial t} + \vec{v}_g \cdot \nabla\right) q = 0 \qquad (2)$$

where $q = f_0 + \beta y + \varsigma - \frac{f_0(\eta - h_T)}{h_0}$, $\vec{v}_g = \frac{g}{f_0}\vec{k} \times \nabla\eta$ and $\varsigma = \frac{g}{f_0}\nabla^2\eta$

Linearizing about a zonal basic flow $\bar{u}$ independent of x, y and t, one can obtain the following perturbation equation by the topography. The perturbed waves are represented by *.

$$\frac{\partial q^*}{\partial t} + \bar{u}\frac{\partial q^*}{\partial x} + v^*\frac{\partial \bar{q}}{\partial y} = 0 \qquad (3)$$

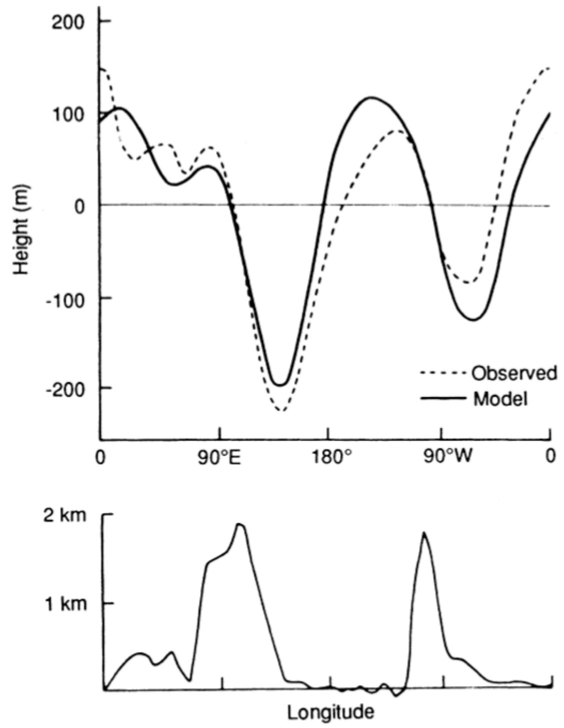
where $\bar{q} = f_0 + \beta y + \bar{\xi} - \frac{f_0\bar{\eta}}{h_0}$, and $q^* = \zeta^* - \frac{f_0}{h_0}(\eta^* - h_T)$ (4)

$$\frac{\partial \bar{q}}{\partial y} = \beta + \frac{\bar{u}}{\lambda^2} \quad ; \quad \lambda^2 = \frac{gh_0}{f_0^{\,2}}$$

An alternative form of Eq. (3) can be expressed as

$$\frac{\partial}{\partial t}\left(\varsigma^* - \frac{f_0}{h_0}\eta^*\right) + \bar{u}\frac{\partial \varsigma^*}{\partial x} + \beta v^* = -\bar{u}\frac{f_0}{h_0}\frac{\partial h_T}{\partial x} \qquad (5)$$

*** Homework.** *Derive eq. (5) from eq. (4).*



(Top) Longitudinal variation of the disturbance geopotential height ($\equiv f_0\Psi/g$) in the Charney–Eliassen model for the parameters given in the text (solid line) compared with the observed 500-hPa height perturbations at 45°N in January (dashed line). (Bottom) Smoothed profile of topography at 45°N used in the computation. (After Held, 1983.)

It may be important to note that the tendency term consists of two terms, vorticity and free surface height variations. It is noted that the rigid upper-boundary condition will eliminate the tendency term of the free surface. Also the steady state condition (eliminating the first term) results in Eq. (5) going back to the non-divergent vorticity equation considered in the previous section.

Substituting a plane wave solution $\eta^* = \mathrm{Re}\left[\tilde{\eta} e^{i(kx+ly-\omega t)}\right]$ and the topography $h_T = \mathrm{Re}\left[\tilde{h}_T e^{i(kx+ly)}\right]$ into Eq. (10), we obtain

$$\left[\omega(k^2+l^2+\lambda^{-2}) - k\left\{\bar{u}(k^2+l^2)-\beta\right\}\right]\tilde{\eta} = -\bar{u}\lambda^{-2}k\,\tilde{h}_T \qquad (6)$$

For free waves (away from the topography) $\tilde{h}_T = 0$,

$$\omega = \frac{k\{\bar{u}(k^2 + l^2) - \beta\}}{k^2 + l^2 + \lambda^{-2}} \qquad (7)$$

In the above, the system cannot provide a stationary Rossby wave $(\omega = 0)$ for a negative zonal-mean flow. But for a positive zonal-mean flow, the stationary wave number is defined as

$$K_s^2 = k_s^2 + l_s^2 = \frac{\beta}{\bar{u}}$$

This stationary wave number is independent of λ, therefore, if one is interested only in the stationary quasi-geostrophic flow, there is no advantage in using the divergent rather than the non-divergent vorticity equation.

For topographically forced stationary waves, the wave amplitude is from (6)

$$\tilde{\eta} = \frac{\tilde{h}_T}{\lambda^2 (K^2 - K_s^2)} \qquad (12)$$

For $K > K_s$ $(K < K_s)$, the wave response is exactly in (out of) phase with respect to topography, with the vorticity forcing by topography balanced primarily by the zonal (meridional) advection of relative (planetary) vorticity. ***It is important to note that the mountain induced (geopotential) height depends not only on the mountain height but also on the stationary wavenumber determined by the zonal mean flow. In particular, the singularity (infinity amplitude) appears at the wave of the stationary wavenumber. This wavenumber is resulted from the resonance of the wave with the zonal mean flow. Thus, the mountain induced waves are largely determined by the given basic state (zonal mean flow) and are thus different for different seasons.***

Following *Charney and Eliassen (1949)*, the resonance singularity ($K = K_s$) is removed by adding a linear damping term $(-\kappa\zeta^*)$ in the vorticity equation. In that case, the solution of $\tilde{\eta}$ is

$$\tilde{\eta} = \frac{\tilde{h}_T}{\lambda^2 (K^2 - K_s^2 - i\varepsilon)}, \quad where \quad \varepsilon \equiv \frac{\kappa K^2}{k\bar{u}}.$$

Then, the final solution of the free surface (height) forced by topography is

$$\eta^* = \mathrm{Re}\left[\frac{\tilde{h}_T}{\lambda^2 (K^2 - K_s^2 - i\varepsilon)} e^{i(kx+ly-\omega t)}\right] \sim \mathrm{Re}\left[\frac{\tilde{h}_T}{\lambda^2 (K^2 - K_s^2 - i\varepsilon)} \sin(ly) e^{i(kx-\omega t)}\right].$$

5. Multiple equilibrium

In their landmark paper on multiple-equilibrium, *Charney and Devore (1979)* (hereafter, *CD*) argued that in the presence of topography, a forced zonal flow (the zonal flow interacted with the waves forced by topography) can satisfy the zonal momentum balance in more than one way. For an incompressible fluid between rigid lids, the zonal momentum balance is

$$\frac{\partial[u]}{\partial t} = \left[v^*\zeta^*\right] + \frac{f}{H_0}\left[v^* h_T^{\ *}\right] - \kappa\left(\left[u\right]-\left[u\right]_e\right) \qquad (1)$$

where the square bracket denotes the zonal mean, and the asterisk the deviation from the zonal mean. In the *CD'* s argument, a relative large value of the second term on the R.H.S. of Eq. (1) can be expressed as a function of the zonal mean flow $[u]$, which is balanced by the third term. Note that the first term plays no role in their quasi-linear argument. Note that we used a same damping coefficient for the waves and zonal mean flow, which could be different from each other.

Eq. (1) can be obtained from the potential vorticity equation <u>with a rigid top condition</u>.

$$\frac{dq}{dt} = 0, \qquad q = f_0 + \beta y + \varsigma + \frac{f_0 h_T}{h_0}$$

Taking the zonal mean, $\frac{\partial[q]}{\partial t} = -\frac{\partial}{\partial y}[v^* q^*]$ (1a)

$\frac{\partial[q]}{\partial t} = \frac{\partial}{\partial t}[\zeta] = \frac{\partial}{\partial t}(-\frac{\partial[u]}{\partial y})$, $\quad -[v^* q^*] = -[v^*\zeta^*] - \frac{f_0}{h_0}[v^* h_T]$ from (4) in previous section.

Substituting the above equations into Eq. (1a) and taking out the y derivative, and adding the Newtonian damping term with respect to the equilibrium basic state $-\kappa\left(\left[u\right]-\left[u\right]_e\right)$, we can obtain Eq. (1).

The purpose of this section is to present multiple states of the balance in the barotropic vorticity equation of Eq. (1) in a steady state. Now express the eddy forcing terms as

$$D([u]) = -[v^*\zeta^*] - \frac{f_0}{h_0}[v^* h_T] \qquad (2)$$

$$v^*\zeta^* = \frac{g}{f}\frac{\partial \eta^*}{\partial x}\cdot\frac{g}{f}\left(-K^2\right)\eta^* = \frac{g^2}{f^2}\left(-K^2\right)\frac{1}{2}\frac{\partial \eta^{*2}}{\partial x} \quad \Rightarrow \quad [v^*\zeta^*] = 0$$

Therefore, the first term of Eq. (2) in R.H.S. is zero, meaning that the zonal-mean meridional vorticity flux does play “no” role in the change of zonal mean wind. The second term can be obtained as follows. Assuming that the topography has a function of $\sin(ly)\exp(ikx)$, the wave has a form below (from the last equation of the previous section).

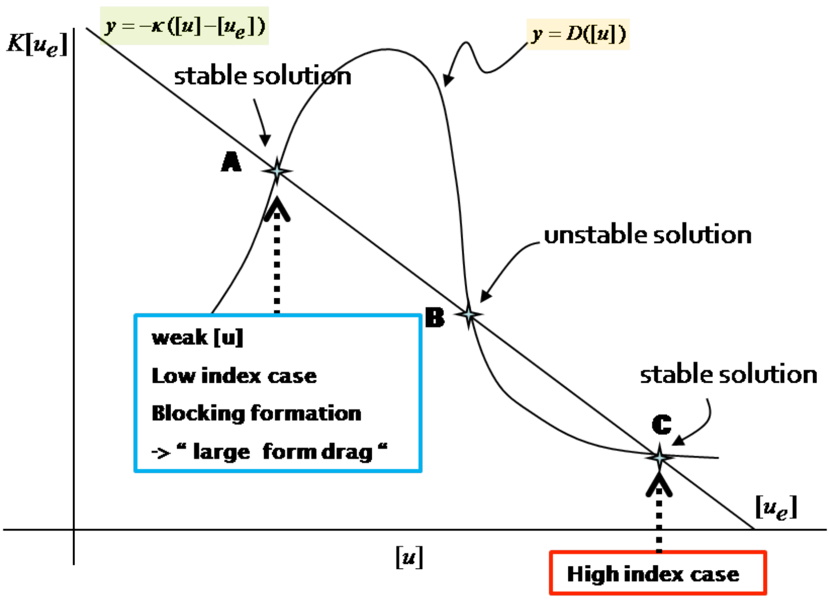
$$\eta^* = \mathrm{Re}\left\{\frac{\tilde{h}_T}{\left[\lambda^2\left(K^2 - K_s^2 - i\varepsilon\right)\right]}\sin ly\, e^{ikx}\right\} = \mathrm{Re}\left\{\frac{\tilde{h}_T\left[\left(K^2 - K_s^2\right) + i\varepsilon\right]}{\lambda^2\left[\left(K^2 - K_s^2\right)^2 + \varepsilon^2\right]}\sin ly\, e^{ikx}\right\}$$

Then using $v^* = \frac{g}{f_o}\frac{\partial \eta^*}{\partial x}$,

$$-\frac{f_0}{h_0}[v^* h_T] = -\frac{f_0}{h_0}\mathrm{Re}\left\{\begin{array}{l}\frac{g}{f_0}\frac{1}{2}\left[\frac{ik\tilde{h}_T\left[\left(K^2 - K_s^2\right) + i\varepsilon\right]}{\lambda^2\left[\left(K^2 - K_s^2\right)^2 + \varepsilon^2\right]}\sin ly e^{ikx} - \frac{ik\tilde{h}_T^*\left[\left(K^2 - K_s^2\right) - i\varepsilon\right]}{\lambda^2\left[\left(K^2 - K_s^2\right)^2 + \varepsilon^2\right]}\sin ly e^{-ikx}\right] \\ \times\frac{1}{2}\left[\tilde{h}_T \sin ly e^{ikx} + \tilde{h}_T^* \sin ly\, e^{-ikx}\right]\end{array}\right\}$$

$$= \frac{\kappa f_0^2 K^2 \left|\tilde{h}_T\right|^2 \sin^2(ly)}{2[u]h_0^2\left[\left(K^2 - K_s^2\right)^2 + \varepsilon^2\right]}, \qquad \lambda^2 = \frac{g h_0}{f_0^2}, \ \varepsilon = \frac{\gamma K^2}{\kappa[u]}$$

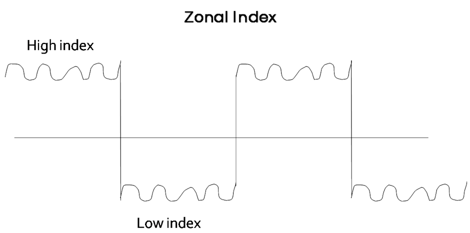
Interestingly, the form drag term by topography represented by - $D([u])$ is determined not only by the topography but also largely by the zonal mean flow. This term can be computed by using observation data for boreal winter and be drawn in the figure below as a function of [u].



A,B,C : solutions (**B** is unstable case !!)

$$0 = -\kappa([u]-[u_e]) - D([u])$$

The steady solution of $[u]$ can be obtained by the graphic representation, that is, the two equations $y = -\kappa([u]-[u_e])$ and $y = D([u])$ produce several points of intersection, those points correspond to the solutions. From the figure, we can obtain the two solutions for the zonal mean flow, which are high and low states. The stability analysis gives the two



stable solutions and one solution is unstable which is hard to exist in the nature. For a strong zonal flow, the stationary wave is relatively small and results in little form drag on the zonal flow so that it is kept near the forced value. A second state occurs with the zonal flow in a resonance with the forced wave, where a large form drag balances a large departure from the equilibrium zonal mean state. Although the argument is based on the linear wave response for a given zonal flow, *CD* also showed multiple stable states in their numerical simulations of the model on a beta-plane channel.

The state A: The difference between the damping term and the form drag, $-\kappa([u]-[u_e]) - D([u])$, becomes to have a negative value as the zonal mean wind increases at the state A, therefore the tendency of zonal wind has a negative sign as the zonal wind increase, and the tendency has a positive value when it decreases. Therefore, the zonal mean wind tends to go back to the equilibrium state when it is perturbed, which is in a “stable” condition. This state is a *low index* case of the zonal mean flow but with a large form drag by large amplitude waves, $'Blocking'$ state.

The state B: This state is “unstable” since with the same argument above, the zonal mean state is departing from the state B further with time after a small perturbation.

The state C: This state is “stable” and has a *high index* of zonal flow with small amplitude waves, $'Normal\ Flow'$ state.