Structure of Tropical and Mid-latitude Atmospheres

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Contents

Very loosely

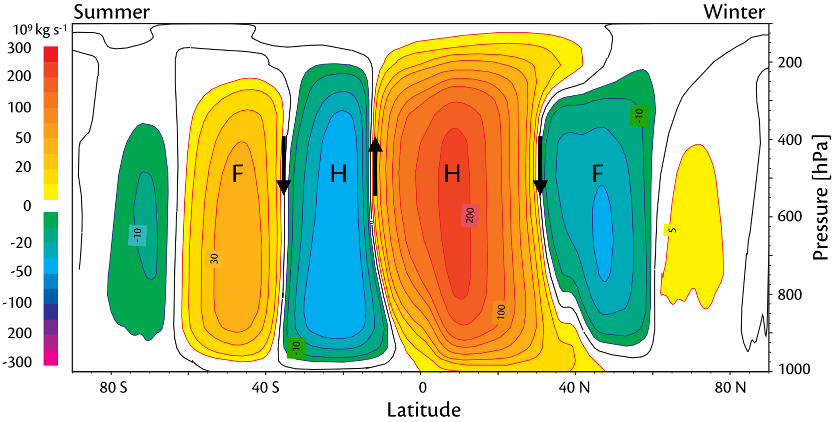
The large-scale structure of the atmosphere in tropics and extra-tropics.

1. General Circulation of the Atmosphere (in brief)
2. Theory of the Hadley Cell.
3. Tropical dynamics
    - Radiative convective equilibrium.
    - Moisture, runaway greenhouse and multiple equilibrium
4. Scale/intensity of motion in tropics and midlatitudes ('weak temperature gradient')
5. Mid-latitude westerlies
6. Ferrel Cell
7. Matsuno Gill Solution.

Short book: **http://tiny.cc/Vallis/essence**

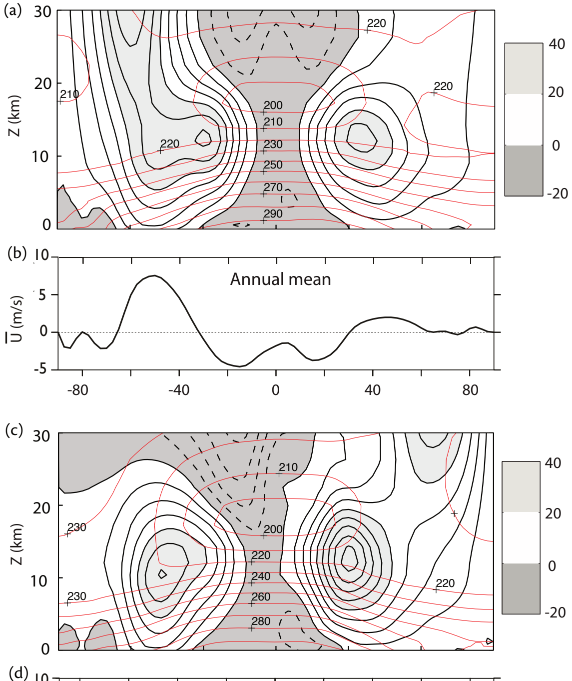
Long book: **http://tiny.cc/Vallis/aofd**

The MOC



Note strong winter Hadley Cell, and Ferrel Cells.

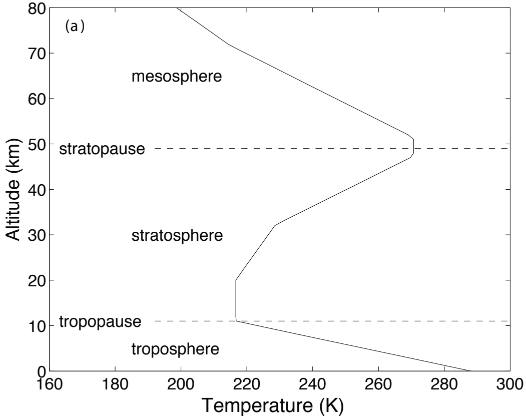
Zonal Average Zonal Wind



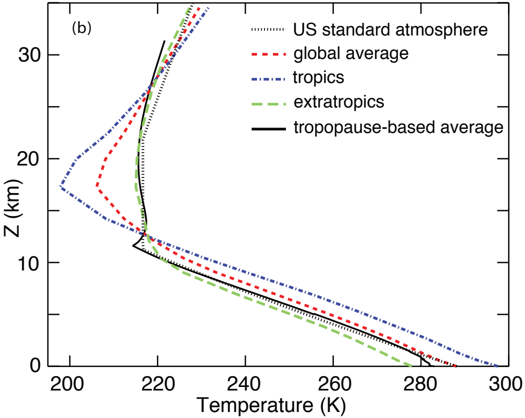
(a) Annual mean, zonally-averaged zonal wind (heavy contours and shading) and the zonally-averaged temperature (red, thinner contours). (b) Annual mean, zonally averaged zonal winds at the surface.

(c) Same except for northern hemisphere winter (DJF).

Temperature Profiles



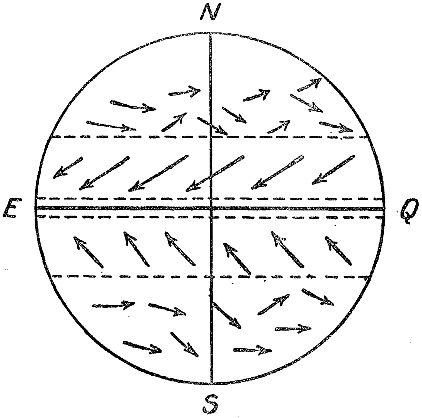
Temperature profile of US standard atmosphere.

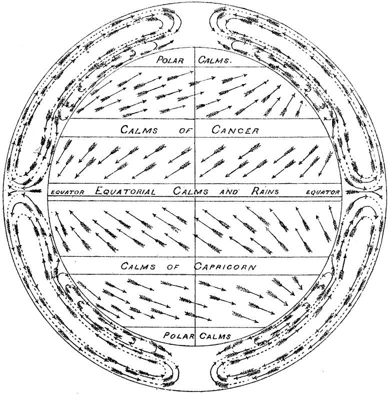


Observed profiles.

Hadley Cell

George Hadley (1685–1768); J. J. Thomson and William Ferrel (19th century); Lorenz (1967, review); Ed Schneider (1977); Held and Hou (1980); Hou (1984) and others.

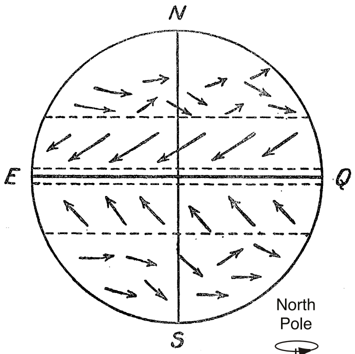


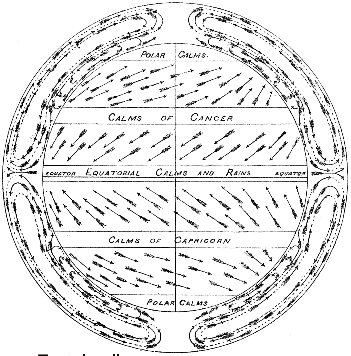


Thomson (1892) (Brother of Lord Kelvin)

Note Pole to equator Hadley Cell, *underneath* which is a shallow indirect cell, the precursor of the Ferrel Cell,

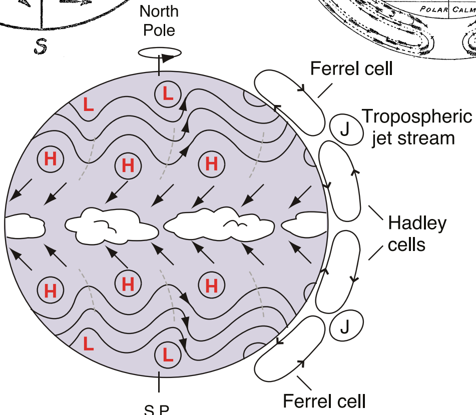
Hadley Cell





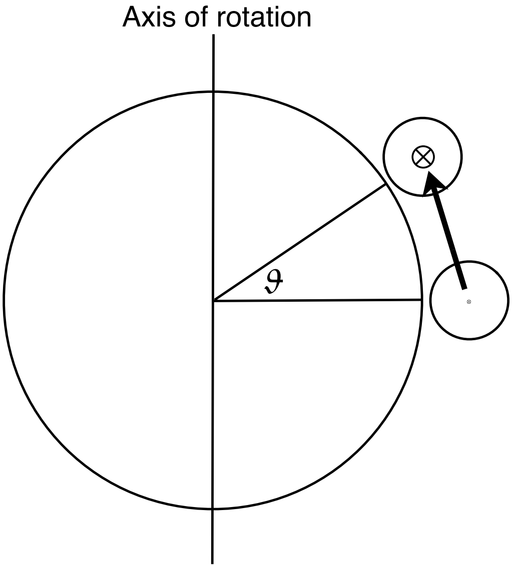
Old View

Thomson (1892), Ferrel (c. 1860)



Modern(ish) view, (Wallace and Hobbs)

Angular Momentum Conservation



Angular momentum conserving wind.

$$m = (u + \Omega a \cos\vartheta) a \cos\vartheta . \qquad (1)$$

If $u = 0$ at equator then $m = \Omega a^2$ so that

$$\Omega a^2 = (u + \Omega a \cos\vartheta) a \cos\vartheta . \qquad (2)$$

and

$$u = \Omega a \frac{\sin^2\vartheta}{\cos\vartheta} \qquad (3)$$

Modern Theory of Hadley Cell

The Hadley Cell cannot extend all the way to the pole on a rotating planet like Earth, for at least two reasons. (It almost can on Venus.)

Zonally-Symmetric Equations of Motion

$$\frac{\partial u}{\partial t} - (f + \zeta)\, v + w \frac{\partial u}{\partial z} = 0. \tag{4}$$

Steady solution

$$(f + \zeta) v = 0. \tag{5}$$

Equivalently, on the sphere, (ϑ = latitude),

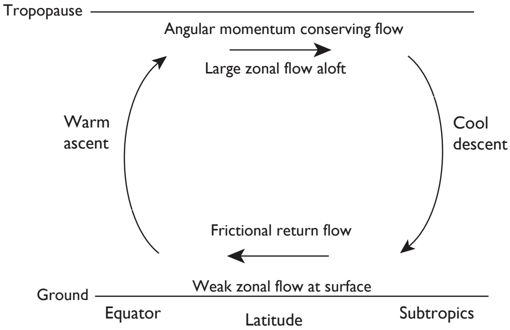
$$v \left(2\Omega \sin\vartheta - \frac{1}{a}\frac{\partial u}{\partial \vartheta} + \frac{u \tan\vartheta}{a} \right) = 0 \tag{6}$$

Solution: either $v = 0$ or

$$u = \Omega a \frac{\sin^2\vartheta}{\cos\vartheta}. \tag{7}$$

The angular momentum conserving wind.

Angular Momentum Conservation



Rising air near the equator moves poleward near the tropopause, descending in the subtropics and returning.

By thermal wind the temperature of the air falls as it moves poleward, gets too cold and sinks.

$$2\Omega \sin\vartheta \frac{\partial u}{\partial z} = -\frac{1}{a}\frac{\partial b}{\partial \vartheta}, \qquad (8)$$

where $b = g\,\delta\theta/\theta_0$.
(Informally, b = temperature.)

THERMODYNAMICS

Forcing via a thermal relaxation to radiative equilibrium temperature θ_E:

$$\theta_E = \theta_{E0} - \Delta\theta \left(\frac{y}{a}\right)^2 . \tag{9}$$

Actual temperature from thermal wind:

$$-\frac{1}{a}\frac{\partial b}{\partial \vartheta} = 2\Omega \sin\vartheta \frac{\partial u}{\partial z}, \tag{10}$$

where $b = g\,\delta\theta/\theta_0$. Gives:

$$\frac{1}{a\theta_0}\frac{\partial \theta}{\partial \vartheta} = -\frac{2\Omega^2 a}{gH}\frac{\sin^3\vartheta}{\cos\vartheta}, \tag{11}$$

and

$$\theta = \theta(0) - \frac{\theta_0 \Omega^2 y^4}{2gHa^2}, \tag{12}$$

Actual temperature cannot fall below radiative equilibrium temperature, so extent of Hadley Cell is:

$$\vartheta_M = \frac{y_M}{a} = \left(\frac{2\Delta\theta g H}{\Omega^2 a^2 \theta_0}\right)^{1/2} \tag{13}$$

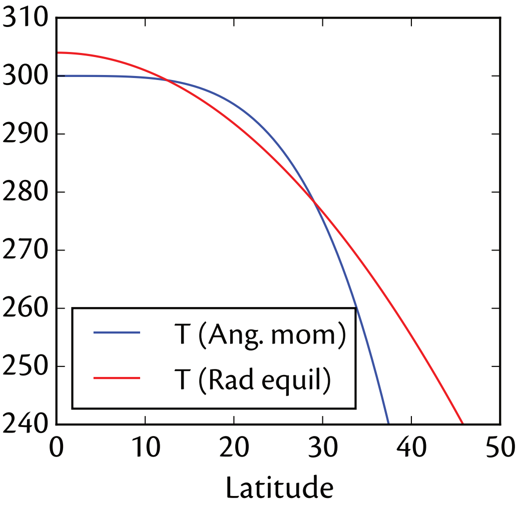
Thermodynamic Budget

- Hadley Cell is thermodynamically self-contained.
- Average forcing temperature = Averaged solution temperature
- Equal area construction gives latitude of edge of Hadley Cell:

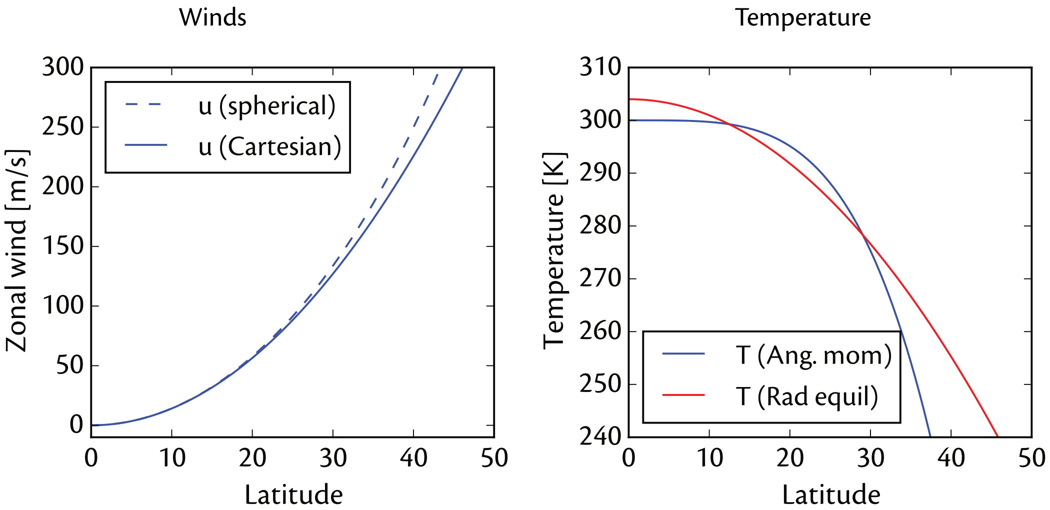
$$\vartheta_H = \left(\frac{5}{3}\frac{gH\Delta\theta_H}{\theta_0\Omega^2 a^2}\right)^{1/2} = a\left(\frac{5}{3}R\right)^{1/2}$$

$$R \equiv \frac{gH\Delta\theta_H}{\theta_0\Omega^2 a^2},$$

Thermal Rossby number.



Ideal Hadley Cell Solution



Blue: Temperature of the angular momentum conserving wind

Red: Radiative equilibrium temperature.

Hadley Cell Strength

$$w \frac{\partial \theta}{\partial z} \approx \frac{\theta_{E0} - \theta}{\tau} \qquad \text{gives} \qquad w \approx \frac{H}{\theta_0 \Delta_V} \frac{\theta_{E0} - \theta}{\tau}. \tag{14}$$

From solution:

$$\frac{\theta_{E0} - \theta}{\tau} = \frac{5R\Delta\theta}{18\tau}. \tag{15}$$

The vertical velocity is then given by

$$w \approx \frac{5R\Delta\theta_H H}{18\tau\Delta\theta_V}. \tag{16}$$

Transform to a streamfunction:

$$\Psi \sim \frac{R^{3/2} a H \Delta_H}{\tau \Delta_V} \propto (\Delta\theta_H)^{5/2}, \tag{17}$$

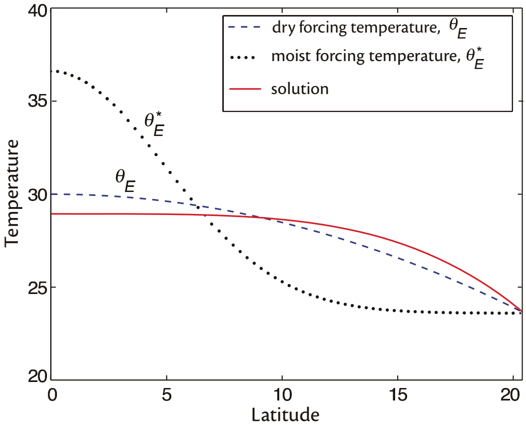
Strength proportional to gradient of radiative-equilibrium meridional temperature gradient.

The Moist Hadley Cell

The Hadley Cell is not 'driven' by convection, or by moisture — but moisture is important!

- Temperature of solution (red line —) unaltered.
- Moist forcing (θ_E^* · · · · · ·) differs more from equilibrium temperature than does dry solution
- So circulation is stronger.
- Moisture is enhancing, not causing, the Hadley Cell. (Also changes the static stability.)

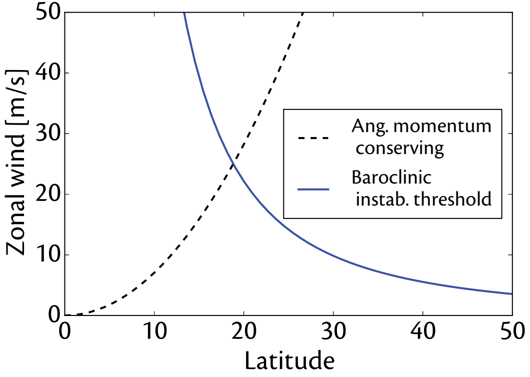
Temperature:



Uh oh!

Baroclinic Instability

The shear is so large it becomes unstable. (The traditional view of Hadley Cell termination.) Use quasi-geostrophic theory::



Critical shear for instability:

$$\Delta U_C = \frac{1}{4}\beta L_d^2 = \frac{H^2 N^2 a}{8\Omega y^2}, \tag{18}$$

Ang mom solution:

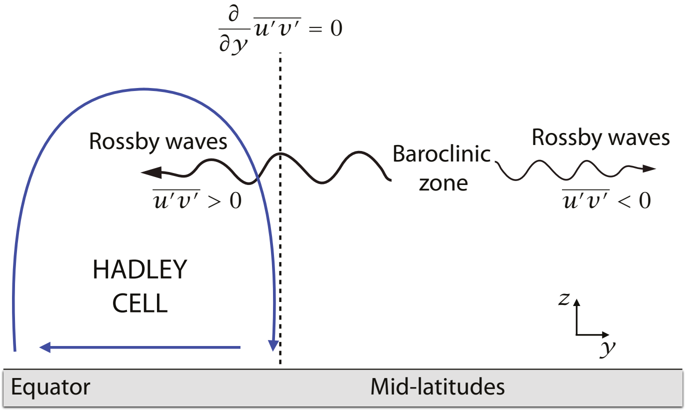
$$\Delta U_M = \frac{\Omega y^2}{2a}, \tag{19}$$

Cross-over latitude (scaling, not exact):

$$\vartheta_C = \frac{y_C}{a} = \left(\frac{NH}{2\Omega a}\right)^{1/2}, \tag{20}$$

On Earth, Hadley Cell is inhibited by baroclinic instability. Not so on Venus.

Baroclinic Instability



Flow satisfies:

$$-(f+\overline{\zeta})\overline{v} = -\frac{\partial}{\partial y}\overline{u'v'}. \qquad (21)$$

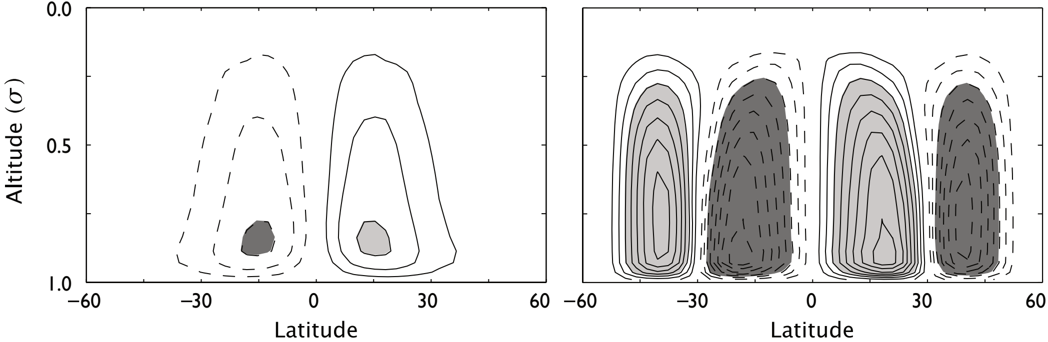
Edge of the Hadley cell where $\overline{v} = 0$ and thus $\partial_y\left(\overline{u'v'}\right) = 0$.
Need not be exactly at onset of baroclinic instability.

Numerical Simulations

Zonally symmetric and 3 D

Courtesy C. Walker

cf., Walker and Schneider (2005)

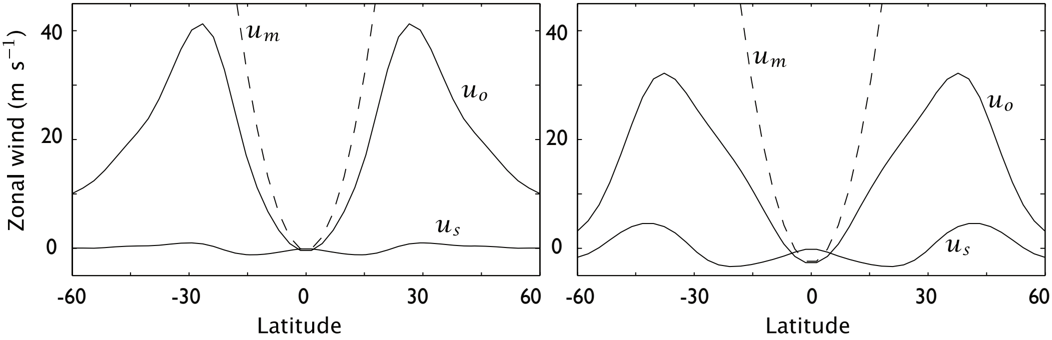


Zonally symmetric 3D

3D simulations have a narrower, stronger Hadley Cell.

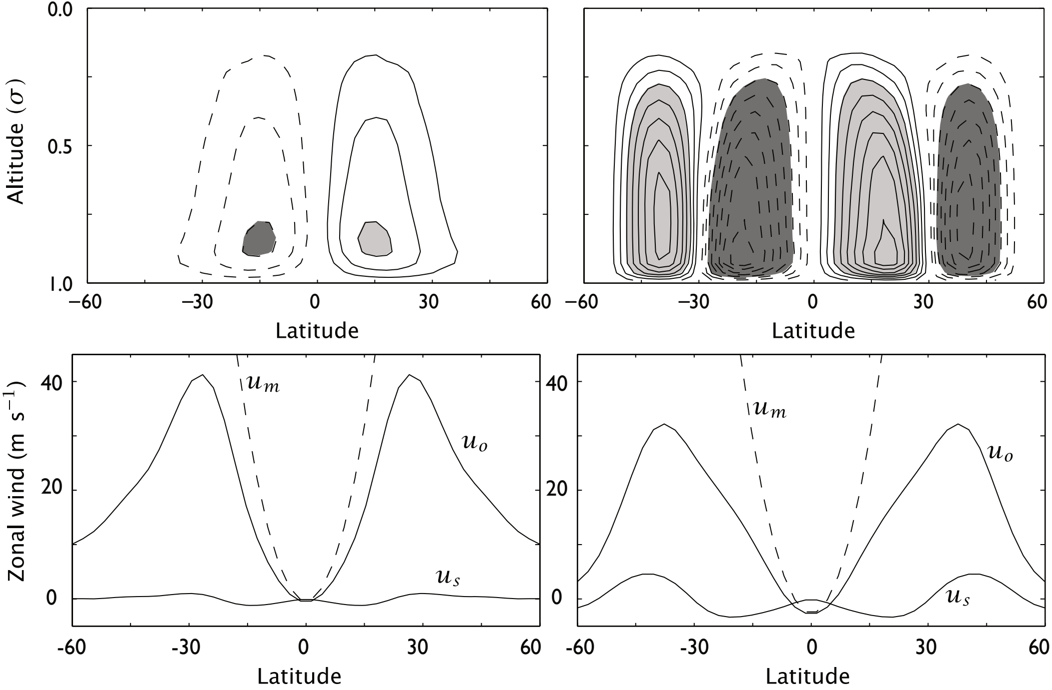
Numerical Simulations

Zonally symmetric and 3 D



Zonally symmetric

3D



Altitude (σ)
0.0
0.5
1.0
−60
−30
0
30
60
Latitude
Zonal wind (m s^{-1})
40
20
0
u_m
u_o
u_s
-60
-30
0
30
60
Latitude

Seasonal Cycle

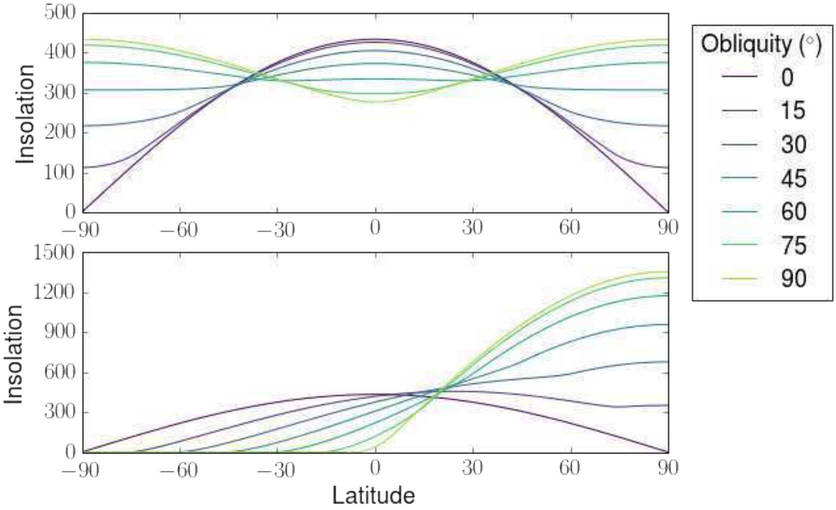
Incoming Solar

Various Obliquities

Top of atmosphere incoming solar radiation

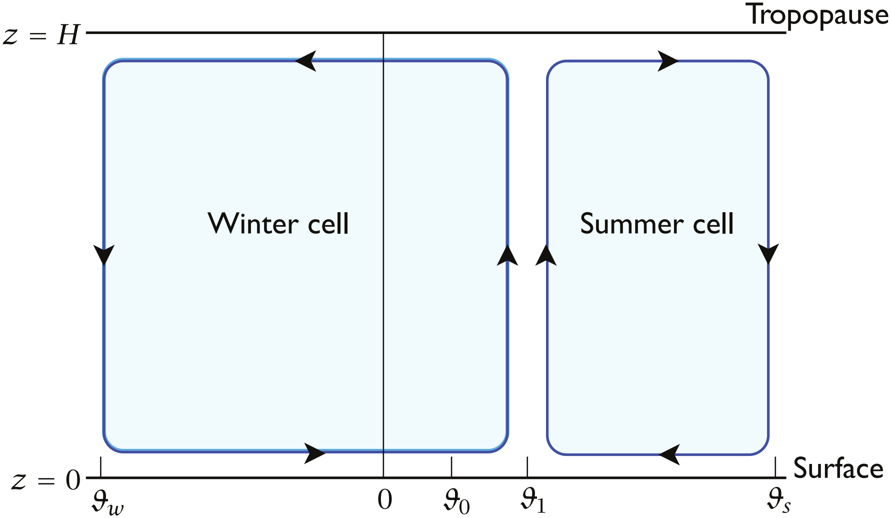
Annual mean

Solstice



The Seasonal Hadley Cell

1. Hadley Cell is *not* centered off the equator.
2. Strong winter cell.



The Seasonal Hadley Cell

'Theory' (following Lindzen and Hou)

1. Quasi-steady (but see Simona's lectures, and monsoon talks next week).
2. Angular momentum conserving.

$$u(\vartheta) = \frac{\Omega a(\cos^2\vartheta_1 - \cos^2\vartheta)}{\cos\vartheta} \,. \tag{22}$$

3. Thermal wind balance

$$f\frac{\partial u}{\partial z} = -\frac{\partial b}{\partial y} \qquad \left(\text{in full:} \quad m\frac{\partial m}{\partial z} = -\frac{g a^2 \cos^2\vartheta}{2\theta_0 \tan\vartheta}\frac{\partial\theta}{\partial\vartheta}\right) \tag{23}$$

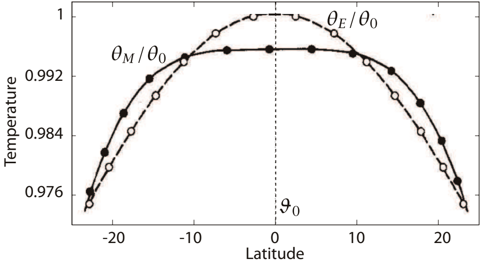
4. Each cell is thermodynamically balanced:

$$\int_{\vartheta_1}^{\vartheta_s} (\theta - \theta_E)\cos\vartheta \, d\vartheta = 0, \qquad \int_{\vartheta_1}^{\vartheta_w} (\theta - \theta_E)\cos\vartheta \, d\vartheta = 0, \tag{24}$$

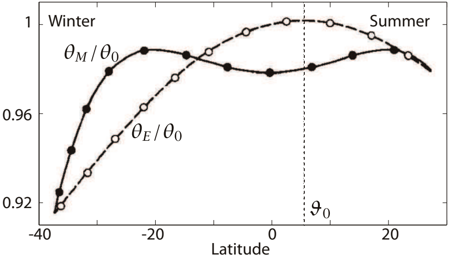
5. Temperature is continuous at edge of Hadley Cell.

Steady seasonal Hadley Cell solution

Lindzen-Hou



Heating centred at the equator.



Heating off the equator. Circulation is dominated by the cell extending from $+18°$ to $-36°$.

Dashed line is the radiative equilibrium temperature and the solid line is the angular-momentum-conserving solution.

Current opinions:

(i) Lindzen–Hou theory is quantitatively *wrong*.

(ii) Winter Hadley cell is more-or-less angular momentum conserving.

(iii) Summer Cell is eddy influenced.

Annual Mean MOC

Simulations, dry Isca, 3D

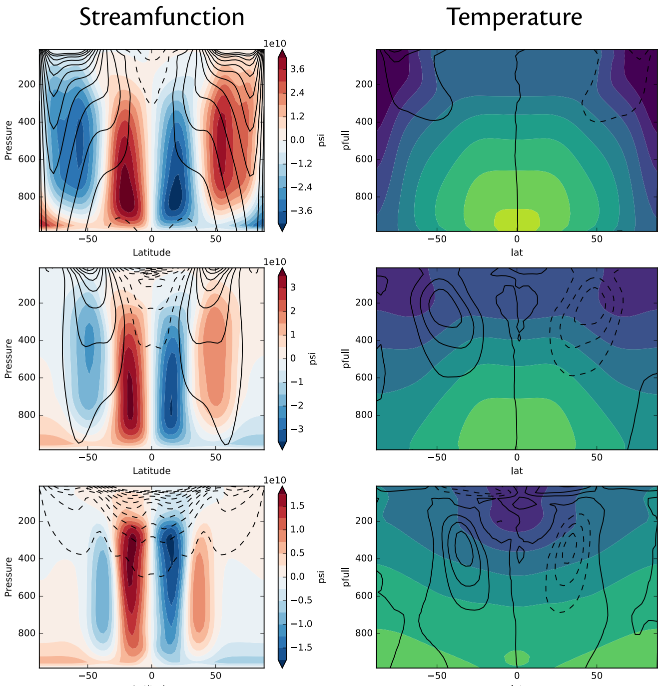
Simulations

by Alex Paterson

Obliquity 0°

Obliquity 30°

Obliquity 60°



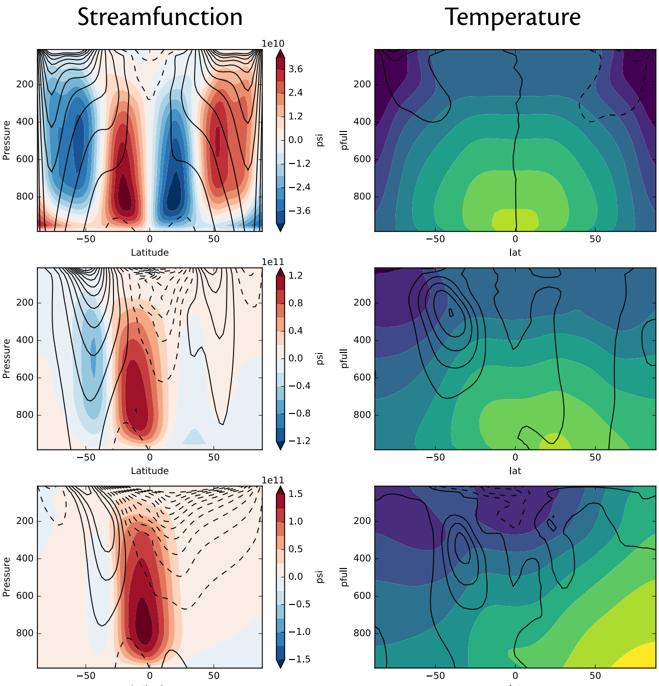
Solstice MOC, 3D

Simulations, dry Isca

Obliquity 0°

Obliquity 30°

Obliquity 60°

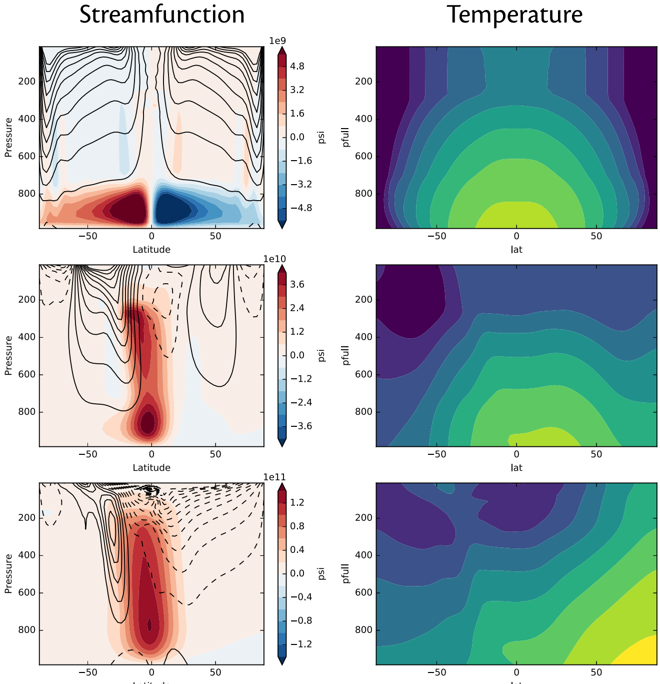


Solstice MOC
Zonally Symmetric

Obliquity 0°

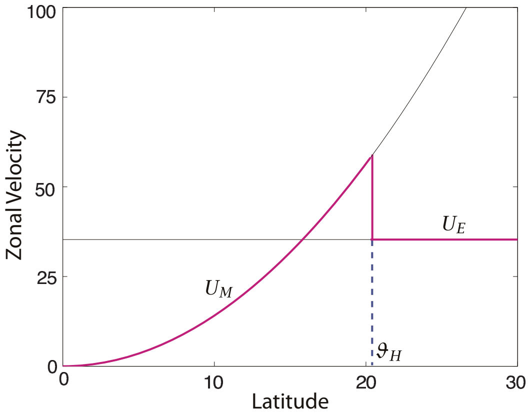
Obliquity 30°

Obliquity 60°



Problems with Hadley Cell

Zonal wind discontinuity

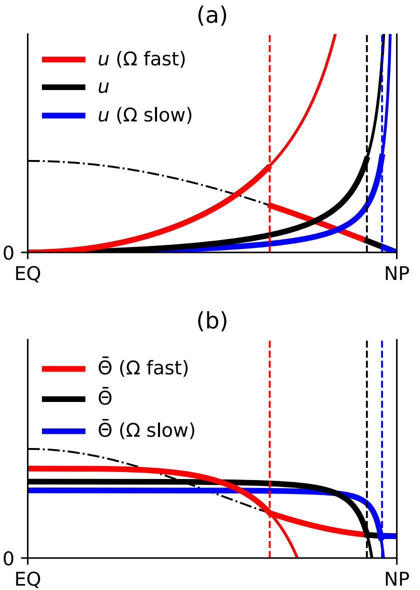


If temperature is continuous, and the high-latitude region is in radiative equilibrium, then zonal wind is discontinuous at the Hadley Cell edge.

Hadley Cell at Various Rotations

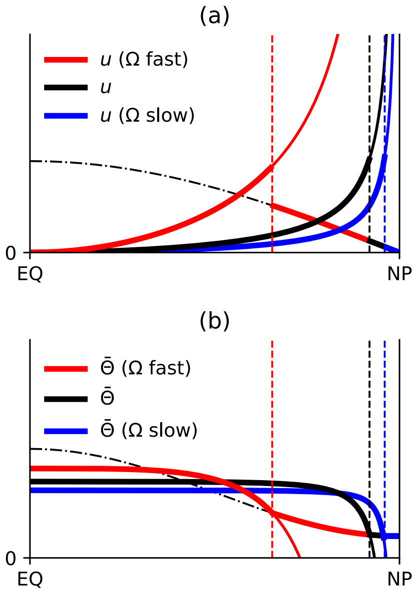
Zonal wind

Temperature

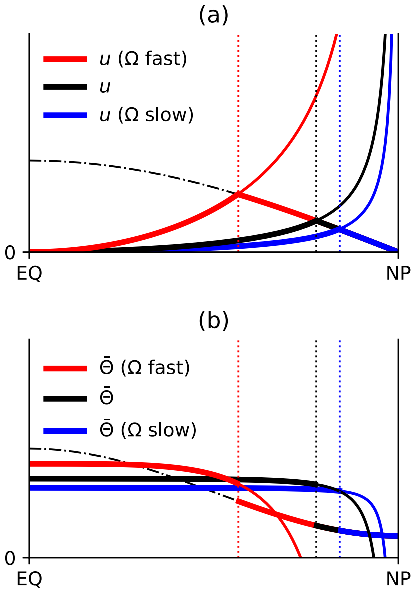


Hadley Cell at Various Rotations (alternate theory)

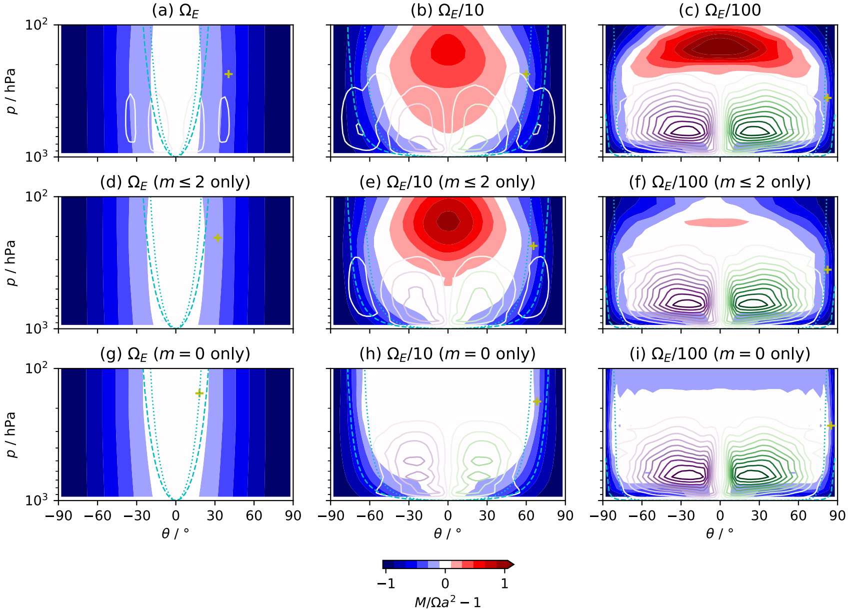
Conventional Theory



Continuous *u*



Hadley Cell with Variable Rotation



Plot of:

$$\frac{M}{\Omega a^2} - 1,$$

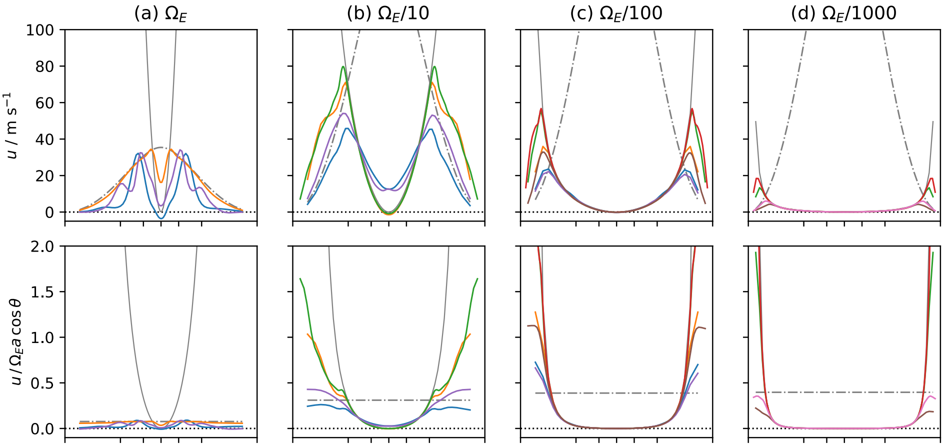
$$M = (u + \Omega a \cos\theta) a \cos\theta$$

Key points

1. Superrotation
2. Angular momentum conservation, especially at low rotation.

Courtesy of Greg Colyer.

Hadley Cell with Variable Rotation



1. — 3D.

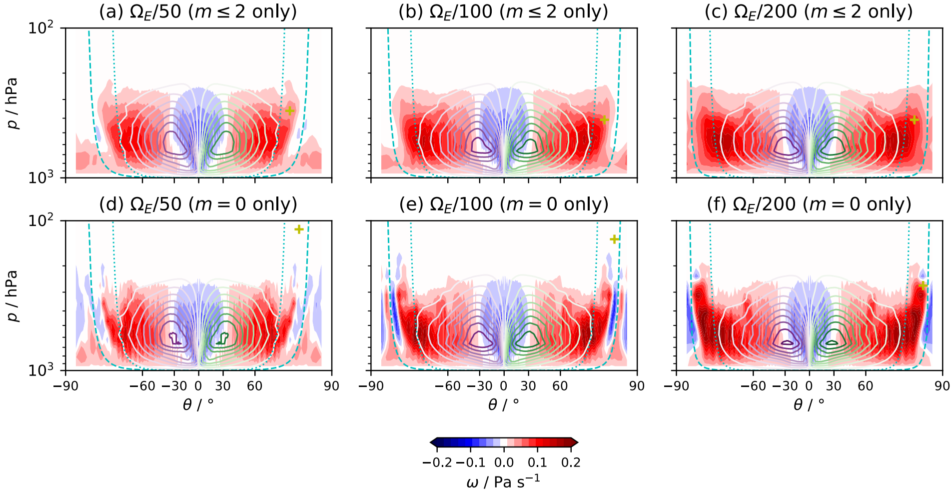
   — Zonally symmetric

2. Hadley Cell extends further as rotation rate falls

3. Zonally-symmetric: – better angular momentum conservation.

Vertical velocity

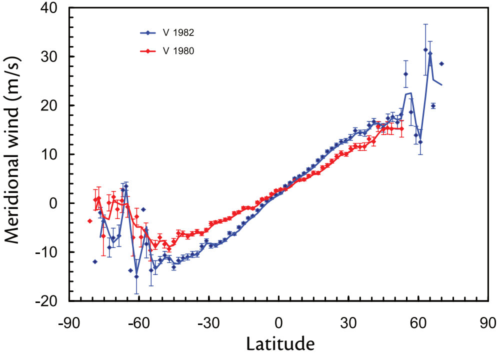
Non-zero overturning circulation even in polar regions.

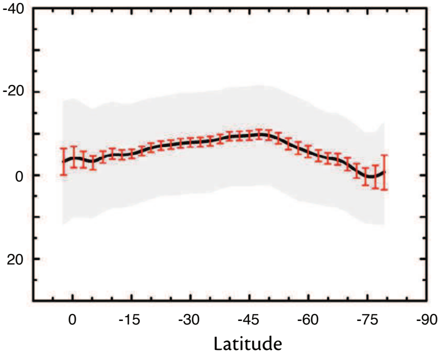


Venus: Hadley Cell

Meridional winds

Sidereal day = 200 Earth days – slow rotation:





(Limaye, 2007 and Khadunstev, 2013)

Venus Hadley Cell extend polewards to about 60°.

Tropical Dynamics

- Radiation
- Convection, quasi-equilibrium.
- Runaway greenhouse
- Weak temperature gradient.

RADIATIVE TRANSFER

$$\mathrm{d}I = I^{\mathrm{in}} - I^{\mathrm{out}} = -\mathrm{d}\tau\ I + \mathrm{d}E. \tag{25}$$

where I is the irradiance, $I\mathrm{d}\tau$ is the absorption and $\mathrm{d}E$ is the thermal emission.
For a gray atmosphere becomes

$$\mathrm{d}I = -\mathrm{d}\tau(I - B) \quad \text{or} \quad \frac{\mathrm{d}I}{\mathrm{d}\tau} = -(I - B). \tag{26}$$

where $B = \sigma T^4$.

Downwards (D) and upwards (U) irradiances are

$$\frac{\mathrm{d}D}{\mathrm{d}\tau} = B - D\,, \qquad \frac{\mathrm{d}U}{\mathrm{d}\tau} = U - B \tag{27}$$

In equilibrium:

$$\frac{\partial(U_L - D_L)}{\partial z} = 0 \quad \text{and} \quad \frac{\partial(U_L - D_L)}{\partial \tau} = 0. \tag{28}$$

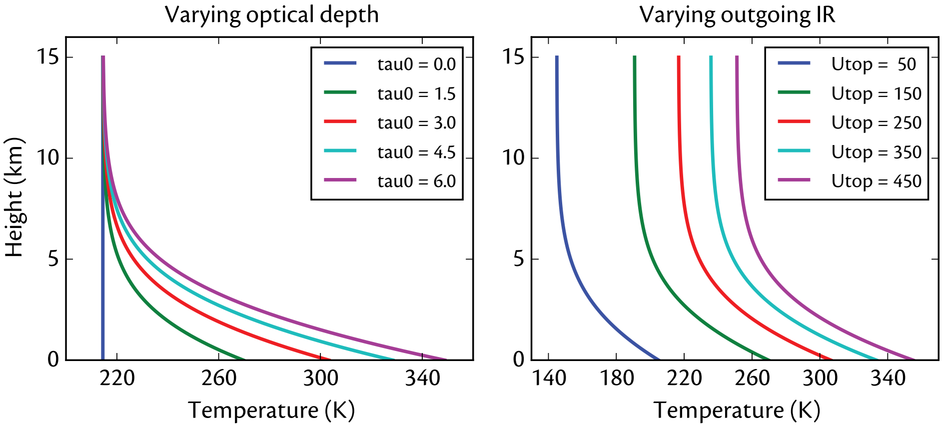
Radiative Equilibrium (Gray)

Solution is:

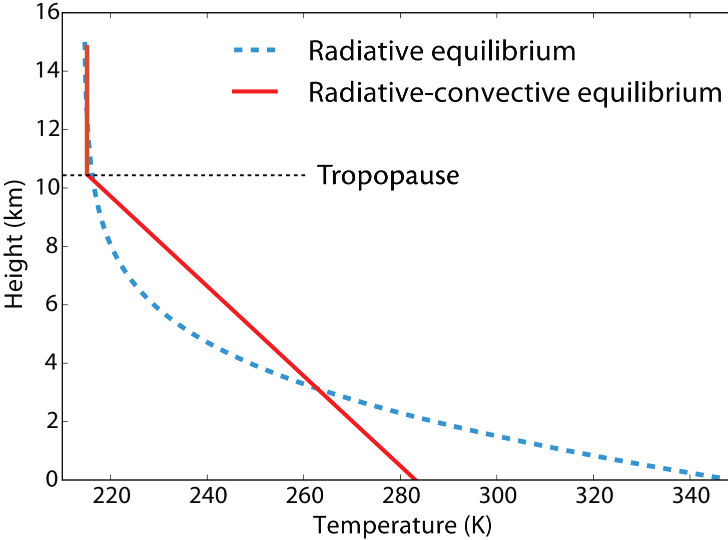
$$D_L = \frac{\tau}{2} U_{Lt}, \qquad U_L = \left(1 + \frac{\tau}{2}\right) U_{Lt}, \qquad B = \left(\frac{1+\tau}{2}\right) U_{Lt}, T = \tag{29}$$

and

$$T^4 = U_{Lt} \left(\frac{1 + \tau_0 e^{-z/H_a}}{2\sigma}\right). \tag{30}$$



Radiative-Convective Equilibrium



Height of tropopause

Analytic calculation: take surface temperature equal to radiative equilibrium.

Tropopause temperature, T_T is (approximately) equal to that at the top of the atmosphere:

$$\sigma T_T^4 = \frac{S_N}{2} = \frac{\sigma T_e^4}{2} \tag{31}$$

Outgoing radiation is determined (to a good approximation) by temperature of tropopause.

The surface temperature, T_S, in radiative equilibrium is

$$\sigma T_S^4 = S_N \left(\frac{1+\tau_0}{2}\right) \qquad \text{or} \qquad T_S = T_T (1+\tau_0)^{1/4}. \tag{32}$$

The height of the tropopause, H_T, is then such that $(T_S - T_T)/H_T = \Gamma$ giving

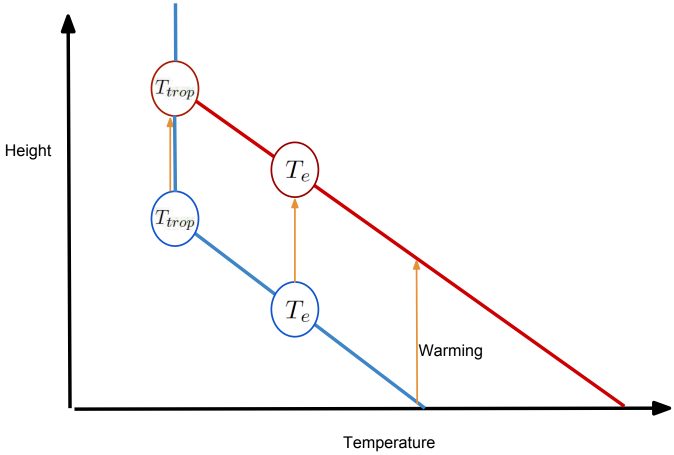
$$H_T = \frac{T_S - T_T}{\Gamma} = \frac{T_T}{\Gamma}\left((1+\tau_0)^{1/4} - 1\right). \tag{33}$$

(Overestimate, but gets the scaling almost right.)

Better:

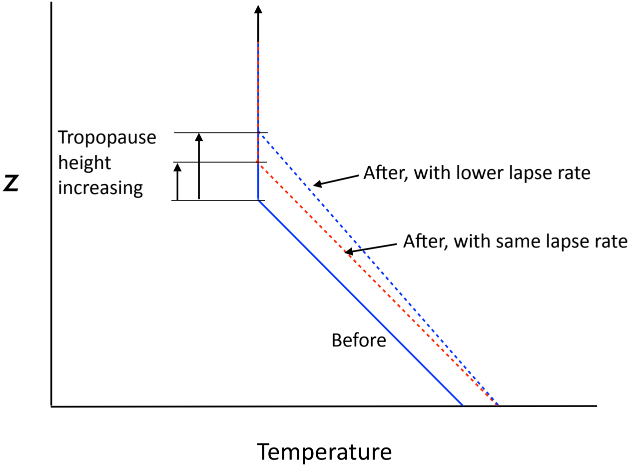
$$H_T = \frac{1}{16\Gamma}\left(C T_T + \sqrt{C^2 T_T^2 + 32 \Gamma \tau_s H_a T_T}\right). \tag{34}$$

Tropopause Height
with global warming



Increase in tropopause height with global warming is unavoidable!

Lapse Rate and Temperature Effects



$$\Delta H_T = \frac{\Delta T}{\Gamma} - \frac{H_T \Delta \Gamma}{\Gamma}$$

$$\Delta H_T = \frac{\Delta T}{\Gamma} - \frac{H_T \Delta \Gamma}{\Gamma}$$

H_T is the tropopause height.

ΔT is the increase in tropospheric temperature.

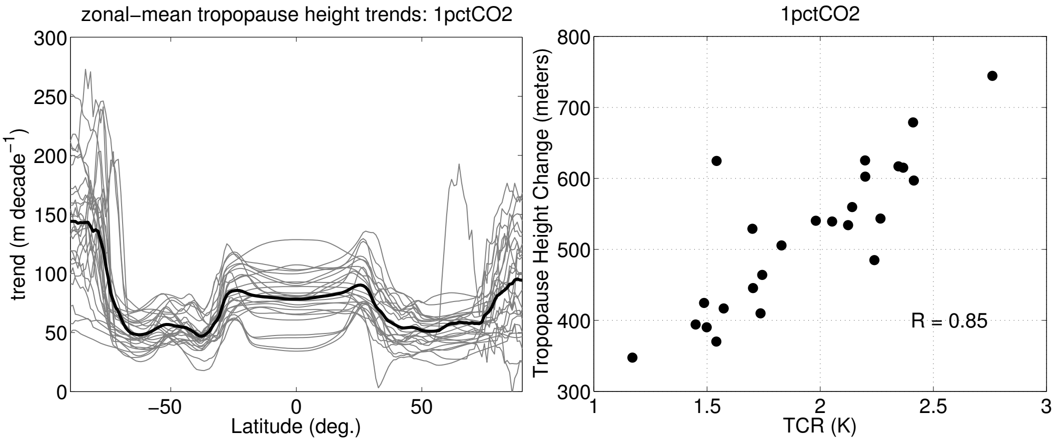
Γ is in the lapse rate.

$\Delta\Gamma$ the change in the lapse rate.

Both effects are comparable.
Predict about 300 m increase per degree Celsius:

$\Delta H_T = 300\,\Delta T$

CMIP5 results about Tropopause height



Tropopause height is projected to increase in all models at about the same rate.

Venus: Tropopause height

- Venus: Surface pressure = 92 bars (Earth surface pressure = 1 bar!)

- Atmosphere almost entirely CO_2 so enormous greenhouse effect! (Almost carbon dioxide rain).

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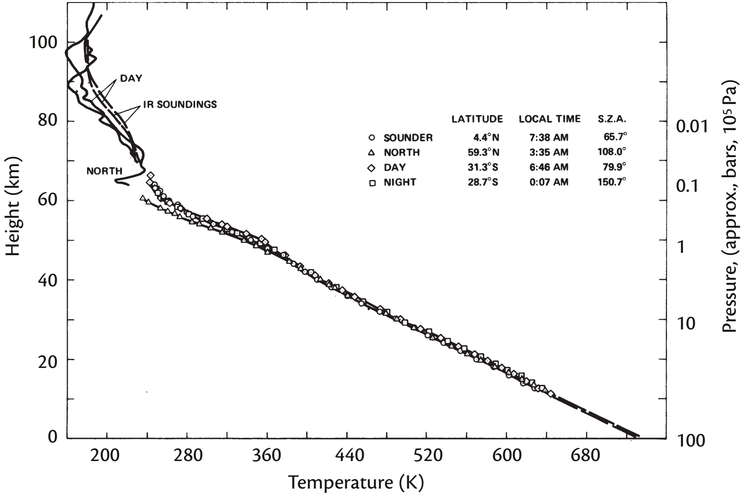
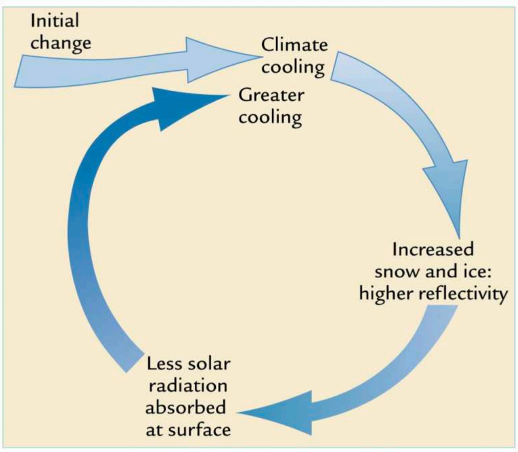


Pioneer mission (Seiff 1979)

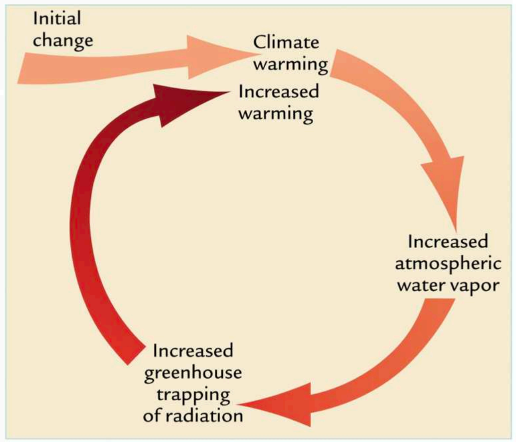
Tropopause height is about 60 km.

Feedbacks and Multiple Equilibrium

Ice-albedo feedback

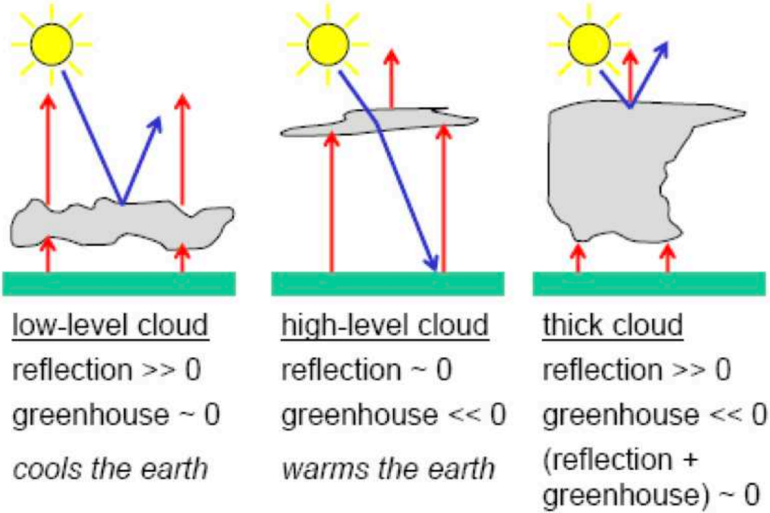


Water-vapor radiative feedback



Diversion
Cloud Feedbacks

There is no nice loop for cloud feedbacks! (Big uncertainty for global warming.)



Multiple Equilibrium due to Water Vapour

Simplest possible EBM. Zero-dimensional

$$\epsilon\sigma T^4 = S(1-\alpha) \tag{35}$$

Ice-albedo feedback makes α a function of temperature.

Water vapour feedback makes ϵ a function of temperature.

Clausius–Clapeyron: water vapour increases approximately exponentially with temperature.

$$e_s \approx e_0 \exp(\gamma T) \tag{36}$$

Simple model of runaway greenhouse

From radiative-equilibrium model, surface temperature is related to TOA temperature by:

$$\begin{aligned}\sigma T_S^4 &= \sigma T_e^4 (1 + \tau_0) \\ &= S(1 - \alpha)(1 + \tau_0)\end{aligned}$$

because $\sigma T_e^4 = S(1 - \alpha)$. (That is, $\epsilon = (1 + \tau_0)^{-1}$.)

Water Vapour Feedback

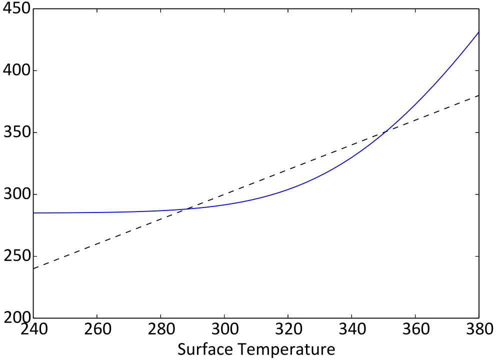
$$\tau_0 = A + Be_s(T_S) \tag{37}$$

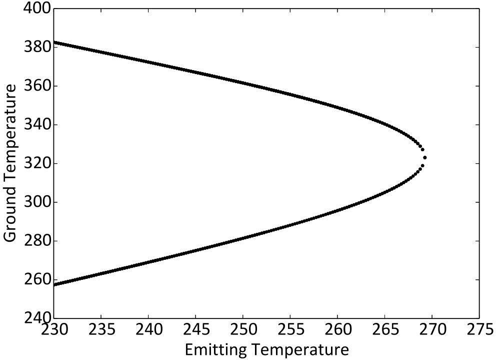
where $e_s = e_0 \exp(\gamma T)$

Surface temperature given by

$$\sigma T_S^4 = S(1 - \alpha)(1 + [A + Be_s(T_S)]) \,. \tag{38}$$

Solutions





Dashes: - - T_S (Surface temperature)

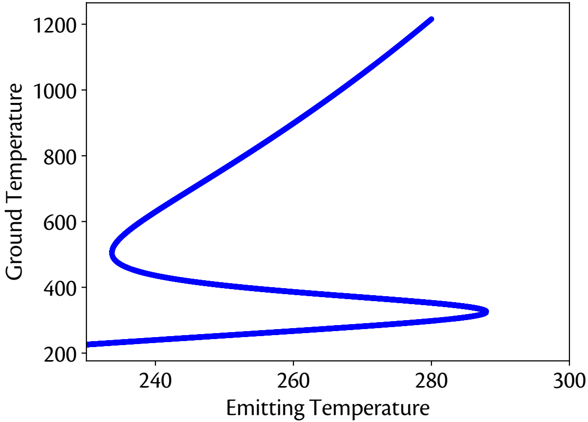
Solid: — $T_e^4(1 + \tau_0(T_g)/2)^{1/4}$

Note that higher temperature state increased emitting temperature $\rightarrow$ *lower* surface temperature.

There is no solution at very high solar constant!!

Solutions

With an IR window



Middle solution is unstable.

Venus may have gone like this (runaway greenhouse).

Scaling of Motion

b is 'temperature'

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \mathrm{f} \times \boldsymbol{u} = -\nabla\phi, \qquad \frac{\partial \phi}{\partial z} = b, \tag{39}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} + N^2 w = 0, \qquad \nabla \cdot \boldsymbol{v} = 0. \tag{40}$$

Think of b as the temperature. Scales are:

$$(x, y) \sim L, \quad z \sim H, \quad (u, v) \sim U, \quad w \sim W, \quad t \sim \frac{L}{U}, \\ \phi \sim \Phi, \quad b \sim B, \quad f \sim f_0. \tag{41}$$

Nondim numbers:

$$Ro = \frac{U}{f_0 L}, \quad Bu = \left(\frac{L_d}{L}\right)^2 = \left(\frac{NH}{f_0 L}\right)^2, \quad Ri = \left(\frac{NH}{U}\right)^2, \tag{42}$$

Tropical vs Midlatitudes

Midlatitudes:

$$\mathbf{f} \times \boldsymbol{u} \approx -\nabla_z \phi, \quad \frac{\partial \phi}{\partial z} = b, \quad \Longrightarrow \quad \boxed{\Phi = f_0 U L, \qquad B = \frac{f_0 U L}{H}.} \tag{43}$$

Tropics

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} \approx -\nabla_z \phi, \quad \frac{\partial \phi}{\partial z} = b, \quad \Longrightarrow \quad \boxed{\Phi = U^2, \qquad B = \frac{U^2}{H}.} \tag{44}$$

Since $U^2 < f_0 U L$, *variations of pressure and temperature are smaller in the tropics than in mid-latitudes.*

Weak temperature gradient approximation. (Charney (1963)

There is a assumption in this argument ...

Tropical vs Midlatitudes

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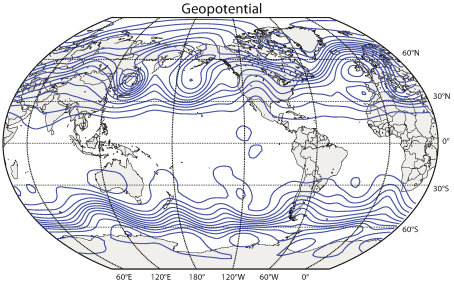
Since $U^2 < f_0 U L$, *variations of pressure and temperature are smaller in the tropics than in mid-latitudes.*

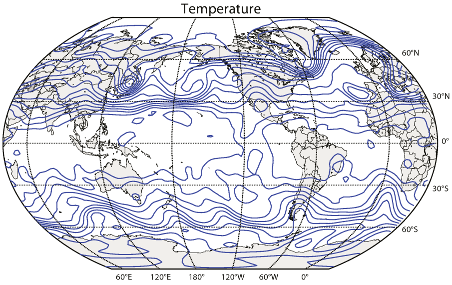
Weak temperature gradient approximation. (Charney (1963)

There is a assumption in this argument …that the winds are similar in tropics and midlatitude. (Might have had same temperature gradient and then have higher winds in the tropics.)

Observations

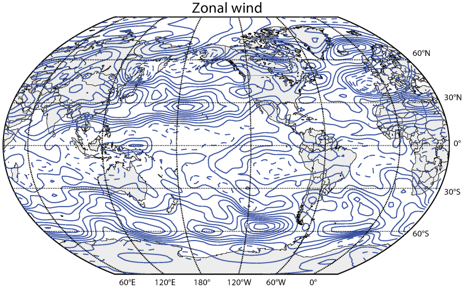
Pressure, Temperature, Wind





↑ Geopotential ('pressure')

Temperature ↑



← Winds

Re-analysis (observations)
Feb 9, 2016

Weak temperature gradient with diabatic sources

Vorticity-divergence form:

$$\frac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{u} h) = Q, \tag{45}$$

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \left[\boldsymbol{u}(\zeta + f_0)\right] = -r\zeta, \tag{46}$$

$$\frac{\partial \delta}{\partial t} + \nabla^2 \left(\frac{1}{2}\boldsymbol{u}^2 + gh\right) - \mathrm{k} \cdot \nabla \times \left[\boldsymbol{u}(\zeta + f_0)\right] = -r\delta, \tag{47}$$

Weak temperature gradient, equations become

$$\nabla \cdot \boldsymbol{u} = \frac{Q}{H} \tag{48}$$

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u} \cdot \nabla(\zeta + f_0) + (\zeta + f)\frac{Q}{H} = -r\zeta, \tag{49}$$

$$g\nabla^2 h = \mathrm{k} \cdot \nabla \times \left[\boldsymbol{u}(\zeta + f_0)\right] - \frac{1}{H}\frac{\partial Q}{\partial t} - r\delta - \nabla^2 \frac{\boldsymbol{u}^2}{2}. \tag{50}$$

Mid-latitudes

1. Jets
2. Ferrel Cell
3. Residual Circulation
4. Ferrel Cell

2 D equations of motion

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}-fv=-\frac{\partial \phi}{\partial x}-ru, \tag{51}$$

Zonal average

$$\frac{\partial \overline{u}}{\partial t}+\frac{\partial \overline{u'v'}}{\partial y}=-r\overline{u}, \tag{52}$$

Since

$$v\zeta=\frac{1}{2}\frac{\partial}{\partial x}\left(v^2-u^2\right)-\frac{\partial}{\partial y}(uv). \tag{53}$$

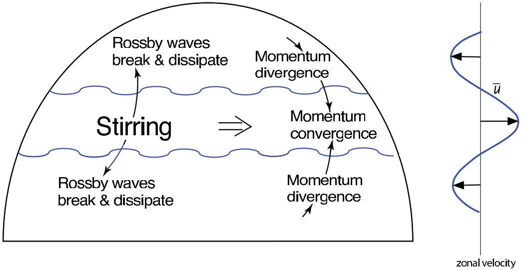
then

$$\overline{v'\zeta'}=-\frac{\partial \overline{u'v'}}{\partial y}, \tag{54}$$

and (52) becomes

$$\frac{\partial \overline{u}}{\partial t}=\overline{v'\zeta'}-r\overline{u}. \tag{55}$$

Rossby waves and jets



Rossby waves generated in mid-latitudes. Must propagate *away* from disturbance.

$$\omega = ck = \overline{u}k - \frac{\beta k}{k^2 + l^2} \equiv \omega_R, \qquad (56)$$

The meridional component of the group velocity:

$$c_g^y = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2 + l^2)^2}\,. \qquad (57)$$

So that $kl >$ north of disturbance and $kl < 0$ south of disturbance.

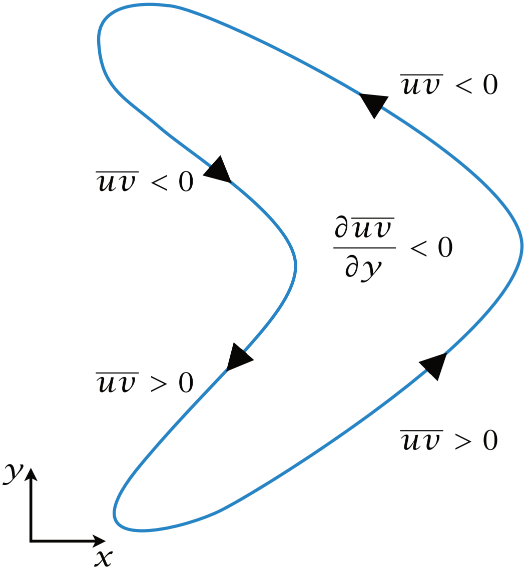
Velocity variations are

$$u' = -\mathrm{Re}\, C\, \mathrm{i} l \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \qquad v' = \mathrm{Re}\, C\, \mathrm{i} k \mathrm{e}^{\mathrm{i}(kx+ly-\omega t)}, \qquad (58)$$

Associated momentum flux is

$$\overline{u'v'} = -\frac{1}{2}C^2 kl. \qquad (59)$$

of opposite sign to group velocity!

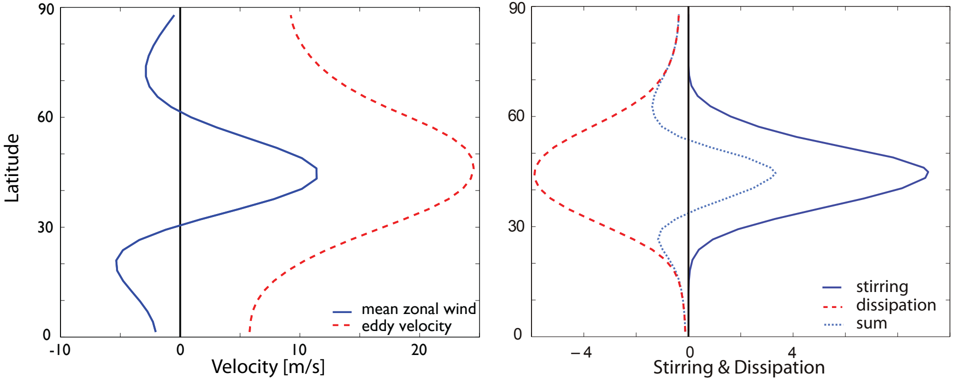


$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \overline{u'v'}}{\partial y} = -r\bar{u}, \tag{60}$$

Since $\partial \overline{u'v'}/\partial y < 0$ in region of forcing, flow accelerates eastward there.

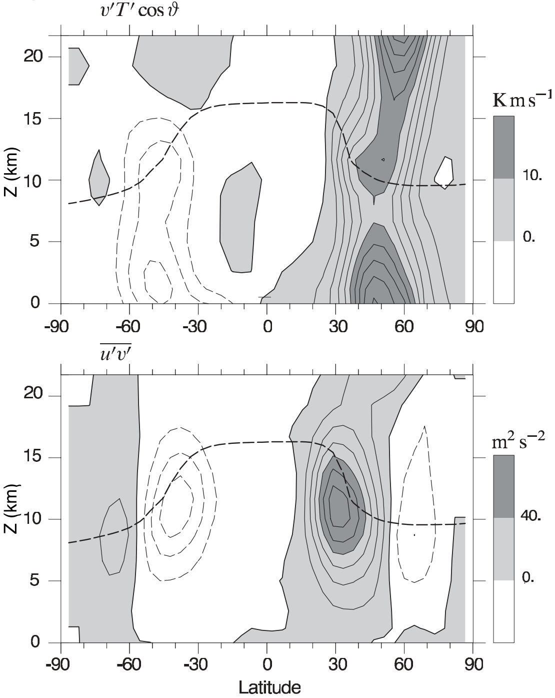
Numerical Simulation

Barotropic model on the sphere



Randomly stirred in midlatitudes.

Observed Eddy Fluxes of Heat and Momentum



$\overline{v''T'}$

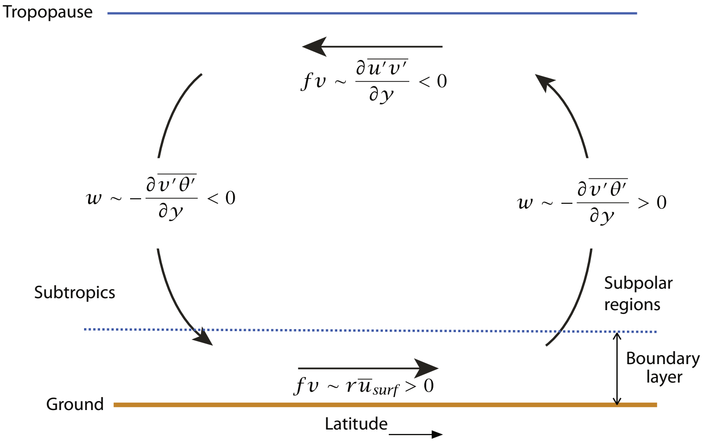
$\overline{u'v'}$

These same heat and momentum fluxes 'drive' the Ferrel Cell.

Ferrel Cell

William Ferrel (American, 1817–1891) got it wrong, but led the way to getting it right.

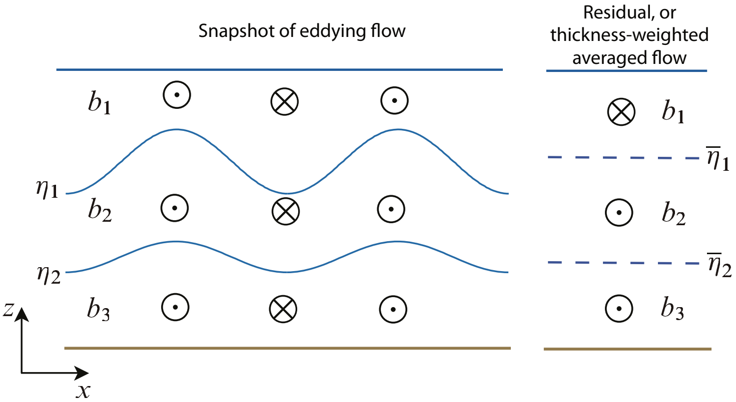
Zonal average u equation:
$$\frac{\partial \overline{u}}{\partial t} - (f + \overline{\zeta})\overline{v} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{\partial}{\partial y}\overline{u'v'} - \frac{\partial}{\partial z}\overline{u'w'} + \frac{1}{\rho}\frac{\partial \tau}{\partial z}. \quad (61)$$



Low Rossby number $|f| \gg \zeta$, steady flow:

$$-f\overline{v} = -\frac{\partial}{\partial y}(\overline{u'v'}) + \frac{1}{\rho}\frac{\partial \tau}{\partial z}. \quad (62)$$

RESIDUAL CIRCULATION



$$\overline{vh} = \overline{v}\,\overline{h} + \overline{v'h'}$$

$$\overline{v'h'} / \overline{h}$$

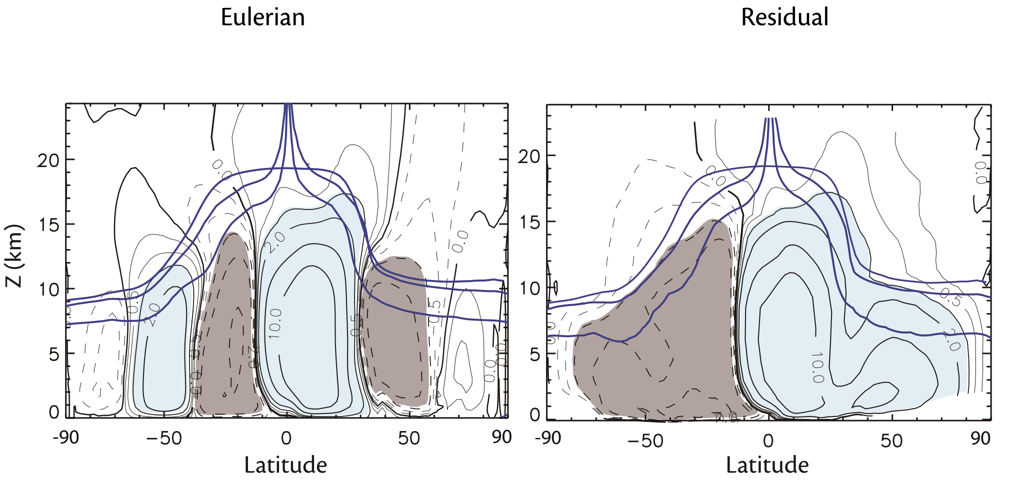
Thickness weighted transport is:

$$\overline{v}^* \equiv \overline{v} + \frac{1}{\overline{h}}\overline{v'h'} = \overline{v} + v_{\mathrm{eddy}}$$

The thickness weighted transport takes into account eddy transport effects.

In a continuous model the thickness is replaced by use of isentropic co-ordinates:
thickness $\propto$ 1/temperature

Residual Overturning Circulation



Quasi-Geostrophic Residual Equations

Eulerian Zonal Average

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v} = \overline{v'\zeta'}, \qquad \frac{\partial \overline{b}}{\partial t} + N^2 \overline{w} = \overline{S}, \tag{63}$$

Residual Equations

The residual velocities are:

$$\overline{v}^* = \overline{v} - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'b'}\right), \qquad \overline{w}^* = \overline{w} + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\overline{v'b'}\right). \tag{64}$$

$$\frac{\partial \overline{u}}{\partial t} - f_0 \overline{v}^* = \overline{v'q'} + \overline{F} \qquad \frac{\partial \overline{b}}{\partial t} + N^2 \overline{w}^* = \overline{S}, \tag{65}$$

Advantages:

(i) Only potential vorticity flux.

(ii) No eddies in thermodynamic equation.
— So the residual circulation is 'direct' – a big pole-equator Hadley Cell similar to the original concept.

Simple Theory

Downgradient diffusion of potential vorticity

$$\overline{v'q'} = -\mathcal{K}\frac{\partial \overline{q}}{\partial y} \approx -\mathcal{K}\beta. \tag{66}$$

Giving, in steady state,

$$-f_0\overline{v}^* = -\mathcal{K}\beta, \qquad \text{or} \qquad \boxed{\overline{v}^* = \frac{\mathcal{K}\beta}{f_0}} \tag{67}$$

Residual polewards flow in upper branch *is a consequence of flux of potential vorticity!*

Steady state

Thermodynamic and Momentum equations need to be consistent:

$$\frac{\partial \overline{v}^*}{\partial y} + \frac{\partial \overline{w}^*}{\partial z} = 0, \qquad \text{gives} \qquad \frac{\partial}{\partial y}\left(\overline{v'q'}\right) = f_0\left(\frac{\partial F}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\overline{S}f_0}{N^2}\right). \tag{68}$$

which is the condition that the steady PV equation is satisfied.

An Equation for the MOC

Use thermal wind to eliminate time dependence in (65)

Residual MOC

$$f_0^2 \frac{\partial^2 \psi^*}{\partial z^2} + N^2 \frac{\partial^2 \psi^*}{\partial y^2} = f_0 \frac{\partial}{\partial z} \overline{v'q'} + f_0 \frac{\partial F}{\partial z} + \frac{\partial Q_b}{\partial y}. \tag{69}$$

This equation holds at all times, even in time-dependent flow.

– PV fluxes and diabatic effects both ‘drive’ the MOC.

– Similar equation can also be applied to Hadley Cell.

Eulerian MOC

By comparison:

$$f_0^2 \frac{\partial^2 \Psi}{\partial z^2} + N^2 \frac{\partial^2 \Psi}{\partial y^2} = f_0 \frac{\partial M}{\partial z} + \frac{\partial J}{\partial y}. \tag{70}$$

where

$$\frac{\partial M}{\partial z} = -\frac{\partial}{\partial z} \left(\frac{\partial (\overline{u'v'})}{\partial y} \right) + \frac{\partial F_u}{\partial z}, \qquad \frac{\partial J}{\partial y} = \frac{\partial Q_b}{\partial y} - \frac{\partial^2}{\partial y^2} (\overline{v'b'}). \tag{71}$$

Back to the Tropics

Matsuno–Gill in brief

What is the response of the atmosphere to an SST anomaly near the equator?

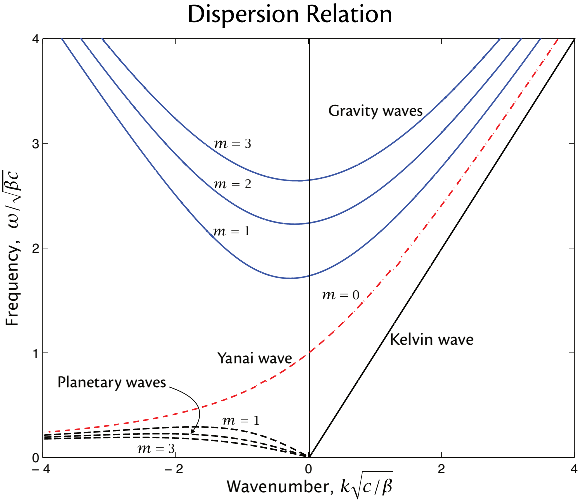
And why do we care?

- Because SST anomalies in the tropics really do affect the atmosphere in both the tropics and the mid-latitudes., and land may warm faster than the ocean on seasonal timescales.

Rossby and Kelvin Waves

Kelvin Waves, eastward: $\omega = +k\sqrt{gH}.$

Rossby Waves, westward: $\omega = -\dfrac{\beta}{k^2 + l^2 + k_d^2}$



Symmetry on the Sphere:

Kelvin waves sit at the equator. Rossby waves sit just *off* the equator.

Suppose we excite waves at the equator, then:

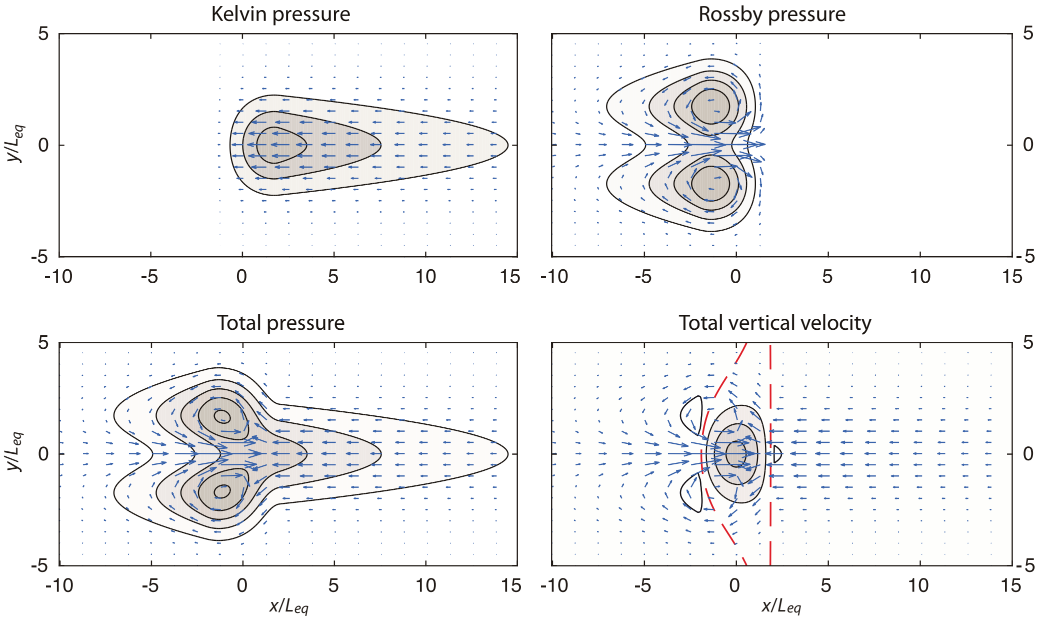
(i) Kelvin waves propagate eastwards at the equator.

(ii) Rossby waves propagate west just off the equator.

(iii) Both may slowed or damped by dissipative effects. The stationary solution with dissipation gives the *Matsuno–Gill* pattern.

Matsuno–Gill solution

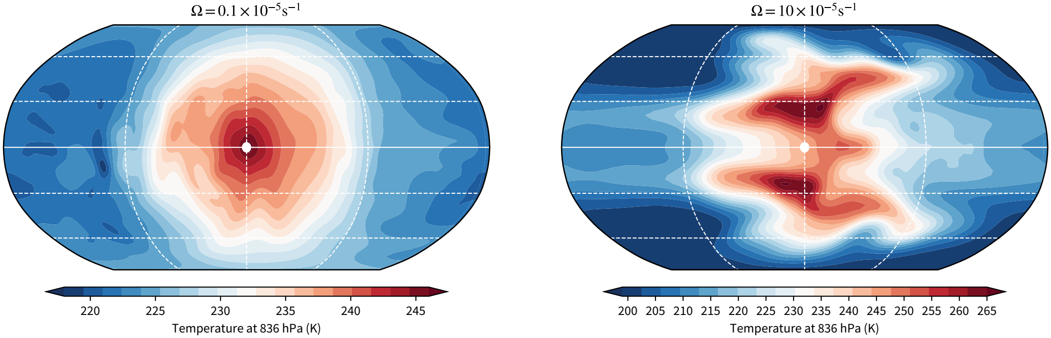
Heating at equator



Matsuno-Gill and Exoplanets!

Tidally-locked planet

Substellar point in center.

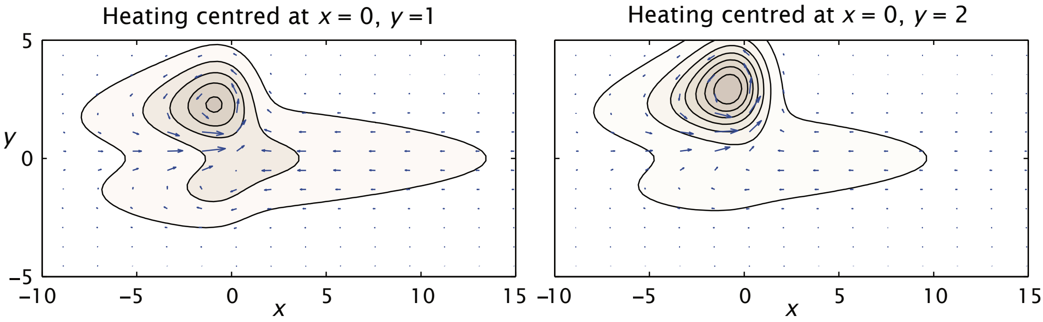


Very slowly rotating

Fast (Earth-like) rotation.

Matsuno-Gill

Heating off the equator



- Stronger Rossby wave response.
- Weaker Kelvin wave response.

The End

Fine

La Fin