Conformal blocks and the moduli space of curves

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Abstract

These are notes from my lectures delivered at the Summer School on the Geometry of Moduli Spaces of Curves, held June 18-22, 2018 at the International Center for Theoretical Physics, in Trieste, Italy. The school was organized by Valentina Beorchia, Ada Boralevi, and Barbara Fantechi.

I plan to make the notes available daily.

Acknowledgement 0.0.1. *These notes have seen earlier iterations, reflecting my lectures at GAeL in Bath, and also to a lesser extent, my talks at the Fields Institute program in Combinatorial Algebraic Geometry. Some of what I have written about here describes work done with my coauthors. I owe a lot of thanks to them, and in particular to Prakash Belkale from whom I learned so much about conformal blocks.*

Contents

Lecture 1

The moduli space of curves

1.1 Introduction

In my lecture today, I will give a very informal description of the moduli space of curves. More complete notes about moduli spaces can be found in Section 1.7. For further reading, I recommend [Kol96, Chapter 1], [EH00, Chapter VI], Kleiman's article on the Picard Scheme in [FGI+05], and [HM98].

I will also try to motivate why one might be interested in studying globally generated vector bundles on the moduli space of curves. In particular, I'll define nef divisors and cones of nef divisors, and the kinds of questions we ask about them. I want to illustrate how these cones tell us about the moduli space of curves, and in turn how by answering such questions we can learn about curves too. Moreover, we can see in this context how these questions for general curves can often be reduced to analogous questions about rational curves.

In Section 1.7 I have included additional information about topics related to the lecture.

1.2 What is M_g and why compactify it?

By M_g, I mean the quasiprojective variety whose closed points correspond to isomorphism classes of smooth curves of genus g. If C is a smooth curve (a 1-dimensional scheme over an algebraically closed field k), its genus is

$$g = \dim H^0(C, \omega_C) = \dim H^1(C, \mathcal{O}_C),$$

where ω_C is the sheaf of regular 1-forms on C. If $k = \mathbb{C}$, then C is a smooth compact Riemann surface, and the algebraic definition of genus is the same as the topological.

In fact M_g is more than just a variety, it is a moduli space. By this I mean that given any flat family $\mathcal{F} \to B$ of curves of genus g, there is a morphism $B \to M_g$, taking a point b to the isomorphism class of the fiber.

In fact M_g is something called a *coarse moduli space*. Because every curve with automorphisms can be used to construct a nontrivial family whose fibers are all isomorphic, one can show that M_g is not a fine moduli space.

Problem Session 1.2.1. *In Section 1.7, look up the definitions of coarse and fine moduli space. Read the example illustrating how* M_g *is not a fine moduli space.*

Intuitively, smooth curves degenerate to singular ones. For example, we can write down the "general curve of genus 2" using the equation:

$$y^2 = x^6 + a_5x^5 + a_4x^4 + \cdots + a_1x + a_0.$$

A general point $(a_0, \ldots, a_5) \in \mathbb{A}^6$ determines a smooth curve. In other words, there is a family of curves parametrized by an open subset of $\mathbb{A}^6$, that includes the general smooth curve of genus 2. Certainly you can see that as the coefficients change, the curves will change, and some choices of coefficients will result in singular curves.

To usefully parametrize families of curves like this one, it really pays to work with a proper space that parametrizes curves that have singularities. The space we will talk about today is denoted $\overline{M}_g$, and it parametrizes stable curves of genus g.

Definition 1.2.2. *A* ***stable curve*** C *of (arithmetic) genus g is a reduced, connected, one dimensional scheme such that*

1. *C has only ordinary double points as singularities.*

2. *C has only a finite number of automorphisms.*

Remark 1.2.3. *To say that* C *has only a finite number of automorphisms, comes down to requiring that if* C_i *is a nonsingular rational component,* C_i *meets the rest of the curve in at least three points, and if* C_i *is a component of genus one, then it meets the rest of the curve in at least one point.*

Definition 1.2.4. *Let* $\overline{M}_g$*, the moduli space of stable curves of genus g be the variety whose points are in one-to-one correspondence with isomorphism classes of stable curves of genus* g*.*

That such a variety[1] $\overline{M}_g$ exists is nontrivial. This was proved by Deligne and Mumford who constructed $\overline{M}_g$ using Geometric Invariant Theory [DM69]. There are other choices of compactifications of M_g, and some of these compactifications receive birational morphisms from $\overline{M}_g$; other compactifications receive rational maps from $\overline{M}_g$. In Section 1.7.3 I discuss some of these alternative compactifications of M_3.

[1]In fact, we may regard the moduli space of curves as a variety, or a scheme, or a stack. For today and this week, we consider it as a variety.

Figure 1.1: Components of the boundary of $\overline{M}_g$

1.3 Components of the boundary

The moduli space $\overline{M}_g$, has dimension $3g-3$ and the set of curves with at least k-nodes has codimension k. So for example:

1. $\Delta^1(\overline{M}_g)$ = the set of curves having at least one node. Δ^1 has codimension one. This is the boundary $\overline{M}_g \setminus M_g$, and it is a union of irreducible boundary components.

2. $\Delta^{3g-4}(\overline{M}_g)$ = the set of curves having $3g-4$ nodes. Δ^{3g-4} is 1-dimensional, composed of a union of curves whose numerical equivalence classes we call F-curves.

Even more specifically, we can describe the co-dimension one boundary of $\overline{M}_g$ as a union of components:

$$\overline{M}_g \setminus M_g = \cup_{i=0}^{\lfloor \frac{g}{2} \rfloor} \Delta_i,$$

- Δ_0 is the closure of the locus of curves with a single non-separating node, and
- for $i > 0$, Δ_i is the closure of the locus of curves with a single separating node whose normalization consists of a curve of genus i and a curve of genus $g-i$.

As one can see in the images pictured in Figure 1.1, moduli of pointed curves come up naturally even if one is only interested in studying $\overline{M}_g$: Each of these co-dimension one boundary components is the image of a morphism from moduli of pointed curves: By normalizing curves representing elements of these components, one can see examples of pointed curves, and we are led naturally to study moduli of such objects.

Definition 1.3.1. *A **stable n-pointed curve** is a complete connected curve C that has only nodes as singularities, together with an ordered collection $p_1, p_2, \ldots, p_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C; p_1, \ldots, p_n)$ has only a finite number of automorphisms.*

Definition 1.3.2. *Let $\overline{M}_{g,n}$, the moduli space of stable n-pointed curves of genus g, be the variety whose points are in one-to-one correspondence with isomorphism classes of stable n-pointed curves of genus* g.

In fact every component of the boundary is the image of a map from a moduli space of stable n-pointed curves, or a product of them:

$$\overline{M}_{g-1,2} \twoheadrightarrow \Delta_0, \quad \text{and for } 1 \leqslant i \leqslant \lfloor \frac{g}{2} \rfloor, \quad \overline{M}_{i,1} \times \overline{M}_{g-i,1} \twoheadrightarrow \Delta_i.$$

One can represent the F-curves as images of maps from $\overline{M}_{0,4}$ and $\overline{M}_{1,1}$.

1.4 The cones of nef and pseudo effective divisors

One way to study a projective variety X, like the moduli space of curves, is to regard it as an object in the category of projective varieties. From this point of view, it is natural to try to understand not only the objects in the category, but also the morphisms. In other words, to understand X, we would like to identify all the morphisms from X to other projective varieties. Such morphisms give rise to nef divisors on X.

Nef divisors and the nef cone

Given a projective variety Y, and a morphism $f : X \longrightarrow Y \hookrightarrow \mathbb{P}^N$, then for any ample divisor $A = \mathcal{O}(1)|_Y$ on Y, one has the pullback divisor $D = f^*A$ on X is base point free. In fact, this divisor D is not only base point free, it has the much weaker property that it is nef. For if C is a curve on our projective variety X, then by the projection formula

$$D \cdot C = f_*(D \cdot C) = A \cdot f_*C,$$

which is zero if the map f *contracts* C, and otherwise, as A is ample, it is positive.

Definition 1.4.1. *A divisor* D *on* X *is nef if* D *nonnegatively intersects all curves* C *on* X.

Definition 1.4.2. *The Nef cone* Nef(X) *is the set of all nef divisors on* X.

The nef cone lives in a finite dimensional vector space called the Néron Severi space, which I briefly take a moment to describe. Let X be a projective, not necessarily smooth variety defined over an algebraically closed field. Good references for the concepts below are [Laz04a],[Laz04b].

Definition 1.4.3. *A variety* X *is called* $\mathbb{Q}$*-factorial if every Weil divisor on* X *is* $\mathbb{Q}$*-Cartier. We assume today that* X *is a* $\mathbb{Q}$*-factorial normal, projective variety over the complex numbers. The moduli spaces* $\overline{M}_{g,n}$ *have these properties.*

Remark 1.4.4. *It is not true that every nef divisor on an arbitrary proper variety* X *has an associated morphism; To have such a property would be very special (a dream situation). But as we saw above, the divisors that give rise to maps do live in the nef cone, and for that reason the nef cone can be used a tool to understand the birational geometry of the space.*

Sufficiently high and divisible multiples of any effective divisor D on X will define a rational map (although not necessarily a morphism) from X to a projective variety Y. The stable base locus of D is the locus where the associated rational map will not be defined.

Definition 1.4.5. *For a* $\mathbb{Q}$*-Cartier divisor* D *on a proper variety* X*, we define the stable base locus of* D *to be the union (with reduced structure) of all points in* X *which are in the base locus of the linear series* $|nmD|$*, for all* n*, where* m *is the smallest integer* $\geqslant 1$ *such that* mD *is Cartier.*

The pseudo-effective cone may be divided into chambers having to do with the stable base loci [ELMta+09], [ELMta+06], [ELMta+05].

Digression: using the effective cone to illustrate that by studying $\overline{M}_g$ one can learn about curves

We started our discussion today by considering a family of curves parametrized by an open subset of $\mathbb{A}^6$, that included the general smooth curve of genus 2.

Generally speaking, if there is a family of curves parametrized by an open subset of A^{N+1} that includes the general curve of genus g, then one would have a dominant rational map from $\mathbb{P}^N$ to our compactification $\overline{M}_g$. In other words, $\overline{M}_g$ would be unirational. This would imply that there are no pluricanonical forms on $\overline{M}_g$. Said otherwise still, the canonical divisor of $\overline{M}_g$ would not be effective.

On the other hand, one of the most important results about the moduli space of curves, proved almost 40 years ago, is that for $g >> 0$ the canonical divisor of $\overline{M}_g$ lives in the interior of the cone of effective divisors (for $g = 22$ and $g \geqslant 24$, by EH, HM, and for by $g = 23$ [Far00]). Once the hard work was done to write down the classes of the canonical divisor, and an effective divisor called the Brill-Noether locus, to prove this famous result, a very easy combinatorial argument can be made to show that the canonical divisor is equal to an effective linear combination of the Brill-Noether and boundary divisors when the genus is large enough.

The upshot is that by shifting focus to the geometry of the moduli space of curves, we learn something basic and valuable about the existence of equations of smooth general curves. Moreover, for these values of g for which $\overline{M}_g$ is known to be of general type, one can consider the canonical ring

$$R_\bullet = \bigoplus_{m \geqslant 0} \Gamma(\overline{M}_g, m\, K_{\overline{M}_g}),$$

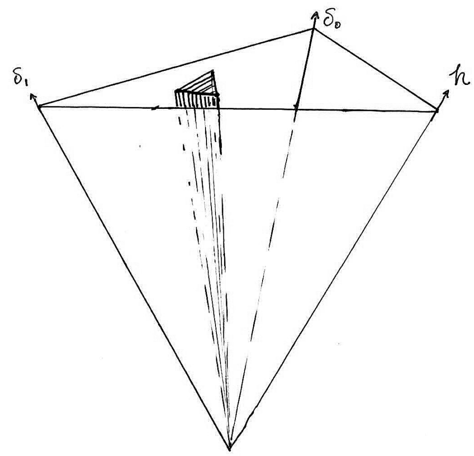


Figure 1.2: $\mathrm{Nef}^1(\overline{M}_3) \subset \overline{\mathrm{Eff}}^1(\overline{M}_3)$ with generators λ, $12\lambda - \delta_0$, and $10\lambda - \delta_0 - 2\delta_1$.

which is now known to be finitely generated by the celebrated work of [BCHM10]. In particular, the canonical model $\mathrm{Proj}(R_\bullet)$, is birational to $\overline{M}_g$.

It is still an open problem to construct this model, and efforts to achieve this goal have both furthered our understanding of the birational geometry of the moduli space of curves, as well as giving a highly nontrivial example where this developing theory can be experimented with and better understood.

Morphisms from $\overline{M}_g$

A simple example illustrates how even very crude information about the location of the cone of nef divisors with respect to the effective cone tells us valuable information about the geometry of the variety X, as we see for $\overline{M}_g$.

Theorem 1.4.6. *Every nef divisor on $\overline{M}_g$ is big. In particular, there are no morphisms, with connected fibers from $\overline{M}_g$ to any lower dimensional projective varieties other than a point.*

Theorem 1.4.6 says that the nef cone of $\overline{M}_g$ sits properly inside of the cone of effective divisors– and their extremal faces only touch at the origin of the Nerón Severi space.

Morphisms from $\overline{M}_{g,n}$

The question of what morphisms are admitted by $\overline{M}_{0,n}$ is still open! We can say something fairly simple in case $g > 0$:

Theorem 1.4.7. *For $g \geqslant 2$, any nef divisor is either big, or is numerically equivalent to the pullback of a big divisor by composition of projection morphisms. In particular, for $g \geqslant 2$, the only morphisms with connected fibers from $\overline{M}_{g,n}$ to lower dimensional projective varieties are compositions of projections given by dropping points, followed by birational maps.*

In all known examples for $n = 0$, the nef and effective cones are polyhedral, and the extremal rays are generated by semi-ample divisors. It turns out that in case $g = 0$, and $g = 1$ the effective cones for $\overline{M}_{g,n}$ are **not** polyhedral for n large enough. In particular, there is less hope than one might like for some kind of description of all the maps (maps with base loci), say given by combinations of extremal rays of the pseudo-effective cone.

One can still hope that the nef cone is tractable, and ask:

Question 1.4.8. *1. Is* $\mathrm{Nef}(\overline{M}_{g,n})$ *polyhedral?*

2. Is every nef divisor on $\overline{M}_{g,n}$ semi-ample?

1.5 Reduction of a problem for $g > 0$ to $g = 0$

As was mentioned earlier in the lecture, on $\overline{M}_{g,n}$, the locus

$$\Delta^k(\overline{M}_{g,n}) = \{(C, \vec{p}) \in \overline{M}_{g,n} \ : \ C \text{ has at least } k \text{ nodes } \}$$

has codimension k. For each k, the locus $\Delta^k(\overline{M}_{g,n})$ decomposes into irreducible components indexed by dual graphs Γ with k edges. Moreover, the closure of the component corresponding to Γ contains components consisting of curves whose corresponding dual graph Γ' contracts to Γ. This gives rise to a stratification of the space which is both reminiscent and analogous to the combinatorial structure determined by the torus invariant loci of a toric variety.

On a complete toric variety, every effective cycle of dimension k can be expressed as a linear combination of torus invariant cycles of dimension k. Fulton compared the action of the symmetric group S_n on $\overline{M}_{0,n}$ with the action of an algebraic torus a toric variety. Following this analogy, he asked whether a variety of dimension k could be expressed as an effective combination of boundary cycles of that dimension. As $\overline{M}_{0,n}$ is rational, of dimension $n-3$, this is true for points and cycles of codimension $n-3$. For the statement to be true for divisors, it would say that every effective divisor would be in the cone spanned by the boundary divisors. This was proved false by Keel [GKM02, page 4] and Vermeire, who found effective divisors not in the convex hull of the boundary divisors. For the statement to be true for curves, it would say that the Mori cone of curves is spanned by irreducible components of $\delta^{n-4}(\overline{M}_{0,n})$: whose dual graph is distinctive: the only vertex that isn't trivalent has valency four. In particular this says a divisor is nef if and only if it nonnegatively intersects those curves that can be described as images of attaching or clutching maps from $\overline{M}_{0,4}$.

This question could be asked for higher genus, and Faber did this independently (as an intermezzo in his thesis), proving the statement for $\overline{M}_3$ and $\overline{M}_4$.

In honor of Faber and Fulton, the numerical equivalence classes of the irreducible components of $\delta^{3g-4+n}(\overline{M}_{g,n})$ are called F-Curves. One can ask the following question:

Question 1.5.1. *(The* F*-Conjecture [GKM02]) Is every effective curve numerically equivalent to an effective combination of* F*-Curves? Otherwise said, is a divisor is nef, if and only if it nonnegatively intersects all the* F*-Curves?*

In [GKM02], using the flag map (see Definition 1.5) we showed that in fact a positive solution to this question for S_g-invariant nef divisors on $\overline{M}_{0,g+n}$ would give a positive answer for divisors on $\overline{M}_{g,n}$. In particular, there is the potential that the cone of nef divisors on $\overline{M}_{0,g+n}$ can tell us about the cone of nef divisors on $\overline{M}_{g,n}$. We know now that the answer to this question is true on $\overline{M}_{0,n}$ for $n \leqslant 7$ KeelMcKernan, and on $\overline{M}_g$ for $g \leqslant 24$ [Gib09].

The flag map

The flag map is defined as follows. Fix a point $(E, q) \in M_{1,1}$ and define the morphism $f : \overline{M}_{0,g+n} \longrightarrow \overline{M}_{g,n}$, which takes a stable $g+n$-pointed rational curve $(C; \{q_1, \dots, q_g\} \cup \{p_1, \dots, p_n\})$ to a stable n-pointed curve of genus g by attaching g copies of (E, p) to C by gluing C and E by identifying q and q_i for $1 \leqslant i \leqslant g$. In [GKM02], we showed that an F-divisor D on $\overline{M}_{g,n}$ is nef if and only if f^*D is nef. An F-divisor is, by definition, any divisor that nonnegatively intersects all the F-curves. Moreover, by [GKM02], every S_g-symmetric nef divisors on $\overline{M}_{0,g+n}$ is equal to the pullback of a nef divisor on $\overline{M}_{g,n}$.

1.6 Why globally generated vector bundles?

Vector bundles of covacua for affine Lie algebras give rise to elements of the cone of nef divisors: each bundle on $\overline{M}_{0,n}$ is globally generated, and so has base point free first Chern class (ie. is of the form f^*A for some morphism $f : \overline{M}_{0,n} \to Y$ where Y is a projective variety, and A is an ample line bundle on it). There are a lot of these bundles: They generate a full dimensional sub-cone of the nef cone.

The F-Conjecture, if true, would give a positive answer to Question 1.4.8 Part (1). Therefore, Question 1.4.8 and the F-Conjecture motivates our interest in vector bundles of conformal blocks. If every nef divisor on $\overline{M}_{0,n}$ is a conformal blocks divisor, then the answer to Question 1.4.8 Part (2) will hold for $g = 0$. If this is true and the cone generated by conformal blocks is polyhedral, then the answer to Question 1.4.8 Part (2) is true and we have more evidence for the F-Conjecture. If If this is true and the cone generated by conformal blocks is not polyhedral, then both the answer posed by Question 1.4.8 Part (2) and the F-Conjecture are false.

Of course it may be that the nef cone is not generated by these bundles, and there is something more to the story.

There are a lot of questions, and in trying to answer just a few, we've learned new things about vector spaces of conformal blocks and the moduli space of curves, some of which I hope to share this week.

1.7 Appendices to Lecture One

1.7.1 What is a moduli space, technically speaking?

The functor of points

Definition 1.7.1. *Let X be a scheme over a field k. The **functor of points** of a scheme X is the contravariant functor*

$$h_X : (\mathcal{S}ch_k) \to (\mathcal{S}ets),$$

from the category $(\mathcal{S}ch_k)$ of schemes over k to the category $(\mathcal{S}ets)$ of sets which takes a scheme $Y \in \mathrm{Ob}(\mathcal{S}ch_k)$ to the set $h_X(Y) = \mathcal{M}or_{\mathcal{S}ch_k}(Y, X)$, and takes maps of schemes $f : Y \to Z$, to maps of sets:

$$h_X(f) : h_X(Z) \to h_X(Y), \;\; [g : Z \to X] \mapsto [g \circ f : Y \to X].$$

Definition 1.7.2. *We say that a contravariant functor*

$$F : (\mathcal{S}ch_k) \to (\mathcal{S}ets),$$

***is representable** if it is of the form h_X for some scheme X. By Yoneda's Lemma (below), if X exists, then it is unique, and we say that X represents the functor F.*

For a proof of **Yoneda's Lemma**, which we next state, see for example [EH00, pages 252-253]

Lemma 1.7.3 (Yoneda)**.** *Let $\mathcal{C}$ be a category and X, and let $X' \in \mathrm{Obj}(\mathcal{C})$.*

1. *If F is any contravariant functor from $\mathcal{C}$ to the category of sets, the natural transformations from $\mathcal{M}or(, X)$ to F are in natural correspondence with the elements of $F(X)$;*

2. *If functors $\mathcal{M}or(\,, X)$ and $\mathcal{M}or(\,, X')$ are isomorphic, then $X \cong X'$.*

Fine moduli spaces

See also [Kol96, Chapter 1], [EH00, Chapter VI, page], Kleiman's article on the Picard Scheme in [FGI$^+$05], and [HM98].

Definition 1.7.4. *Given a reasonable[2] collection of objects $\mathcal{S}$, we define a (contravariant) moduli functor from the category $(\mathcal{S}ch_k)$ of schemes over k to the category $(\mathcal{S}ets)$ of sets*

$$\mathcal{F}_{\mathcal{S}} : (\mathcal{S}ch_k) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{F}_{\mathcal{S}}(T),$$

where $\mathcal{F}_{\mathcal{S}}(T)$ is equal to the set of flat families of objects in $\mathcal{S}$ parametrized by T up to isomorphism over T.

The question one then asks is whether there is a scheme which we can call $\mathrm{Mod}_{\mathcal{S}}$, or better said, a flat morphism of schemes:

$$u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}},$$

which is a fine moduli space for the moduli functor. This means that for every object $T \in \mathrm{Obj}(\mathcal{S}ch_k)$, pulling back, gives an equivalence of sets:

$$\mathcal{F}_{\mathcal{S}}(T) = \mathcal{M}or_{\mathcal{S}ch}(T, \mathrm{Mod}_{\mathcal{S}}).$$

For example, taking $T = \mathrm{Mod}_{\mathcal{S}}$, we obtain the universal family $u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}}$ which corresponds to the identity element $\mathrm{id} \in \mathcal{M}or_{\mathcal{S}ch}(\mathrm{Mod}_{\mathcal{S}}, \mathrm{Mod}_{\mathcal{S}})$. And taking $T = \mathrm{Spec}(k)$, we see that the set of k-points of $\mathrm{Mod}_{\mathcal{S}}$ corresponds to the fibers of the family $u : \mathcal{U}_{\mathcal{S}} \to \mathrm{Mod}_{\mathcal{S}}$.

Another more formal way to say this is the following.

Definition 1.7.5. *The functor $\mathcal{F}_{\mathcal{S}}$ from Definition 1.7.4 is represented by the scheme $\mathrm{Mod}_{\mathcal{S}}$ if there is a natural isomorphism between $\mathcal{F}_{\mathcal{S}}$ and the functor of points $\mathcal{M}or_{\mathcal{S}ch}(\quad, \mathrm{Mod}_{\mathcal{S}})$. In this case we say $\mathrm{Mod}_{\mathcal{S}}$ is **a fine moduli space** for the functor $\mathcal{F}_{\mathcal{S}}$.*

Example: The Grassmannian

Let S be a scheme of finite type over a field k, and let $(\mathcal{S}ch_S)$ denote the category of schemes of finite type over S. Fix two integers $0 < d < r$. We will consider the contravariant functor from $(\mathcal{S}ch_S)$ to the category $(\mathcal{S}ets)$ of sets:

$$\mathfrak{g}_S^{r,d} : (\mathcal{S}ch_S) \to (\mathcal{S}ets), \quad T \mapsto \mathfrak{g}_S^{r,d}(T),$$

[2] As part of being a reasonable collection of objects, we require that $\mathcal{S}$ is closed under base extension. So for example, if objects X in $\mathcal{S}$ are defined over $\mathrm{Spec}(k)$, where k is a field, and if $k \hookrightarrow k$ is a field extension, then $X_k = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k)$ is also in $\mathcal{S}$.

such that

$$\mathfrak{g}_S^{r,d}(T) = \{\ q : \mathcal{O}_T^r \twoheadrightarrow \mathcal{F} :\ \mathcal{F} \text{ a coherent locally free } \mathcal{O}_T \text{ -module of rank } d\ \}/\sim,$$

where two quotients $q_1 : \mathcal{O}_T^r \twoheadrightarrow \mathcal{F}_1$ and $q_2 : \mathcal{O}_T^r \twoheadrightarrow \mathcal{F}_2$ in $\mathfrak{g}_S^{r,d}(T)$ are equivalent if there is an isomorphism $f : \mathcal{F}_1 \to \mathcal{F}_2$, making the diagram

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{f} & \mathcal{F}_1 \\ & {\scriptstyle q_1} \nwarrow & \uparrow {\scriptstyle q_2} \\ & & \mathcal{O}_T^r \end{array}$$

commute. Grothendieck proved that there is a projective scheme $G_S^{r,d}$ of finite type over S (ie an object in $(\mathcal{S}ch_S)$) that represents the functor $\mathfrak{g}_S^{r,d}$.

One can generalize the Grassmannian, forming Hilbert schemes, and Quot schemes for example.

Example: Hilbert schemes

If X is a projective scheme of finite type over S, we can consider the contravariant functor

$$h_{X/S} : (\mathcal{S}ch_S) \to (\mathcal{S}ets), \quad T \mapsto h_{X/S}(T),$$

and for $X_T = X \times_S T$, one has $h_{X/S}(T) = \{q : \mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F} :\ \mathcal{F} \text{ satisfying (1) and (2)}\}/\sim$.

1. is a coherent sheaf of $\mathcal{O}_T$-modules; and
2. is flat and with compact support with respect to the projection $p_2 : X \times_S T \to T$.

Notice here that $r = 1$, and as $\mathcal{O}_{X_T}$ is the ring of regular functions for $X_T = X \times_S T$, by taking kernels of the maps $q : \mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F}$, we get that the set $h_{X/S}(T)$ is in bijection with the set of closed subschemes of X parametrized by T. Grothendieck showed this functor is representable by the Hilbert scheme $\mathrm{Hilb}_{X/S}$, which while not of finite type over S, is a union of schemes of finite type, parametrized by Hilbert polynomials, each of which represents a moduli functor. We'll speak more about these.

Example: Quot schemes

A common generalization of the previous two examples are the following two contravariant functors.

$$\mathcal{Q}_{\mathcal{O}_X^r/X/S} : (\mathcal{S}ch_S) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{Q}_{\mathcal{O}_X^r/X/S}(T),$$

such that, for $X_T = X \times_S T$, the set $\mathcal{Q}_{\mathcal{O}^r_X/X/S}(T)$ is equal to

$$\{q : \mathcal{O}^r_{X_T} \twoheadrightarrow \mathcal{F} : \ \mathcal{F}\text{coherent } \mathcal{O}_{X_T}\text{-module, flat with compact support over T}\}/\sim .$$

More generally, if $\mathcal{E}$ is a locally free sheaf on X, we define a contravariant functor

$$\mathcal{Q}_{\mathcal{E}/X/S} : (\mathcal{S}ch_S) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{Q}_{\mathcal{E}/X/S}(T),$$

where for $p_1 : X_T = X \times_S T \to X$ the projection onto the first factor, the set $\mathcal{Q}_{\mathcal{E}/X/S}(T)$ is

$$\{q : p_1^*\mathcal{E} \twoheadrightarrow \mathcal{F} : \ \mathcal{F}\text{coherent } \mathcal{O}_{X_T}\text{-module, flat with compact support over T}\}/\sim .$$

Grothendieck proved that $\mathcal{Q}_{\mathcal{E}/X/S}$ is represented by the so-called Quot-scheme $\mathrm{Quot}_{\mathcal{E}/X/S}$, which while not finite type over S, again is a union of schemes of finite type over S, parametrized by Hilbert polynomials.

Not an example: the moduli space of smooth curves

Consider, for $g = \dim H^1(C, \mathcal{O}) \geqslant 2$:

$$\mathcal{M}_g : (\mathcal{S}ch_k) \to (\mathcal{S}ets), \quad T \mapsto \mathcal{M}_g(T),$$

where $\mathcal{M}_g(T)$ is the set of proper flat maps $\pi : \mathcal{F} \to T$ such that every fiber $\mathcal{F}_t$ is a smooth projective curve of genus g modulo isomorphism over T. This functor is not represented by a fine moduli space: every curve with nontrivial automorphisms creates issues.

Example 1.7.6. *We will consider a nontrivial family of hyperelliptic curves parametrized by $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$. To describe this family, let $X = Z(y^2 - f(x))$ be any smooth hyperelliptic curve of genus g with $\mathrm{Aut}(X) \cong C_2 =< \tau >$. The cyclic group C_2 acts on X and on $\mathbb{G}_m$:*

$$C_2 \times X \to X, \ (\tau, (x, y)) \mapsto (x, -y), \quad \textit{and} \quad C_2 \times \mathbb{G}_m \to \mathbb{G}_m, \ (\tau, z) \mapsto -z;$$

and we can form the contracted product

$$\mathcal{F} = \mathbb{G}_m \times_{C_2} X = (\mathbb{G}_m \times X)/\sim, \quad \textit{where} \quad (\tau \cdot \alpha, p) \sim (\alpha, \tau \cdot p).$$

We'll set

$$\pi : \mathcal{F} \to \mathbb{G}_m \ \ [(\alpha, p)] \mapsto \alpha^2,$$

which is well defined since by this prescription $(\tau \cdot \alpha, p) = (-\alpha, p) \mapsto \alpha^2$, and $(\alpha, \tau \cdot p) \mapsto \alpha^2$. To see that fibers of π are isomorphic to X, notice that one can view the set of points lying over $\alpha^2 \in \mathbb{G}_m$ as all points lying on two copies of X that are identified by the equivalence relation $\sim$. In *particular if the functor $\mathcal{M}_g$ were represented by a fine moduli space M_g with a universal family $u : U_g \to M_g$, then there would be a constant map*

$$\mu_\pi : \mathbb{G}_m \to M_g, \quad \alpha \mapsto [X],$$

and so $\mathcal{F}$ would be equal to the constant family, giving a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{F} & \mathbb{G}_m \times X \\ & \searrow^{\pi} & \downarrow p_1 \\ & & \mathbb{G}_m. \end{array}$$

But the map $F : \mathcal{F} \to \mathbb{G}_m \times X$ could simply not be well defined, for all points $[(\alpha, p)] \in \mathcal{F}$, and so this is impossible.

However, there is a scheme M_g with the following properties:

1. for an algebraically closed field k, the k-points of M_g are in one to one correspondence with the set of isomorphism classes of smooth curves of genus g defined over k;

2. if $\pi : \mathcal{F} \to T$ is a flat family of curves of genus g, then there is a map $\mu_\pi : T \to M_g$ such that if $t \in T$ is a geometric point, then $\mu_\pi(t)$ is the point $[\mathcal{F}_t]$ in M_g corresponding to the isomorphism class of the fiber $\mathcal{F}_t = \pi^{-1}(t)$.

Coarse moduli spaces

Definition 1.7.7. *We say that a scheme $\mathrm{Mod}_\mathcal{S}$ is a* ***coarse moduli space*** *for the functor $\mathcal{F}_\mathcal{S}$ (from Definition 1.7.4), if*

1. *there is a natural transformation of functors $\mathcal{F}_\mathcal{S} \to \mathcal{M}or_{\mathcal{S}ch}(\ \ , \mathrm{Mod}_\mathcal{S})$;*

2. *the scheme $\mathrm{Mod}_\mathcal{S}$ is universal for (1);*

3. *for any algebraically closed field extension $k \hookrightarrow K$,*

$$\mathcal{F}_\mathcal{S}(K) \cong \mathcal{M}or_{\mathcal{S}ch}(\mathrm{Spec}(K), \mathrm{Mod}_\mathcal{S}) = \mathrm{Mod}_\mathcal{S}(K),$$

is an isomorphism of sets.

In particular, the scheme M_g is a coarse moduli space for the functor $\mathcal{M}_g$ described in Section 1.7.1.

Definition 1.7.8. *For* $g = \dim H^1(C, \mathcal{O}_C) \geqslant 2$*, consider the contravariant functor:*

$$\overline{\mathcal{M}}_g : (\mathcal{S}ch_k) \to (\mathcal{S}ets), \quad T \mapsto \overline{\mathcal{M}}_g(T),$$

where $\overline{\mathcal{M}}_g(T)$ *is the set of flat proper morphisms* $\pi : \mathcal{F} \to T$ *such that every fiber* $\mathcal{F}_t$ *is a stable curve of genus* g *modulo isomorphism over* T*.*

Theorem 1.7.9. *[DM69] There exists a coarse moduli space* $\overline{M}_g$ *for the moduli functor* $\overline{\mathcal{M}}_g$*; Moreover,* $\overline{M}_g$ *is a projective variety that contains* M_g *as a dense open subset.*

Remark 1.7.10. *Let* T *be any smooth curve and* $p \in T$ *a (geometric) point on* T*. Suppose there is a regular map*

$$\mu^* : T^* = T \setminus \{p\} \to \overline{M}_g.$$

By definition of coarse moduli space, this map corresponds to a family $\pi : X \to T^*$ *of stable curves of genus g, parametrized by* T^**. Now by Theorem 1.7.9, the moduli space* $\overline{M}_g$ *is proper, and so by the valuative criterion for properness, there is an extension of* μ^* *giving a morphism* $\mu : T \to \overline{M}_g$*. But by Theorem 1.7.9,* $\overline{M}_g$ *is also separated, and one can use this to show this extension* μ *is unique. So this says that there is a unique extension to a family* $\pi : X \to T$ *parametrized by* T*. This is the content of* ***the stable reduction theorem****.*

1.7.2 Tautological maps

We have, and will often continue to refer this week to the following tautological maps

1. projection maps:
$$\pi_i : \overline{M}_{g,n} \longrightarrow \overline{M}_{g,n-1},$$
given by dropping the i-th marked point (and stabilizing, if necessary).

2. attaching maps:
$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \longrightarrow \overline{M}_{g_1+g_2,n_1+n_2},$$
given by glueing pointed curves together;

3. clutching maps:
$$c : \overline{M}_{g-k,n+2k} \longrightarrow \overline{M}_{g,n},$$
given by attaching marked points in pairs.

and combinations of these. It can be beneficial to think of the moduli spaces as a unified system, and ultimately many questions even about $\overline{M}_{g,n}$ and structures like vector bundles on $\overline{M}_{g,n}$, can be reduced to analogous questions on $\overline{M}_{0,N}$, for suitable N.

Underpinning the well-definedness of the projection maps is the Nodal Reduction Theorem:

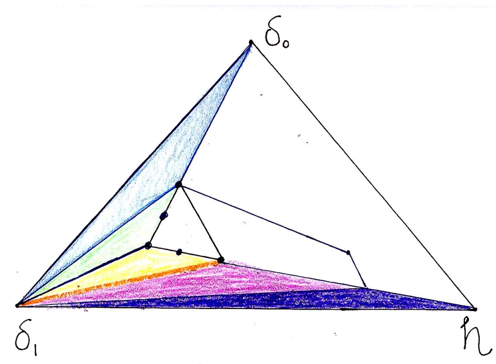


Figure 1.3: A partial chamber decomposition of

$$\mathrm{Nef}^1(\overline{M}_3) \subset \mathrm{Mov}(\overline{M}_3) \subset \overline{\mathrm{Eff}}^1(\overline{M}_3)$$

seen in a cross section.

Theorem 1.7.11. *(Nodal Reduction) Let* T *be a smooth curve,* p *a point of* T *and* $T^* = T\setminus\{p\}$*. Let* $X \to T^*$ *be a flat family of nodal curves of genus* g*,* $\psi : X \to Z$ *any morphism to a projective scheme* Z*, and* $D \subset X$ *any divisor finite over* T^**. Then there exists a branched cover* $T' \to T$ *and a family* $X' \to T'$ *of nodal curves, extending the fiber product* $X \times_{T^*} T'$ *with the following properties:*

1. *The total space* X' *is smooth;*
2. *The morphism* $\pi_X \circ \psi : X \times_{T^*} T' \to Z$ *extends to a regular morphism on all of* X'*;*
3. *The closure of* $\pi_X^{-1}(D)$ *in* X' *is a disjoint union of sections of* $X' \to T'$*.*

Any two such extensions are dominated by a third and so have special fibers whose stable models are isomorphic.

1.7.3 Chambers of the pseudo-effective cone of $\overline{M}_3$

The first work done to understand the nef and effective cones for the moduli space of curves was done by Mumford in [Mum83], where everything was worked out for $\overline{M}_2$, and where it was checked that the intersection theory could be done on $\overline{M}_g$ in general. By [Fab90], we know that $\overline{\mathrm{NE}}^1(\overline{M}_3)$ is spanned by the classes $\delta_0 = [\Delta_0]$, $\delta_1 = [\Delta_1]$ and the class h of the hyperelliptic locus $\mathcal{H}_3$. The hyperelliptic locus $\mathcal{H}_g$ on $\overline{M}_g$ is isomorphic to $\widetilde{\mathcal{M}}_{0,2g+2}$ under the map

$$h : \widetilde{\mathcal{M}}_{0,2g+2} \xrightarrow{\cong} \mathcal{H}_g \subseteq \overline{M}_g,$$

given by taking a double cover branched at the marked points. For $g = 2$, the map is an isomorphism, for $g = 3$ the image has codimension one, and for $g \geqslant 4$ the image has higher codimension and isn't a divisor.

There is a partial chamber decomposition of $\mathrm{Nef}(\overline{M}_3) \subset \mathrm{Mov}(\overline{M}_3) \subset \overline{\mathrm{NE}}^1(\overline{M}_3)$. Two chambers have to do with different compactifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties: The classical Torelli map

$$\mathcal{M}_g \xrightarrow{t} \mathcal{A}_g,$$

which takes a smooth curve X of genus g to its Jacobian, doesn't extend to a morphism on $\overline{M}_g$. But there are extensions to various compactifications of $\mathcal{A}_g$.

The Satake Chamber

Let $\overline{\mathcal{A}}_g^{Sat}$ be the Satake compactification of the moduli space $\mathcal{A}_g$. The classical Torelli map extends to a regular map

$$t^{Sat} : \overline{M}_g \longrightarrow \overline{\mathcal{A}}_g^{Sat}.$$

This morphism is given by the divisor λ. In other words, $\lambda = (t^{Sat})^*(A)$, where A is an ample divisor $\overline{\mathcal{A}}_g^{Sat}$.

The 2nd Voronoi Chamber

We let $\overline{\mathcal{A}}_g^{Vor}$: be the toroidal compactification of $\mathcal{A}_g$ for the $2nd$ Voronoi fan. The Torelli map is known to extend to the regular map

$$\bar{t}_g : \overline{M}_g \xrightarrow{t^{Sat}} \mathcal{A}_g^{Vor(2)}.$$

This morphism is given by a divisor which lies on the (interior of the) face of the nef cone spanned by λ and $12\lambda - \delta_0$.

The Shepherd-Barron Unknown (SBU) Chamber

There is a morphism

$$f : \overline{M}_g \longrightarrow X,$$

given by the base point free extremal nef divisor $12\lambda - \delta_0$. As far as I know, there isn't a modular interpretation for X.

The Pseudo-Stable Chamber

Let $\overline{M}_g^{ps}$ be the moduli stack of pseudo stable curves. Replacing elliptic tails with cusps gives the divisorial contraction

$$T : \overline{M}_g \longrightarrow \overline{M}_g^{ps}.$$

T is given by a divisor that lies on the face of the nef cone spanned by $12\lambda - \delta_0$ and $10\lambda - \delta_0 - 2\delta_1$.

The C-Stable Chamber

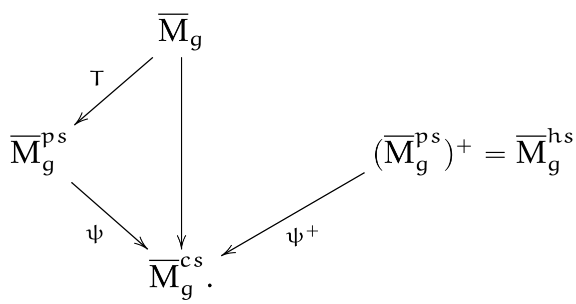
Let $\overline{M}_g^{cs}$ be the moduli space of c-stable curves. Contracting elliptic bridges to tacnodes defines the small modification $\psi : \overline{M}_g^{ps} \longrightarrow \overline{M}_g^{cs}$, and composing with T defines a regular map

$$\overline{M}_g \xrightarrow{T} \overline{M}_g^{ps} \xrightarrow{\psi} \overline{M}_g^{cs},$$

given by the extremal divisor $10\lambda - \delta_0 - 2\delta_1$.

The First Flip: H-Semistable Curves in the Moving Cone

We can also see the first flip: Let $\overline{M}_g^{hs}$ be the moduli space of h-semistable curves. There is a morphism $\psi^+ : \overline{M}_g^{hs} \longrightarrow \overline{M}_g^{cs}$ which is a flip of ψ:



We can see the chamber of the effective cone of $\overline{M}_3$ corresponding to $\overline{M}_g^{hs}$. It doesn't touch the Nef cone of $\overline{M}_3$ because there isn't a morphism from $\overline{M}_3$ to $\overline{M}_g^{hs}$. Instead, there is a rational map, which for $g = 3$ is given by the moving divisors pictured.

There is another chamber of the moving cone, as we can see in the picture. This corresponds to the pullback of the nef cone of the second flip.

Lecture 2

Conformal blocks, Co-invariants, Factorization, and Propagation of Vacua

2.1 Introduction

Today I will describe fibers of the Verlinde bundles, and their duals, the vector spaces of conformal blocks. First, I will state a theorem, which describes conformal blocks in terms of $\mathrm{Bun}_G(C)$. We will return to this later in the week. Second, I will give a construction of vector spaces of coinvariants using the action of certain Lie algebra attached to the pointed curve. I will also briefly describe two other important theorems: Factorization and Propagation of Vacua, used in extending the definition of the fibers of the bundles at smooth pointed curves, to fibers at stable curves with no marked points and to curves with singularities. These are fundamental, and play a role in many results obtained about the bundles.

The description I give can be modified to work in families, and used to define sheaves of coinvariants and conformal blocks. Fibers can be shown to be finite dimensional, and the sheaves of coinvariants coherent. These properties were originally proved in [TUY89].

In case it's new, you may want to read the background sections I've prepared with definitions from representation theory or consult the references given there. There are many references on the topics I'll cover in tomorrow's lecture. I particularly like [Bea96], [BK01, Chapter 7], [Fak12], and the original, [TUY89].

Each of the descriptions given today has its own advantages. For instance, the second allows one to prove the bundles are globally generated in case $g = 0$, as well as to give Beauville's Quotient Construction, which gives a simple proof of Propagation of Vacua, and the finiteness results necessary for the proofs that the sheaves produced are vector bundles. The first approach, gives contact with more geometric descriptions, as we'll see later in the fourth Lecture.

If there is time, I'd like to say something about computing the ranks of these bundles.

2.2 Conformal blocks and Bun_G

The fibers of the vector bundles we study are determined by $(n+1)$-tuples consisting of a simple Lie algebra $\mathfrak{g}$, and for every marked point, a $\mathfrak{g}$-module. We begin with the case of no marked points, and no $\mathfrak{g}$-modules.

Statement with no marked points

For G be a simple, simply connected, complex linear algebraic group, C a stable curve of arithmetic genus $g \geqslant 2$, let $\mathrm{Bun}_G(C)$ be the smooth algebraic stack whose fiber over a scheme T is the groupoid of principal G-bundles on $C \times T$ (Def 2.5.2). Principal G-Bundles are defined in Section 2.5.2. To any representation $G \to GL(V)$, there corresponds a distinguished line bundle on $\mathrm{Bun}_G(C)$, the determinant of cohomology line bundle $\mathcal{D} = \mathcal{D}(V)$, described below in Def 2.2.2.

Theorem 2.2.1. *For* $G = SL(r)$ *and* $\rho : G \to Gl(V)$ *the standard representation of* G,

$$\mathbb{V}(\mathfrak{g}, \ell)|^*_{(C)} = \mathcal{H}^0(\mathrm{Bun}_G(C), \mathcal{D}^{\otimes \ell})$$

History

Theorem 5.3.1 goes back to the 90's for smooth curves, and first evidence of it appeared in the work of Aaron Bertram.

In [Ber93], Bertram considered the moduli *space* $M(X, D, \ell)$ of semistable parabolic bundles of rank 2 with trivial determinant on pairs (X, D) where X is a smooth curve, $D = \sum_{i=1}^{n} d_i p_i$, and d_i is the multiplicity of the divisor D at the point p_i on X. Here $\ell \geqslant \max d_i$, and one can define a notion of stability associated to the weights $\frac{d_i}{\ell}$ at each point p_i. The moduli space has a determinant line bundle Δ, and Bertram proved that $h^0(M(X, D, \ell), \Delta)$ is given by the Verlinde formula in case the degree of Δ is even. In the proof he uses a geometric interpretation of the sections of the moduli spaces in terms of classical projective geometry of curves. Later, in [Tha94], Michael Thaddeus gave a very different proof of Bertram's result.

Bertram's initial result for SL_2-bundles on a smooth curve C, ultimately indicated the very general relationship in Theorem 5.3.1between $\mathcal{H}^0(\mathrm{Bun}_G(C), \mathcal{D}^{\otimes \ell})$ and conformal blocks, which was subsequently proved for smooth curves by the following authors:

- $n = 0$, and $G = SL_r$ Beauville-Laszlo [BL94];
- $n = 0$, arbitrary G Faltings [Fal94], and Kumar-Narasimhan-Ramanathan [KNR94];

- arbitrary n and $G = SL_r$ Pauly [Pau96]; and
- full generality by Laszlo and Sorger in [LS97].

The analogous result that holds for curves with marked points replaces $\mathrm{Bun}_G(C)$ by $\mathrm{Parbun}_G(C)$ and the determinant of cohomology by a different line bundle.

Only recently, in [BF15], Belkale and Fakhruddin proved that the result holds for *stable curves with singularities*. The proof of Belkale and Fakhruddin holds in families and can be used to define the sheaves of conformal blocks.

The determinant of cohomology line bundle

Following [Fal93], we describe the determinant of cohomology of a vector bundle on a curve.

Definition 2.2.2. *For any vector bundle $\mathcal{E}$ on a curve C, the determinant of cohomology of $\mathcal{E}$ on C is the one dimensional vector space given by*

$$\mathcal{D}(C, \mathcal{E}) = \left(\Lambda^{max} H^0(C, \mathcal{E})\right)^* \otimes \left(\Lambda^{max} H^1(C, \mathcal{E})\right). \tag{2.1}$$

$\mathrm{Bun}_G(C)$ is the smooth algebraic stack whose fiber over a scheme T is the groupoid of principal G-bundles on $C \times T$. Following [LS97], we define the determinant of cohomology line bundle on $\mathrm{Bun}_G(C)$ for $G = SL(r)$.

Definition 2.2.3. *Let $G = SL(r)$, and $\rho : G \to Gl(V)$ be a representation of G. If E is a family of G-bundles on C parameterized by a scheme T, then given a point $t \in T$, one has that E_t is a G-bundle on C, and one can form a vector bundle $\mathcal{E}_t(V)$ on C by taking the contracted product $\mathcal{E}_t(V) = E_t \times_G V$. The determinant of cohomology line bundle $\mathcal{D}_E(V)$ is the line bundle on T whose fiber over a point $t \in T$ is the line $\mathcal{D}(C, \mathcal{E}_t(V))$, described in Def 2.2.2.*

Lemma 2.2.4. *Let G be any semisimple group. Given a principal G-bundle $\mathcal{E}$, and any representation $\rho : G \to GL(V)$, by the contracted product $E = \mathcal{E} \times_G V$, has trivial determinant.*

Proof. To see that $\det(E)$ is trivial, we note that since G is semisimple, $[G, G] = G$, and so the image $\rho(G)$ is contained in the kernel of the determinant map which is $SL(V)$. In particular, E has transition functions given by matrices with trivial determinant. These are the transition functions of the line bundle $\det(E)$, and so $\det(E)$ is necessarily trivial. □

2.3 Coinvariants and affine Lie algebras

To begin, we describe a fiber of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ on $\overline{M}_{g,n}$ at a point $(C, \vec{p}) \in \overline{M}_{g,n}$, such that $U = C \setminus \{p_1, \ldots, p_n\}$ is affine. This is the case for instance, if C is a smooth curve of genus g with at least one marked point, but can also be true more generally (for instance if there is at least one marked point on each component of a stable curve C with singularities).

Remark 2.3.1. *The Propagation of Vacua theorem will enable one to show that the bundles are defined on $\overline{M}_g$. The Factorization Theorem allows for the extension of this definition to curves with singularities.*

Since we have marked points to work with, we also have $\mathfrak{g}$-modules. As will be evident when we see the formulas for Chern classes in the fourth Lecture, the combinatorial data of dominant integral weights provides a convenient language to index these modules, and reflects many of their properties. We therefore find it convenient to say that vector bundles of coinvariants $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ on $\overline{M}_{g,n}$ are determined by collections of data including:

- a simple Lie algebra $\mathfrak{g}$;
- a positive integer ℓ; and
- an n-tuple $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of dominant integral weights for $\mathfrak{g}$ at level ℓ;

For the bundles to have nontrivial rank, the triples should satisfy a compatibility criterion, which will be described.

Problem Session 2.3.2. *In case you aren't familiar with the language from representation theory, I have included basic definitions here in Section 2.5.4, and you can reference the terms I refer to there. Many other better references can be found (for instance [FH91]). In the problem session today you can ask questions about this material.*

As we shall see in the third Lecture, in case $g = 0$, the bundles $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ are quotients of a constant bundle $\mathbb{A}(\mathfrak{g}, \vec{\lambda})$. Fibers of $\mathbb{A}(\mathfrak{g}, \vec{\lambda})$, the vector spaces of coinvariants $[V_{\vec{\lambda}}]_{\mathfrak{g}}$ are easy to describe, and involve many of the same elements as fibers of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$.

Warmup: Fibers of the constant bundle $\mathbb{A}(\mathfrak{g}, \vec{\lambda})$

Finite dimensional $\mathfrak{g}$-modules correspond to weights λ_i, and we write V_{λ_i} for such $\mathfrak{g}$-modules. Set $V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, and consider the diagonal action of $\mathfrak{g}$ on $V_{\vec{\lambda}}$

$$\mathfrak{g} \times V_{\vec{\lambda}} \to V_{\vec{\lambda}}, \quad (g, v_1 \otimes \cdots v_n) \mapsto \sum_{i=1}^{n} v_1 \otimes \cdots v_{i-1} \otimes (g \cdot v_i) \otimes v_{i+1} \otimes \cdots v_n.$$

We write $[V_{\vec{\lambda}}]_{\mathfrak{g}}$ for the **space of coinvariants of** $V_{\vec{\lambda}}$: The largest quotient of $V_{\vec{\lambda}}$ on which $\mathfrak{g}$ acts trivially. That is, the quotient of $V_{\vec{\lambda}}$ by the subspace spanned by the vectors $X \cdot v$ where $X \in \mathfrak{g}$ and $v \in V_{\vec{\lambda}}$.

The fibers $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}$ are also vector spaces of coinvariants, analogous to $[V_{\vec{\lambda}}]_{\mathfrak{g}}$, only they have something to do with the point $(C, \vec{p}) \in \overline{M}_{g,n}$, as we next explain.

Affine Lie Algebras $\hat{\mathfrak{g}}_i$ and their modules

Problem Session 2.3.3. *A Lie algebra is a vector space with a bracket. For example, $\mathbb{C}((\xi_i))$, the field of Laurant power series over $\mathbb{C}$ in the variable ξ_i, and $\mathcal{O}_C(U)$ the ring of regular functions of C on U can be considered Lie algebras. In Section 2.5.4, definitions and examples of Lie algebras are given, including the Lie algebras commonly used in the constructions here, including Lie algebras given by taking central extensions. Consult Section 2.5.4 or the references there for further details.*

For each $i \in \{1, \dots, n\}$, let ξ_i be a local parameter of p_i on C. A local parameter on C at p_i is a holomorphic function with a simple pole at p_i. Consider the affine Lie algebra

$$\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C} \cdot c,$$

be the one-dimensional central extension of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}((\xi_i))$, where $\mathbb{C}((\xi_i))$ is the field of Laurent series, and c is in the center of $\hat{\mathfrak{g}}_i$. To define the bracket, we note that elements in $\hat{\mathfrak{g}}_i$ are tuples $(a_i, \alpha c)$, with $a_i = \sum_j X_{ij} \otimes f_{ij}$, with $f_{ij} \in \mathbb{C}((\xi_i))$. We define the bracket on simple tensors:

$$[(X \otimes f, \alpha c), (Y \otimes g, \beta c)] = [X, Y] \otimes fg, c(X, Y) \cdot \text{Res}_{\xi_i=0}(g(\xi_i) df(\xi_i)).$$

Problem Session 2.3.4. *Check that $\hat{\mathfrak{g}}_i$ is a Lie algebra (this is done in Section 2.5.4)*

The affine Lie algebra $\widehat{\mathfrak{g}}_i$ has a triangular decomposition

$$\widehat{\mathfrak{g}}_i = (\widehat{\mathfrak{g}}_i)_{<0} \oplus \mathfrak{g} \otimes \mathbb{C}c \oplus (\widehat{\mathfrak{g}}_i)_{>0},$$

where

$$(\widehat{\mathfrak{g}}_i)_{<0} = \mathfrak{g} \otimes \xi_i^{-1}\mathbb{C}[\xi_i^{-1}], \ \ (\widehat{\mathfrak{g}}_i)_{>0} = \mathfrak{g} \otimes \xi_i \mathbb{C}[[\xi_i]],$$

and we write

$$(\widehat{\mathfrak{g}}_i)_{\geqslant 0} = \mathfrak{g} \otimes \mathbb{C}c \oplus \mathfrak{g} \otimes \xi_i \mathbb{C}[[\xi_i]].$$

To every $\mathfrak{g}$-module V_{λ_i}, one can form the Verma module, a $\widehat{\mathfrak{g}}_i$-module

$$\mathcal{V}_{\lambda_i} = U((\widehat{\mathfrak{g}}_i)_{<0}) \otimes_{\mathbb{C}} V_{\lambda_i} = U(\widehat{\mathfrak{g}}_i) \otimes_{U((\widehat{\mathfrak{g}}_i)_{\geqslant 0})} V_{\lambda_i},$$

where to do this, we extend the action of $\mathfrak{g}$ on V_{λ_i} to an action of $(\widehat{\mathfrak{g}}_i)_{\geqslant 0} = \mathfrak{g} \otimes \mathbb{C}c \oplus (\widehat{\mathfrak{g}}_i)_{>0}$ on V_{λ_i} by declaring that $(\widehat{\mathfrak{g}}_i)_{>0}$ act by zero, and the central element c by $\ell \cdot \mathrm{id}_{V_{\lambda_i}}$. One has simple $\widehat{\mathfrak{g}}_i$-modules $\mathcal{H}_{\lambda_i} = \mathcal{V}_{\lambda_i}/Z_{\lambda_i}$, where $Z_{\lambda_i} \subset \mathcal{V}_{\lambda_i}$ is the unique maximal submodule.

In fact, the Lie algebra

$$\widehat{\mathfrak{g}}_{[n]} = \left(\bigoplus_{i=1}^{n} \mathfrak{g} \otimes \mathbb{C}((\xi_i))\right) \oplus \mathbb{C}c$$

acts diagonally on the tensor product $\mathcal{H}_{\vec{\lambda}} = \otimes_{i=1}^{n} \mathcal{H}_{\lambda_i}$.

For $U = C \backslash \{p_1, \ldots, p_s\}$, and $\mathfrak{g}(U) = \mathfrak{g} \otimes \mathcal{O}_C(U)$, one can show there is a homomorphism of Lie algebras:

$$\mathfrak{g}(U) \longrightarrow \widehat{\mathfrak{g}}_{[n]}, \quad (X \otimes f) \mapsto (X \otimes f_{p_1}(\xi_1), \ldots, X \otimes f_{p_n}(\xi_n), 0). \tag{2.2}$$

This is important, so that one can restrict the action of $\widehat{\mathfrak{g}}_{[n]}$ on $\mathcal{H}_{\vec{\lambda}}$ to the image $\mathfrak{g}(U)$ under this map

$$\mathfrak{g}(U) \times \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\vec{\lambda}},$$

$$((X \otimes f), (v_1 \otimes \cdots \otimes v_n)) \mapsto \sum_{i=1}^{n} v_1 \otimes \cdots v_{i-1} \otimes (X \otimes f_{p_i}) \cdot v_i \otimes \cdots \otimes v_n. \tag{2.3}$$

Definition 2.3.5. *The fibers of the bundles of covacua are the vector spaces of coinvariants:*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})} \cong [\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(U)} = \mathcal{H}_{\vec{\lambda}} / \mathfrak{g}(U) \mathcal{H}_{\vec{\lambda}}.$$

Problem Session 2.3.6. *Check that Eq* (2.2) *is actually a homomorphism of Lie algebras, and that Eq* (2.3) *is an action (See Claim 2.5.33).*

2.4 Propagation of Vacua and Factorization

Notice in the construction above, we assumed that $U = C \backslash \{p_1, \ldots, p_n\}$ is affine. In particular, we assumed that $n > 1$. To remove this condition, a theorem is required. In the third Lecture we will rearrange the coinvariants so that the following result can be easily proved:

Theorem 2.4.1. *Let* $q \in C \backslash \vec{p}$. *Then* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})} = \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \{0\}, \ell)|_{(C,\vec{p} \cup \{q\})}$.

The Factorization Theorem, originally proved by Tsuyshiya, Ueno and Yamada [TUY89, Prop 2.2.6], explains how a vector bundle of conformal blocks at a point on the moduli space where the underlying curve has a node, factors into sums and products of bundles on the normalization of the curve where the sum is taken over all possible weights at points over which the normalization is "glued" to make the original curve. Applications of Factorization include inductive formulas for the rank and Chern classes of the bundle. In fact, Beauville, in [Bea96] gives an elementary proof of Factorization using this quotient construction.

Definition 2.4.2. *Given a weight* $\mu \in \mathcal{P}_\ell(\mathfrak{g})$*, let* $\mu^\star \in \mathcal{P}_\ell(\mathfrak{g})$ *be the element with the property that* $-\mu^\star$ *is the lowest weight of the weight space* V_μ*.*

Theorem 2.4.3 (Factorization)**.** *Let* $(C_0; p_1, \dots, p_n)$ *be a stable* n*-pointed curve of genus* g *where* C_0 *has a node* x_0*.*

1. *If* x_o *is a non-separating node,* $\nu : C \to C_0$ *the normalization of* C_0 *at* x_0*, and* $\nu^{-1}(x_0) = \{x_1, x_2\}$*, then*

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})} \cong \bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \mu \cup \mu^\star\}, \ell)_{(C;\vec{p}\cup\{x_1,x_2\})}.$$

2. *If* x_0 *is a separating node,* $\nu : C_1 \cup C_2 \to C_0$ *the normalization of* C_0 *at* x_0 *and* $\nu^{-1}(x_0) = \{x_1, x_2\}$*, with* $x_i \in C_i$*, then*

$$\begin{aligned} &\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C_0;\vec{p})} \\ &\cong \bigoplus_{\mu \in \mathcal{P}_\ell(\mathfrak{g})} \mathbb{V}(\mathfrak{g}, \lambda(C_1) \cup \{\mu\}, \ell)|_{(C_1;\{p_i \in C_1\}\cup\{x_1\})} \otimes \mathbb{V}(\mathfrak{g}, \lambda(C_2) \cup \{\mu^\star\}, \ell)|_{(C_2;\{p_i \in C_2\}\cup\{x_2\})}, \end{aligned} \tag{2.4}$$

   where $\lambda(C_i) = \{\lambda_j | p_j \in C_i\}$*.*

Example 2.4.4. *If* $\mu \in \mathcal{P}_\ell(\mathfrak{sl}_2)$*, then* $\mu^\star = \mu$*.*

Example 2.4.5. *For* $\mathfrak{g} = \mathfrak{sl}_{r+1}$ *we express a weight* λ_i *as a sum* $\lambda_i = \sum_{j=1}^r c_j \omega_j$*, and* λ_i *has a corresponding Young diagram that fits into an* $(r+1) \times \ell$ *sized grid, where since* λ_i *is normalized, the last row is empty. In terms of Young diagrams, the level is the number of "filled in" boxes across the top, and* $|\lambda_i|$ *means the total number of boxes "filled in" altogether. To find the Young diagram corresponding to* $\lambda^\star$ *we fill in the boxes in the diagram directly below the boxes corresponding to* λ*, and then rotate by* 180 *degrees to get the Young diagram associated to the weight* λ^**. For example, if* $r+1 = 4$*, and* $\ell \geqslant 5$ *for the weight* λ *pictured in white on the left below, then the dual weight* $\lambda^\star$ *is pictured in green on the right.*

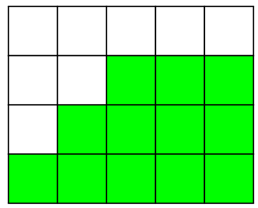
Example 2.4.6. *[BGM15a] We will factorize the bundle* $\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_r, \ell\omega_r\}, \ell)$ *on* $\overline{M}_{0,4}$ *at the two types of points* $(C; p_1, \dots, p_4)$*, where the curve* C *has one node: the first type* $X_1 = (C_{11} \cup C_{12}; p_1, \dots, p_4)$ *where* C_{11} *is labeled by* p_1 *and* p_2 *and* C_{12} *by* p_3 *and* p_4*; and the second type of curve* $X_2 = (C_{21} \cup C_{22}; p_1, \dots, p_4)$ *where* C_{21} *is labeled by* p_1 *and* p_3 *and* C_{22} *by* p_2 *and* p_4*.*

1. *If* $r+1 = 2$ *this is* $\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_1\}, \ell)$*, and we obtain:*

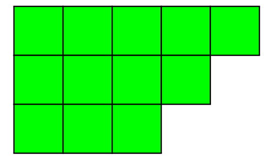
$$\begin{aligned} &\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_r\}, \ell)|_{X_1} \\ &\cong \bigoplus_{\substack{m \geqslant 0 \\ \text{even}}} \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, m\omega_1\}, \ell)|_{(C_{11}, p_1, p_2, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_2, \{\ell\omega_1, \ell\omega_r, m\omega_1\}, \ell)|_{(C_{12}, p_3, p_4, x_2)}. \end{aligned}$$

Figure 2.1

$\lambda = 3\omega_1 + \omega_2 + \omega_3$
for $\mathfrak{sl}_4$,
and level $\ell(\lambda) \geqslant 5$.



$\lambda^\star = \omega_1 + \omega_2 + 3\omega_3$.



As we'll see later, the only term in the sum above that gives bundles of nonzero rank occurs when $m = 0$*, and that both bundles have rank one.*

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \omega_1, \ell\omega_1, \ell\omega_r\}, \ell)|_{X_2} \\ \cong &\bigoplus_{\substack{m \geqslant 0 \\ m+\ell \equiv 1 (\mathrm{mod}\, 2)}} \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \ell\omega_1, m\omega_1\}, \ell)|_{(C_{21}, p_1, p_3, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_2, \{\omega_1, \ell\omega_1, m\omega_1\}, \ell)|_{(C_{22}, p_2, p_4, x_2)}.\end{aligned}$$

Again, we'll see that the only term above that gives two bundles of nonzero rank occurs when $m = (\ell - 1)$*, and has rank one in this case.*

2. *If* $r + 1 = 3$ *this is* $\mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_2, \ell\omega_1\}, \ell)$*, and we obtain, for*

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_2, \ell\omega_1\}, \ell)|_{X_1} \\ \cong &\bigoplus_{\substack{\mu = c_1\omega_1 + c_2\omega_2 \\ c_1 + 2c_2 \equiv 1 (\mathrm{mod}\, 3)}} \mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, \mu\}, \ell)|_{(C_{11}, p_1, p_2, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_3, \{\ell\omega_1, \ell\omega_r, \mu^\star\}, \ell)|_{(C_{12}, p_3, p_4, x_2)}.\end{aligned}$$

We'll later see that the only summand on the right hand side with nonzero rank is the one with $\mu = \omega_1$ *(so* $c_1 = 1$*, and* $c_2 = 0$*).*

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_2, \ell\omega_1\}, \ell)|_{X_2} \\ \cong &\bigoplus_{\substack{\mu = c_1\omega_1 + c_2\omega_2 \\ \ell + c_1 + 2c_2 \equiv 1 (\mathrm{mod}\, 3)}} \mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \ell\omega_1, \mu\}, \ell)|_{(C_{21}, p_1, p_3, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_3, \{\omega_1, \ell\omega_2, \mu^\star\}, \ell)|_{(C_{22}, p_2, p_4, x_2)}.\end{aligned} \tag{2.5}$$

We'll later see that the only summand on the right hand side with nonzero rank is the one with $\mu = (\ell-1)\omega_2$ *(so* $c_1 = 0$*, and* $c_2 = (\ell - 1)$*).*

3. *In general:*

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_r, \ell\omega_r\}, \ell)|_{X_1} \\ \cong &\bigoplus_{\substack{\mu = \sum_{i=1}^r c_i\omega_i \\ \sum_{i=1}^r i \cdot c_i + 2 \equiv 0 (\mathrm{mod}(r+1))}} \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, \mu\}, \ell)|_{(C_{11}, p_1, p_2, x_1)} \otimes \mathbb{V}(\mathfrak{sl}_{r+1}, \{(\ell-1)\omega_1 + \omega_r, \ell\omega_r, \mu^\star, \ell).\end{aligned}$$

Moreover, one can show that the only summand on the right hand side with nonzero rank is the one with $\mu = \omega_{r-1}$.

$$\begin{aligned}&\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \omega_1, (\ell-1)\omega_1 + \omega_r, \ell\omega_r\}, \ell)|_{X_2} \\ &\cong \bigoplus_{I} \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, (\ell-1)\omega_1+\omega_r, \mu\}, \ell)|_{(C_{21},p_1,p_3,x_1)} \otimes \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \ell\omega_r, \mu^\star, \ell)|_{(C_{22},p_2,p_4,x_2)},\end{aligned}$$

where we sum over the set

$$I = \{\mu = \sum_{i=1}^{r} c_i\omega_i \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1}) : \sum_{i=1}^{r} i \cdot c_i + \ell + r \equiv 0 (\mathrm{mod}(r+1))\}.$$

We will eventually show that the only summand on the right hand side with nonzero rank is the one with $\mu = (\ell-1)\omega_r$ *and* $\mu^\star = (\ell-1)\omega_1$. *We'll see that:*

$$\mathrm{rk}\,\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, (\ell-1)\omega_1 + \omega_r, (\ell-1)\omega_r\}, \ell) = \mathrm{rk}\,\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1, \ell\omega_r, (\ell-1)\omega_1, \ell) = 1.$$

Remark 2.4.7. *This example exhibits the potential for the use of factorization to compute ranks, which is the idea behind the proof of the Verlinde formula. The comments made also indicate that there is a lot of vanishing happening – which is a foreshadowing of one of the open problems in the subject: that is to determine given* $\mathfrak{g}$ *and* ℓ *necessary and sufficient conditions which will guarantee that the first Chern class of the bundle* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *is not zero. One indication is that it's rank is nonzero, which is actually enough for* $\mathfrak{sl}_2$*, but this is not in general. For example, while the rank of* $\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3, 2\omega_1 + \omega_3\}, 3)$ *is one, the first Chern class of this bundle is zero [BGM16]. We'll discuss this problem.*

2.5 Appendix

2.5.1 Witten's Dictionary

To compute the ranks in Example 2.4.6 one can use the Verlinde formula, or one may use the following cohomological form of Witten's Dictionary, which expresses ranks of the bundles as the intersection numbers of particular classes (depending on the bundle) in the small quantum cohomology ring of certain Grassmannian varieties.

Theorem 2.5.1. *Let* $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ *be a vector bundle on* $\overline{\mathcal{M}}_{g,n}$ *such that* $\sum_{i=1}^{n} |\lambda_i| = (r+1)(\ell+s)$ *for some integer* s.

1. *If* $s > 0$, *then let* $\lambda = \ell\omega_1$. *The rank of* $\mathbb{V}$ *is the coefficient of* $q^s\sigma_{\ell\omega_{r+1}}$ *in the quantum product*

$$\sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_n} \star \sigma_\lambda^s \in QH^*(Gr(r+1, r+1+\ell)).$$

2. *If* $s \leqslant 0$, *then the rank of* $\mathbb{V}$ *is the multiplicity of the class of a point* $\sigma_{k\omega_{r+1}}$ *in the product*

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdot \dots \cdot \sigma_{\lambda_n} \in H^*(\mathrm{Gr}(r+1, r+1+k)),$$

where $k = \ell + s$.

The relation to quantum cohomology follows from [Wit95] and the twisting procedure of [Bel08a], see Eq (3.10) from [Bel08a]. Examples of such rank computations were done using Witten's Dictionary in [BGM15b], [BGM16], and [Kaz16].

2.5.2 Background reading: Principal G-bundles

Definition 2.5.2. *Let* G *be an algebraic group,* X *a variety, and* T *a Grothendieck topology. A principal* G*-bundle on* X *with respect to* T*, is a morphism* $\pi : P \to X$ *together with an action* $P \times G \xrightarrow{a} P$ *such that the following properties hold:*

1. *The diagrams*

$$\begin{array}{ccc} P \times G & \xrightarrow{a} & P \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ P & \xrightarrow[\pi]{} & X \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} P \times G \times G & \xrightarrow{id \times \mu} & P \times G \\ \downarrow{\scriptstyle a \times id} & & \downarrow{\scriptstyle a} \\ P \times G & \xrightarrow[a]{} & P, \end{array}$$

commute, where $\mu : G \times G \to G$ *denotes the multiplication operation on* G.

2. *There exists a covering* $\{\cup_{j \in J} U_j \to X\}$ *of* X *in the* T *topology, for which for each* $j \in J$ *there are* G*-space isomorphisms* $\psi_j : P|_{U_j} \xrightarrow{\cong} U_j \times G$*, meaning that the following two diagrams*

$$\begin{array}{ccc} P|_{U_j} & \xrightarrow{\psi_j} & U_j \times G \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \pi_1} & \\ U_j & & \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} P|_{U_j} \times G & \xrightarrow{a} & P|_{U_j} \\ \downarrow{\scriptstyle \psi_j \times id} & & \downarrow{\scriptstyle \psi_j} \\ U_j \times G \times G & \xrightarrow[id \times \mu]{} & U_j \times G \end{array}$$

commute.

Remark 2.5.3. *If* X *is defined over a field of char* 0*, then the fppf and etale topologies are the same. If* G *is simply connected and* X *is a curve, as in our situation, then this is the same as working with the Zariski topology.*

2.5.3 Background reading: The universal enveloping algebra

From any associative algebra A one can build a Lie algebra $\mathcal{L}(A)$ by taking the Lie bracket to be the commutator. Given a Lie algebra, we can also construct an associative algebra called the universal enveloping algebra – it has many of the features of the Lie algebra we start with but is in some sense easier to work with.

Definition 2.5.4. *For any (possibly infinite dimensional) Lie algebra $\mathfrak{g}$, the* ***universal enveloping algebra*** *of $\mathfrak{g}$ is defined to be any pair (U, i) where U is an associative algebra with unity and $i : \mathfrak{g} \to \mathcal{L}(U)$ is a homomorphism of Lie algebras with the property that, if A is any other associative algebra with unity and if $\phi : \mathfrak{g} \to \mathcal{L}(A)$ is any Lie algebra homomorphism, then there is a unique homomorphism of unital algebras $\psi : U \to A$, so that the following diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ i\ } & \mathcal{L}(U) \\ & \searrow^{\phi} & \downarrow{\scriptstyle \psi_*} \\ & & \mathcal{L}(A). \end{array}$$

commutes. In the diagram, the map ψ_* is equal to ψ, considered as a homomorphism of Lie algebras.

The universal enveloping algebra $(U(\mathfrak{g}), i)$ is constructed from the tensor algebra $\mathcal{T}(\mathfrak{g})$.

Definition 2.5.5. *Given a vector space V over a field k, the* ***tensor algebra*** *$\mathcal{T}(V)$ is defined to be the direct sum*

$$\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} T^k(V), \quad \textit{where} \quad T^k(V) = V^{\otimes k} = V \otimes V \otimes \cdots \otimes V,$$

with multiplication determined by the canonical isomorphism

$$T^k(V) \bigotimes T^m(V) \to T^{k+m}(V),$$

given by the tensor product and extended linearly to all of $T(V)$.

Definition 2.5.6. *Let $\mathfrak{g}$ be a Lie algebra. Then set $U(\mathfrak{g})$ equal to the quotient of $\mathcal{T}(\mathfrak{g})$ by the ideal generated by all elements of the form*

$$X \otimes Y - Y \otimes X - [X, Y],$$

for all X and $Y \in \mathfrak{g}$, and define

$$i : \mathfrak{g} \to U(\mathfrak{g}), \quad X \mapsto X.$$

Check that the relations defining $U(\mathfrak{g})$ ensure that $i : \mathfrak{g} \to U(\mathfrak{g})$ is a morphism of Lie algebras, and that $(U(\mathfrak{g}), i)$ is a universal enveloping algebra. Show that the universal enveloping algebra (U, i) of $\mathfrak{g}$ is unique up to isomorphism.

2.5.4 Background reading: Just enough representation theory

Simple Lie algebras

Throughout, we fix a field k, which will be useful to assume later is algebraically closed, and of characteristic 0.

Definition 2.5.7. *A **Lie algebra** is a k-vector space $\mathfrak{g}$ together with an binary operation called **the Lie bracket***

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (A, B) \to [A, B]$$

which satisfies the following three conditions

1. *bilinearity:* $[A + B, C] = [A, C] + [B, C]$ *and* $[A, B + C] = [A, B] + [A, C]$*;*
2. *anti-symmetry:* $[A, A] = 0$*; or equivalently if* $\mathrm{char}(k) \neq 2$, $[A, B] = -[B, A]$*; and*
3. *the Jacobi identity:* $[A, [B, C]] - [[A, B], C] = [B, [A, C]]$.

Example 2.5.8. *Let V be a k-vector space of dimension n. We let $\mathfrak{gl}(V)$ be the **general linear Lie algebra**, consisting of the set of linear transformations $V \to V$, and Lie bracket given by the commutator $[\phi, \theta] = \phi \circ \theta - \theta \circ \phi$.*

In particular, as is conventional, we denote $\mathfrak{gl}(k^n)$ by $\mathfrak{gl}_n$, taking elements to be $n \times n$ matrices over k, and the Lie bracket to be the commutator:

$$[A, B] = AB - BA.$$

Clearly this is bilinear and anti-symmetric. One may also verify that the Jacobi identity:

$$\begin{aligned}(2.6)\quad [A, [B, C]] - [[A, B], C] &= \big(A(BC - CB) - (BC - CB)A\big) - \big((AB - BA)C - C(AB - BA)\big) \\ &= ABC - ACB - BCA + CBA + ABC + BAC + CAB - CBA \\ &= BAC + CAB - ACB - BCA = [B, [A, C]].\end{aligned}$$

Definition 2.5.9. *A Lie algebra $\mathfrak{g}$ is **Abelian** if $[A, B] = 0$ for every $A, B \in \mathfrak{g}$.*

Definition 2.5.10. *A Lie algebra is* **simple** *if it is not Abelian, and has no nonzero proper ideals.*

Dominant integral weights for $\mathfrak{g}$

To define dominant integral weights for $\mathfrak{g}$ we start with representations of $\mathfrak{g}$.

Definition 2.5.11. *A* **homomorphism of Lie algebras** *is a linear map of vector spaces* $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ *preserving the bracket:*

$$f(\,[A,B]_{\mathfrak{g}_1}) = [f(A), f(B)]_{\mathfrak{g}_2}, \;\forall\, A, B \in \mathfrak{g}_1.$$

Definition 2.5.12. *Let* V *be a vector space, and* $\mathfrak{g}$ *a Lie algebra. A* ***representation*** *of* $\mathfrak{g}$ *on* V *is a Lie algebra homomorphism* $\mathfrak{g} \to \mathfrak{gl}(V)$*. Equivalently, a representation of* $\mathfrak{g}$ *on* V *is a rule* $\mathfrak{g} \times V \to V$*, say* $(A, v) \mapsto A \cdot v$ *such that*

$$[A,B] \cdot v = A \cdot (B \cdot v) - B \cdot (A \cdot v).$$

Remark 2.5.13. *If* $\mathfrak{g} \to \mathfrak{gl}(V)$ *is a representation of* $\mathfrak{g}$ *on* V*, we often abuse language and simply refer to* V *itself as a representation (omitting the homomorphism from the notation).*

Definition 2.5.14. *If* $\mathfrak{g}$ *is a Lie algebra, then it acts on itself via*

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (A,B) \mapsto A \cdot B = [A,B].$$

This gives the homomorphism of Lie algebras

$$\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad A \to \mathrm{ad}_{\mathfrak{g}}(A),$$

where $\mathrm{ad}_{\mathfrak{g}}(A)$ *is the linear transformation on defined by*

$$\mathrm{ad}_{\mathfrak{g}}(A)(B) = [A,B].$$

This very important representation is referred to as ***the adjoint representation****.*

Definition 2.5.15. *We say that a representation of* $\mathfrak{g}$ *on* V *is* ***irreducible*** *if it has no nontrivial proper sub-representations. That is, if there is no non-trivial and proper vector subspace* $W \subset V$ *and representation* $\mathfrak{g} \to \mathfrak{gl}(W)$*, making the natural induced diagram:*

$$\mathfrak{g} \to \mathfrak{gl}(W) \subset \mathfrak{gl}(V),$$

commute.

Definition 2.5.16. *A linear subspace* $\mathfrak{g}_1 \subset \mathfrak{g}_2$ *is a* **Lie subalgebra** *if* $\mathfrak{g}_1$ *is closed under the Lie bracket of* $\mathfrak{g}_2$*:*

$$[A,B]_{\mathfrak{g}_2} \in \mathfrak{g}_1, \;\forall\, A, B \in \mathfrak{g}_1.$$

If $\mathfrak{g} \to \mathfrak{gl}(W)$ is a sub-representation of V, then $\mathfrak{gl}(W) \subset \mathfrak{gl}(V)$ is a Lie subalgebra.

Example 2.5.17. *Let $\mathfrak{sl}(V)$ (resp. $\mathfrak{sl}_n$) denote the Lie subalgebra of $\mathfrak{gl}(V)$ (resp. $\mathfrak{gl}_n$) called the **special linear Lie algebra** consisting of those operators on V of trace 0 (ie. those matrices whose trace is 0).*

Definition 2.5.18. *A **Cartan subalgebra** of a Lie algebra $\mathfrak{g}$ is an Abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is maximal with respect to the property of being Abelian.*

Exercise 2.5.19. *Let $\mathfrak{h} \subset \mathfrak{sl}_n$ be the diagonal matricies. Show $\mathfrak{h}$ is a Cartan subalgebra.*

Definition 2.5.20. *Let $\mathfrak{g}$ be a Lie algebra and V be a representation for $\mathfrak{g}$. Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. We describe the* **weights and roots for** $\mathfrak{g} \to \mathfrak{gl}(V)$ *as follows:*

1. *By an **eigenvalue** for the action of $\mathfrak{h}$, we will mean an element $\alpha \in \mathfrak{h}^*$ such that $H(v) = \alpha(H) \cdot v$, for some nonzero $v \in V$, and all $H \in \mathfrak{h}$. An eigenvalue $\alpha \in \mathfrak{h}^*$ of the action of $\mathfrak{h}$ on the representation V of $\mathfrak{g}$ is called a **weight of the representation**. The weights $\alpha \in \mathfrak{h}^*$ that occur in the adjoint representation are called **roots**. The convention is that $0 \in \mathfrak{h}^*$ is not considered a root.*

2. *By the **eigenspace** V_α associated to the eigenvalue α we mean the subspace of all vectors $v \in V$ such that $H(v) = \alpha(H) \cdot v$. The corresponding eigenvectors in V_α are called **weight vectors** and V_α is called the **weight space**. The eigenspaces $\mathfrak{g}_\alpha$ corresponding to the roots are called **root spaces**.*

Definition 2.5.21. *We define the* **weights and roots for** $\mathfrak{g}$ *as follows.*

1. *The weights for $\mathfrak{g}$ are the weights for all representations $\mathfrak{g} \to \mathfrak{gl}(V)$.*

2. *We denote the set of all roots by $R \subset \mathfrak{h}^*$.*

Definition 2.5.22. *One can define a* **highest weight** *as follows:*

- *We choose a direction in $\mathfrak{h}^*$ which means defining a linear functional $f : \mathfrak{h}^* \to \mathbb{C}$. This gives a decomposition of the set*

  $$R = R^+ \cup R^-, \text{ where}$$

  $R^+ = \{\alpha \in R : f(\alpha) > 0\}$, *called the **positive roots**, and*
  $R^- = \{\alpha \in R : f(\alpha) < 0\}$, *called the **negative roots**.*

- *We say that a positive (resp., negative) root $\alpha \in R$ is primitive or **simple** if it cannot be expressed as a sum of two positive (resp. negative) roots.*

- *A nonzero vector $v \in V$ which is both an eigenvector for the action of $\mathfrak{h}$ and in the kernel of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$ is called a **highest weight vector**.*

Remark 2.5.23. *In Definition 2.5.30 we will describe the Killing form. After that we will be able to define a semisimple Lie algebra over a field of characteristic zero as one whose Killing form is nondegenerate. The following can be shown to be equivalent for a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic 0:*

1. *$\mathfrak{g}$ is semisimple;*

2. *$\mathfrak{g}$ is a finite direct product of simple Lie algebras.*

In particular, if $\mathfrak{g}$ is a finite dimensional simple Lie algebra defined over a field of characteristic 0, then $\mathfrak{g}$ is semisimple. While not necessary for our application, the next statement holds for the broader context of semisimple Lie algebras.

Proposition 2.5.24. *[FH91, 14.13] For any semisimple complex Lie algebra $\mathfrak{g}$,*

1. *every finite dimensional representation V of $\mathfrak{g}$ has a highest weight vector;*

2. *an irreducible representation has a unique highest weight vector up to scalars.*

Definition 2.5.25. *A **dominant integral weight** is an element $\alpha \in \mathfrak{h}^*$ such that $H(v) = \alpha(H) \cdot v$, for all $H \in \mathfrak{h}$, where $v \in V$ is the highest weight vector of an irreducible representation V of $\mathfrak{h}$.*

Definition 2.5.26. *[FH91, Section 14.2] R generates a lattice $\Lambda_R \subset \mathfrak{h}^*$, the **root lattice**, of rank equal to $\dim(\mathfrak{h})$. The free generators for the lattice are **fundamental dominant weights**.*

Remark 2.5.27. *Depending on the author, weights are sometimes called integral weights; dominant integral weights are sometimes referred to as dominant weights.*

Definition 2.5.28. *A character of a Lie algebra $\mathfrak{g}$ is a linear map $\mathfrak{g} \to k$. That is, since $k = \mathfrak{gl}_1$, a character of a Lie algebra $\mathfrak{g}$ is a 1-dimensional representation of $\mathfrak{g}$.*

Example 2.5.29. *Let $\mathfrak{g} = \mathfrak{sl}_2$. We first set $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}$. Then*

$$\mathrm{ad}_{\mathfrak{g}}(A) : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2, \quad B \mapsto AB - BA,$$

so that in particular

$$\mathrm{ad}_{\mathfrak{g}}(A) \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} bz - yc & 2(ay - bx) \\ 2(cx - az) & -(bz - yc) \end{pmatrix}.$$

The Cartan subalgebra $\mathfrak{h}$ *is the set of diagonal matrices in* $\mathfrak{g} = \mathfrak{sl}_2$*. Consider*

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in \mathfrak{h},$$

so that

$$\mathrm{ad}_{\mathfrak{g}}(A) : \mathfrak{sl}_2 \to \mathfrak{sl}_2, \ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mapsto 2a \begin{pmatrix} 0 & y \\ -z & 0 \end{pmatrix}.$$

We shall see that $\mathrm{ad}_{\mathfrak{g}}(A)$ *is a direct sum of three characters of* $\mathfrak{h}^*$*. Namely, one can decompose* $\mathfrak{sl}_2$ *as a direct sum of three one-dimensional vector spaces* $\mathfrak{sl}_2 \cong V_1 \oplus V_2 \oplus V_3$*, where*

$$V_1 = \{\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : x \in k\}; \quad V_2 = \{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in k\};$$

and

$$V_3 = \{\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} : z \in k\}.$$

The sub-vector spaces $V_i \subset \mathfrak{sl}_2$ *are sub-representations of the adjoint representation of* $\mathfrak{h}$ *on* $\mathfrak{sl}_2$ *defined by*

$$\mathfrak{h} \times V_1 \to V_1, \ (\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

$$\mathfrak{h} \times V_2 \to V_2, \ (\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}) \mapsto 2a \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix};$$

$$\mathfrak{h} \times V_3 \to V_3, \ (\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}) \mapsto -2a \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}.$$

The second and third characters $\alpha_1 = 2a$ *and* $\alpha_2 = -2a$*, which are the nonzero representations, are the two roots on* $\mathfrak{sl}_2$*. The root* α_1 *is a simple root. In general, one has* r *simple roots of* $\mathfrak{sl}_{r+1}$*.*

Dominant integral weights for $\mathfrak{g}$ at level ℓ

In order to define the level of a weight, we next define the Killing form $(\mid)$, and the normalized Killing form $(\, , \,)$, which both come from an inner product $< \mid >$ on $\mathfrak{g}$.

Definition 2.5.30. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{h}$ *a Cartan subalgebra. Recall that for* $A \in \mathfrak{g}$*, one has the adjoint representation*

$$\mathrm{ad}_{\mathfrak{g}}(A) : \mathfrak{g} \to \mathfrak{g}, \ C \mapsto \mathrm{ad}_{\mathfrak{g}}(A)(C) = [A, C].$$

In particular, a choice of basis for $\mathfrak{g}$ gives a representation of this linear transformation $\mathrm{ad}_{\mathfrak{g}}(A)$ by a square matrix of $\dim(\mathfrak{g})$. We define an inner product $< | >$ on $\mathfrak{g}$ by setting, for A and $B \in \mathfrak{g}$,

$$< A \mid B >= \mathrm{trace}(\mathrm{ad}_{\mathfrak{g}}(A) \cdot \mathrm{ad}_{\mathfrak{g}}(B)).$$

One can then define a natural morphism from $\mathfrak{h}$ to $\mathfrak{h}^*$ by setting

$$\psi : \mathfrak{h} \to \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, k), \ \ A \mapsto \{B \mapsto < A \mid B >\}.$$

One can check that this is an isomorphism and that this induces an inner product on $\mathfrak{h}^*$:

$$(f|g) :=< \psi^{-1}(f) \mid \psi^{-1}(g) >= \mathtt{trace}(\mathrm{ad}_{\mathfrak{g}}(\psi^{-1}(f)) \cdot \mathrm{ad}_{\mathfrak{g}}(\psi^{-1}(g))).$$

*This natural inner product $(\ |\ )$ is referred to as the **Killing form**.*

Remark 2.5.31. *One can prove that there is a unique positive root $\theta \in R^+$ with the property that $(\theta|\theta) \geqslant (\alpha|\alpha)$ for any other root $\alpha \in R^+$. This root theta is called the* **longest root***. It is conventional to normalize the Killing form, writing $(\ ,\ )$, so that $(\theta, \theta) = 2$.*

Definition 2.5.32. *The* **level** *of any weight α is equal to the value (α, θ), where θ is the longest root, and $(\ ,\ )$ is the normalized Killing form. The level is an integer.*

Action Claim

Claim 2.5.33. *Equation 2.3 defines an action of $\mathfrak{g}(U)_{\mathrm{Out}}$ on $H_{\vec{\lambda}}$.*

Proof. Given $X \otimes f$, and $Y \otimes g \in \mathfrak{g}(U)$, and $v = v_1 \otimes \cdots \otimes v_n \in \mathcal{H}_{\vec{\lambda}}$, we want to check that

$$[X \otimes f, Y \otimes g] \cdot v = (X \otimes f) \cdot \big((Y \otimes g) \cdot v\big) - (Y \otimes g) \cdot \big((X \otimes f) \cdot v\big).$$

The right hand side simplifies as follows:

$$\begin{aligned} (2.7) \quad & (X \otimes f) \cdot \big((Y \otimes g) \cdot v\big) - (Y \otimes g) \cdot \big((X \otimes f) \cdot v\big) \\ &= (X \otimes f) \cdot \Big(\sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \\ &\quad - (Y \otimes g) \cdot \Big(\sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big) \end{aligned}$$

(2.8)

$$= \Big(\sum_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} v_1 \otimes \cdots v_{j-1} \otimes (X \otimes f_{p_j}) \cdot v_j \otimes v_{j+1} \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)$$

$$- \Big(\sum_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} v_1 \otimes \cdots v_{j-1} \otimes (Y \otimes g_{p_j}) \cdot v_j \otimes v_{j+1} \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)$$

$$= \Big(\sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes (X \otimes f_{p_i}) \cdot \big((Y \otimes g_{p_i}) \cdot v_i\big) \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)$$

$$- \Big(\sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes (Y \otimes g_{p_i}) \cdot \big((X \otimes f_{p_i}) \cdot v_i\big) \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)$$

$$= \Big(\sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots v_{j-1} \otimes \cdots \otimes v_{i-1} \otimes \big([X,Y] + (fg)_{p_i}\big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)$$

The left hand side simplifies as follows:

(2.9)

$$\sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \Big([X,Y] \otimes f_{p_i} g_{p_i} + (X,Y) \operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} c \Big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n$$

$$= \sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \Big([X,Y] \otimes f_{p_i} g_{p_i} \Big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n$$

$$+ \sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \Big((X,Y) \operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} c \Big) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n .$$

Now, $c \cdot v_i = \ell \cdot v_i$ for all i, and so we can rewrite the second summand as follows

$$\sum_{1\leqslant i\leqslant n} v_1 \otimes \cdots \otimes v_{i-1} \otimes ((X,Y) \operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} c) \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \tag{2.10}$$

$$= \sum_{1\leqslant i\leqslant n} (X,Y) \operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} \Big(v_1 \otimes \cdots \otimes v_{i-1} \otimes c \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)$$

$$= \sum_{1\leqslant i\leqslant n} (X,Y) \operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} \Big(v_1 \otimes \cdots \otimes v_{i-1} \otimes \ell \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n \Big)$$

$$= \big(\ell \sum_{1\leqslant i\leqslant n} (X,Y) \operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} \big) \Big(v_1 \otimes \cdots \otimes v_n \Big).$$

Since $\sum_{1\leqslant i\leqslant n} (X,Y) \operatorname{Res}_{\xi_i=0} g_{p_i} df_{p_i} = 0$, this contribution is zero. Therefore the left and right hand sides of the expressions are the same, and we have checked that $\mathfrak{g}(U)$ acts on $H_{\vec{\lambda}}$ as claimed. $\square$

Lecture 3

Beauville's quotient construction and global generation in case $g = 0$

3.1 Introduction

Vector bundles of covacua are defined on $\overline{M}_{g,n}$ for all g and all n. As we will see today, they are globally generated in case the genus is zero.

Globally generated vector bundles are useful as they give rise to Chern classes with valuable positivity properties. For instance, the Hodge bundle, defined on $\overline{M}_g$ is a globally generated vector bundle and its Chern classes, called the λ_i classes, first studied by Mumford, are tautological and fundamental to any description of the Chow and Tautological rings of the moduli spaces $\overline{M}_{g,n}$.

Global generation of vector bundles of conformal blocks defined at smooth curves of genus zero can be seen using what is called Beauville's quotient construction. Beauville's Quotient construction is an incredibly useful alternative way to define fibers of vector bundles of covacua, and it holds in all genera. Before stating the result, and sketching the proof, I'll explain how one can use it to prove Propagation of Vacua. I'll also describe how Beauville uses it to see that when restricted to $\mathcal{M}_{0,n}$, the bundles $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ are quotients of the constant bundle $\mathbb{A}(\mathfrak{g}, \vec{\lambda})$, introduced in Lecture 2.5.4.

We will prove global generation of the bundles on $\overline{M}_{0,n}$ following Ueno's original approach, which while stated for fibers at smooth curves, as was pointed out by Fakhruddin, holds for any pointed curve with simple poles.

3.2 Beauville's Quotient Construction

Notation

We recall the definition of the fibers of vector bundles of covacua, described in Lecture 2.5.4, and also define evaluation modules, which will be referred to in Beauville's Quotient Construction stated in Theorem 3.2.1.

Let C be a possibly nodal curve, $p_1, p_2, \ldots, p_n \in C$ smooth points of C, and let ξ_i be a local parameter of C at the points p_i. Recall that the affine Lie algebras $\widehat{\mathfrak{g}}_i$ have triangular decompositions

$$\widehat{\mathfrak{g}}_i = \mathfrak{g} \otimes \mathbb{C}((\xi_i)) \oplus \mathbb{C}c = (\widehat{\mathfrak{g}}_i)_{<0} \oplus \mathfrak{g} \otimes \mathbb{C}c \oplus (\widehat{\mathfrak{g}}_i)_{>0}.$$

To every $\mathfrak{g}$-module V_{λ_i}, one can form the Verma module, a $\widehat{\mathfrak{g}}_i$-module

$$\mathcal{V}_{\lambda_i} = U((\widehat{\mathfrak{g}}_i)_{<0}) \otimes_{\mathbb{C}} V_{\lambda_i} = U(\widehat{\mathfrak{g}}_i) \otimes_{U((\widehat{\mathfrak{g}}_i)_{\geqslant 0})} V_{\lambda_i},$$

where we extend the action of $\mathfrak{g}$ on V_{λ_i} to an action of $(\widehat{\mathfrak{g}}_i)_{\geqslant 0} = \mathfrak{g} \otimes \mathbb{C}c \oplus (\widehat{\mathfrak{g}}_i)_{>0}$ on V_{λ_i} by declaring that $(\widehat{\mathfrak{g}}_i)_{>0}$ act by zero, and the central element c by $\ell \cdot \mathrm{id}_{V_{\lambda_i}}$. One has simple $\widehat{\mathfrak{g}}_i$-modules $\mathcal{H}_{\lambda_i} = \mathcal{V}_{\lambda_i}/Z_{\lambda_i}$, where $Z_{\lambda_i} \subset \mathcal{V}_{\lambda_i}$ is the unique maximal submodule, and the Lie algebra $\widehat{\mathfrak{g}}_{[n]} = \left(\bigoplus_{i=1}^{n} \mathfrak{g} \otimes \mathbb{C}((\xi_i))\right) \oplus \mathbb{C}c$ acts diagonally on the tensor product $\mathcal{H}_{\vec{\lambda}} = \otimes_{i=1}^{n} \mathcal{H}_{\lambda_i}$.

For $U = C \backslash \{p_1, \ldots, p_s\}$, and $\mathfrak{g}(U) = \mathfrak{g} \otimes \mathcal{O}_C(U)$, recall that one can show there is an embedding of Lie algebras:

$$\mathfrak{g}(U) \hookrightarrow \widehat{\mathfrak{g}}_{[n]}, \quad (X \otimes f) \mapsto (X \otimes f_{p_1}(\xi_1), \ldots, X \otimes f_{p_n}(\xi_n), 0),$$

so that one can restrict the action of $\widehat{\mathfrak{g}}_{[n]}$ on $\mathcal{H}_{\vec{\lambda}}$ to $\mathfrak{g}(U)$:

$$\mathfrak{g}(U) \times \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\vec{\lambda}}, \quad ((X \otimes f), (w_1 \otimes \cdots \otimes w_n)) \mapsto \sum_{i=1}^{n} w_1 \otimes \cdots w_{i-1} \otimes (X \otimes f_{p_i}) \cdot w_i \otimes \cdots \otimes w_n.$$

The fibers of the bundles of covacua are the vector spaces of coinvariants:

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C, \vec{p})} \cong [\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(U)} = \mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(U)\mathcal{H}_{\vec{\lambda}}.$$

Given m other points $q_1, \ldots, q_m \in U$, and $\mathfrak{g}$-modules $V_{\mu_1}, V_{\mu_2}, \ldots, V_{\mu_m}$, one can define an action of $\mathfrak{g}(U)$ on the V_{μ_j} by evaluation:

$$\mathfrak{g}(U) \times V_{\mu_j} \to V_{\mu_j}, \quad (x \otimes f, v) \mapsto f(q_j)\, x \cdot v.$$

Theorem 3.2.1. *[Bea96, Prop 2.3]The inclusions $V_{\mu_j} \hookrightarrow \mathcal{H}_{\mu_j}$ induce an isomorphism*

$$[\mathcal{H}_{\vec{\lambda}} \otimes V_{\vec{\mu}}]_{\mathfrak{g}(U)} \xrightarrow{\sim} [\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{\mu}}]_{\mathfrak{g}(U \backslash \vec{q})} \cong \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \vec{\mu}, \ell)|_{(C, \vec{p} \cup \vec{q})}.$$

3.2.1 Application One: Propagation of Vacua

Corollary 3.2.2. *Let* $q \in C\backslash\vec{p}$. *Then* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})} = \mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \{0\}, \ell)|_{(C,\vec{p}\cup\{q\})}$.

Proof. By Theorem 3.2.1, and the definition of vector spaces of covacua:

$$\mathbb{V}(\mathfrak{g}, \vec{\lambda} \cup \{0\}, \ell)|_{(C,\vec{p}\cup\{q\})} = [\mathcal{H}_{\vec{\lambda}} \otimes_{\mathbb{C}} \mathcal{H}_0]_{\mathfrak{g}(C\backslash(\vec{p}\cup\{q\}))} \cong [\mathcal{H}_{\vec{\lambda}} \otimes_{\mathbb{C}} V_0]_{\mathfrak{g}(C\backslash\vec{p})}.$$

Now since $V_0 \cong \mathbb{C}$, one has

$$[\mathcal{H}_{\vec{\lambda}} \otimes_{\mathbb{C}} V_0]_{\mathfrak{g}(C\backslash\vec{p})} \cong [\mathcal{H}_{\vec{\lambda}} \otimes_{\mathbb{C}} \mathbb{C}]_{\mathfrak{g}(C\backslash\vec{p})} \cong [\mathcal{H}_{\vec{\lambda}}]_{\mathfrak{g}(C\backslash\vec{p})} \cong \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}.$$

□

We briefly outline the proof of Theorem 3.2.1 in three steps. The full proof is given in [Bea96, pages 7-8].

3.2.2 Application Two: generic global generation for $g = 0$

Corollary 3.2.3. *For* $(C, \vec{p}) \in \mathcal{M}_{0,n}$, *one has a surjection*

$$\mathbb{A}(\mathfrak{g}, \vec{\lambda})|_{(C,\vec{p})} \twoheadrightarrow \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}.$$

To prove this, in [Bea96, Proposition 4.1], Beauville shows that there are surjections

$$\mathbb{A}(\mathfrak{g}, \vec{\lambda})|_{(C,\vec{p})} = V_{\vec{\lambda}}/\mathfrak{g}V_{\vec{\lambda}} \twoheadrightarrow \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})} = V_{\vec{\lambda}}/\big(\mathfrak{g}V_{\vec{\lambda}} + \operatorname{Im} T^{\ell+1}\big),$$

where

$$T : V_{\vec{\lambda}} \longrightarrow V_{\vec{\lambda}}, \ \ v_1 \otimes \cdots \otimes v_n \mapsto \sum_{i=1}^{n} \xi_i \, v_1 \otimes \cdots \otimes v_{i-1} \otimes x_\theta \cdot v_i \otimes v_{i+1} \otimes \cdots \otimes v_n.$$

Problem Session 3.2.4. *In the problem session, go through the details of the proof for* $\mathfrak{sl}_2$.

We will, in Section 3.3, prove Corollary 3.2.3 with another approach due to Ueno.

We next briefly sketch the proof of Theorem 3.2.1.

Sketch of the proof of Theorem 3.2.1

Proof. (of Theorem 3.2.1) Following the proof in [Bea96], we work by induction: Put $q = q_m$, $\mu = \mu_m$, $U = C\backslash\vec{p}$, and $\mathcal{H} = \mathcal{H}_{\vec{\lambda}} \otimes V_{\mu_1} \cdots \otimes V_{\mu_{m-1}}$. It will be enough to show that the inclusion $V_\mu \hookrightarrow \mathcal{H}_\mu$ induces an isomorphism

$$[\mathcal{H} \otimes V_\mu]_{\mathfrak{g}(U)} \xrightarrow{\sim} [\mathcal{H} \otimes \mathcal{H}_\mu]_{\mathfrak{g}(U\backslash q)}.$$

Step One.

Show that the inclusion of $V_\mu \hookrightarrow \mathcal{H}_\mu$ is equivariant with respect to the action of $\mathfrak{g}(U)$ so that it induces a linear map

$$[\mathcal{H} \otimes V_\mu]_{\mathfrak{g}(U)} \to [\mathcal{H} \otimes \mathcal{H}_\mu]_{\mathfrak{g}(U \backslash q)}.$$

Step Two.

We prove the result when we replace $\mathcal{H}_\mu$ by the Verma module $\mathcal{V}_\mu$:

Claim 3.2.5.

$$[\mathcal{H} \otimes V_\mu]_{\mathfrak{g}(U)} \xrightarrow{\sim} [\mathcal{H} \otimes \mathcal{V}_\mu]_{\mathfrak{g}(U \backslash q)}.$$

Proof. (Outline) Choose a local coordinate z at q so that $z^{-1} \in \mathcal{O}_C(U \backslash q)$, and write

$$\mathfrak{g}(U \backslash q) = \mathfrak{g} \otimes \mathcal{O}_C(U \backslash q) = \mathfrak{g} \otimes \Big(\sum_{n \geqslant 1} \mathbb{C} z^{-n} \Big) = \mathfrak{g} \otimes \mathcal{O}_C(U) \oplus \Big(\sum_{n \geqslant 1} \mathfrak{g} z^{-n} \Big) = \mathfrak{g}(U) \oplus \widehat{\mathfrak{g}}_{<0},$$

where we identify the Lie algebra $\sum_{n \geqslant 1} \mathfrak{g} z^{-n}$ with its image $\widehat{\mathfrak{g}}_{<0}$ in $\widehat{\mathfrak{g}}$. We will see

$$[\mathcal{H} \otimes V_\mu]_{\mathfrak{g}(U)} \xrightarrow{\sim} [\mathcal{H} \otimes \mathcal{V}_\mu]_{\mathfrak{g}(U) \oplus \widehat{\mathfrak{g}}_{<0}}.$$

We first prove that

$$[\mathcal{H} \otimes \mathcal{V}_\mu]_{\widehat{\mathfrak{g}}_{<0}} \cong \mathcal{H} \otimes V_{\vec{\lambda}}.$$

After doing so, taking coinvariants by the action of $\mathfrak{g}$ will give the result.

By definition, $[\mathcal{H} \otimes \mathcal{V}_\mu]_{\widehat{\mathfrak{g}}_{<0}}$ is isomorphic to the tensor product $\mathcal{H} \otimes_{U(\widehat{\mathfrak{g}}_{<0})} \mathcal{V}_\mu$. Now by definition of $\mathcal{V}_\mu$,

$$\mathcal{H} \otimes_{U(\widehat{\mathfrak{g}}_{<0})} \mathcal{V}_\mu \cong \mathcal{H} \otimes_{U(\widehat{\mathfrak{g}}_{<0})} U(\widehat{\mathfrak{g}}_{<0}) \otimes_{\mathbb{C}} V_{\vec{\lambda}} \cong \mathcal{H} \otimes_{\mathbb{C}} V_{\vec{\lambda}}.$$

□

Step Three.

For Z_μ such that $\mathcal{H}_\mu = \mathcal{V}_\mu / Z_\mu$, one has the exact sequence:

$$\mathcal{H} \otimes Z_\mu \to [\mathcal{H} \otimes \mathcal{V}_\mu]_{\mathfrak{g}(U \backslash q)} \to [\mathcal{H} \otimes \mathcal{H}_\mu]_{\mathfrak{g}(U \backslash q)} \to 0.$$

Claim 3.2.6. *The image of $\mathcal{H} \otimes Z_\mu$ in $[\mathcal{H} \otimes \mathcal{V}_\mu]_{\mathfrak{g}(U \backslash q)}$ is zero.*

Proof. (Outline) Using that by definition, $[\mathcal{H} \otimes \mathcal{V}_\mu]_{\mathfrak{g}(U\setminus q)}$ is the same as $\mathcal{H} \otimes_{U(\mathfrak{g}(U\setminus q))} \mathcal{V}_\mu$, and that as a $U(\hat{\mathfrak{g}})$-module, Z_μ is generated by the element

$$(X_\theta \otimes z^{-1})^{\ell-(\theta,\mu)+1} \otimes v_\mu,$$

where v_μ is the highest weight vector associated to μ and this vector is annihilated by $\hat{\mathfrak{g}}_{>0}$, It is enough to show that $h \otimes ((X_\theta \otimes z^{-1})^{\ell-(\theta,\mu)+1} \cdot v_\mu) = 0$ for all $h \in \mathcal{H}$. This is done in [Bea96].

□

Problem Session 3.2.7. *Go through the details of the last step for $\mathfrak{sl}_2$ in the problem session (tomorrow).*

3.3 Ueno's Quotient Theorem

Theorem 3.3.1. *[Uen08, Proposition 6.1] and [Fak12] For $(C, \vec{p}) \in \overline{\mathcal{M}}_{0,n}$, the map*

$$\mathbb{A}(\mathfrak{g}, \vec{\lambda})|_{(C,\vec{p})} = V_{\vec{\lambda}}/\mathfrak{g}V_{\vec{\lambda}} \longrightarrow \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}$$

is surjective. In particular, $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ is the quotient of the constant bundle $\mathbb{A}(\mathfrak{g}, \vec{\lambda})$.

Approach to the proof

To show the map is surjective, we argue that the dual map

$$j : \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|^{\dagger}_{(C,\vec{p})} = \left(\mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(U)\mathcal{H}_{\vec{\lambda}}\right)^{\dagger} \longrightarrow \left(V_{\vec{\lambda}}/\mathfrak{g}V_{\vec{\lambda}}\right)^{\dagger},$$

is injective.

We will use a filtration $F_k(\mathcal{H}_\lambda)$ of $\mathcal{H}_\lambda$ to show that given $\phi \in \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|^{\dagger}_{(C,\vec{p})}$ such that $j(\phi) = 0$, then $\phi|_{F_k(\mathcal{H}_\lambda)} = 0$ for all k. We show this by induction on k.

Two descriptions of the filtration

There are a couple of equivalent ways to describe the filtration. First, for

$$\hat{\mathfrak{g}}_{[n]} = \left(\mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{i=1}^{n} \mathbb{C}((\xi_i))\right) \oplus \mathbb{C}c,$$

let

$$F_k(\widehat{\mathfrak{g}}_{<0}) = \begin{cases} \left(\mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{i=1}^{n} \mathbb{C}[\xi_i^{-1}]\xi_i^{-k}\right) \oplus \mathbb{C}c & \text{if } k \geqslant 0 \\ \mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{i=1}^{n} \mathbb{C}[\xi_i^{-1}]\xi_i^{-k} & k < 0. \end{cases}$$

Then $\mathcal{H}_\lambda$ has a natural filtration induced from that on $\widehat{\mathfrak{g}}_{[n]}$. Namely:

$$F_k\,\mathcal{H}_\lambda = F_k U(\widehat{\mathfrak{g}}_{<0})V_\lambda,$$

where

$$F_k U(\widehat{\mathfrak{g}}_{<0}) = \sum_{k_1+\cdots+k_n \leqslant k} F_{k_1}(\widehat{\mathfrak{g}}_{<0}) \otimes \cdots F_{k_n}(\widehat{\mathfrak{g}}_{<0}).$$

We can alternatively describe this in terms of a filtration given by the Casimir operator on V_{λ_i}. For each $\mathfrak{g}$-module V_λ there is an operator

$$L_0 : V_\lambda \to V_\lambda, \;\; \nu \mapsto L_0(\nu) = \Delta_\lambda \nu.$$

To get elements of $\mathcal{H}_\lambda$ we will multiply elements of ν with elements of the form $x(-d) = x \otimes \xi^{-d}$, with $x \in \mathfrak{g}$.

Now L_0 satisfies a Liebnitz rule: Namely, if one takes $x(-d) = x \otimes \xi^{-d} \in \mathfrak{g}(U)$, and $\nu \in V_\lambda$, then

$$L_0(x(-d)\cdot\nu) = L_0(x(-d))\cdot\nu + x(-d)L_0(\nu) = dx(-d)\cdot\nu + x(-d)\Delta_\lambda(\nu) = (d+\Delta_\lambda)x(-d)\cdot\nu.$$

So it makes sense to define, for each $i \in \{1,\ldots,n\}$,

$$\mathcal{H}_{\lambda_i}(d) = \{\omega \in \mathcal{H}_{\lambda_i} : L_0(\omega) = (d+\Delta_{\lambda_i})\omega\}.$$

Then

$$F_k(\mathcal{H}_{\vec{\lambda}}) = \{\omega_1 \otimes \cdots \otimes \omega_n \in \mathcal{H}_{\vec{\lambda}} : \omega_i \in \mathcal{H}_{\lambda_i}(k_i), \sum_i k_i \leqslant k\}.$$

3.3.1 Proof

We know the base case holds:

$$\phi|_{F_0\mathcal{H}_{\vec{\lambda}}} = \phi|_{V_{\vec{\lambda}}} = 0,$$

by assumption. We assume for induction that $\phi|_{F_k\mathcal{H}_{\vec{\lambda}}} = 0$.

We can express any element in $F_{k+1}\mathcal{H}_{\vec{\lambda}}$ as a sum of elements of the form

$$\widetilde{\omega} = \omega_1 \otimes \cdots \otimes \omega_{j-1} \otimes x(-m)\widetilde{\omega}_j \otimes \cdots \otimes \omega_n,$$

with

$$\omega = \omega_1 \otimes \cdots \otimes \omega_{j-1} \otimes \tilde{\omega}_j \otimes \cdots \otimes \omega_n \in F_k \mathcal{H}_{\vec{\lambda}},$$

where here we write

$$x(-m) = x_1(-m_1), \;\; \tilde{w}_j = x_2(-m_2) \cdots x_k(-m_k) v_j \in F_{k_j - m}(\mathcal{H}_{\vec{\lambda}}) \subset F_{k_j - 1}(\mathcal{H}_{\vec{\lambda}}),$$

and $k_i = \deg(\omega_i) = \sum_{i=1}^k m_i$.

We consider in particular, the element $f = \frac{1}{(z-z_i)^n}$, and write

$$\sum_{i=1}^n \rho_i(x \otimes f)\omega = \sum_{i=1}^n \omega_1 \otimes \cdots \otimes \omega_{i-1} \otimes (x \otimes f) \cdot \omega_i \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n \in \mathfrak{g}(U) \cdot \mathcal{H}_{\vec{\lambda}}. \tag{3.1}$$

Note that $\rho_j(x \otimes f)\omega = \tilde{\omega}$, and because f is holomorphic at z_i, for $i \neq j$, we have

$$\rho_i(x \otimes f)\omega \in F_k(\mathcal{H}_{\vec{\lambda}}). \tag{3.2}$$

Since

$$\phi \in \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|^{\dagger}_{(C,\vec{p})} = \left(\mathcal{H}_{\vec{\lambda}} / \mathfrak{g}(U)\mathcal{H}_{\vec{\lambda}}\right)^{\dagger},$$

and since ϕ will be zero when evaluated on any element of $\mathfrak{g}(U)\mathcal{H}_{\vec{\lambda}}$, we have that $\phi(\sum_j \rho_j(x \otimes f)\omega) = 0$. We therefore have that

$$\tilde{\omega} = \phi(\rho_j(x \otimes f)\omega) = -\sum_{i \neq j} \phi(\rho_i(x \otimes f)\omega) = 0,$$

by Eq 3.2.

Lecture 4

Chern classes: vanishing, identities and open questions

4.1 Introduction

For $g = 0$, the vector bundle $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ is a quotient of the constant bundle:

$$\mathbb{A}(\mathfrak{g}, \vec{\lambda}) = [V_{\vec{\lambda}}]_{\mathfrak{g}} \times \overline{M}_{0,n} \twoheadrightarrow \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell).$$

So for every point $(C, \vec{p}) \in \overline{M}_{0,n}$, there is a surjective map of vector spaces

$$[V_{\vec{\lambda}}]_{\mathfrak{g}} \twoheadrightarrow \mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})}.$$

In other words, for $a = \mathrm{rk}(\mathbb{A}(\mathfrak{g}, \vec{\lambda}))$, and $r = \mathrm{rk}(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$, there is a composition of morphisms

$$\overline{M}_{0,n} \xrightarrow{\phi} \mathrm{Gr}^{quo}([V_{\vec{\lambda}}]_{\mathfrak{g}}, r) \xrightarrow{p=\text{Plücker}} \mathbb{P}^{\binom{a}{r}-1}, (C, \vec{p}) \mapsto [\Lambda^a [V_{\vec{\lambda}}]_{\mathfrak{g}} \twoheadrightarrow \Lambda^r(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|_{(C,\vec{p})})].$$

The first Chern class of $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$, a globally generated divisor class, is the pullback of $\mathbb{O}_P(1)$, an effective divisor from the projective space $\mathbb{P}$ in which the Grassmannian is embedded. The analogous notion in higher Chern classes, is not to take the determinant, but to pull back effective cycle classes from the Grassmann variety instead.

Lehmann and Fulger in defined the pliant cone $\mathrm{PL}^k(X)$ to be the closure of the cone generated by products Chern classes of globally generated vector bundles (maybe for different bundles) with total codimension k. We'd like to say as much as we can about these classes. In tomorrow's lecture, I'll try to say a little bit about what we know, and to give some open problems.

4.2 Three Families of first Chern classes

Example 4.2.1. *In [Fak12], Fakhruddin proved that the set of nonzero level one $\mathfrak{sl}_2$ bundles*

$$\beta_1 = \{c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, 1)) : \mathrm{rk}\, \mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, 1) > 0\},$$

gives a basis for $\mathrm{Pic}(\overline{M}_{0,n})$. *In particular, the cone spanned by the elements in* β_1 *forms a full dimensional subcone of* $\mathrm{Nef}(\overline{M}_{0,n}$. *Swinarski proved in [Swi] that the elements of* β *do not cover the nef cone, at least for* $n \geqslant 6$. *There are 3190 extremal extremal rays of* $\mathrm{Nef}(\overline{M}_{0,6})$, *which Faber first showed can be classified into 28 orbits, using the symmetric group* S_6 *(these are listed in a table in [Swi]). On his list, Swinarski finds conformal block descriptions, mainly in terms of* $\mathfrak{sl}_2$ *for 11 of these orbits. We have found further descriptions of these, listed in Section 4.7.4. While* β *doesn't fill up the nef cone, it is worth noting that Fakhruddin did remark that the* $\mathfrak{sl}_3$ *divisors at level one seem to form a larger subcone than the* $\mathfrak{sl}_2$ *at level one (see [Fak12]).*

One can also get full dimensional subcones of nef cones of higher codimension using β_1*: By [Kee92], one has that* $A^1(\overline{M}_{0,n})$ *generates* $A^k(\overline{M}_{0,n})$ *for all* k, *and so products of the classes determine full dimensional subcones of* $\mathrm{Nef}^k(\overline{M}_{0,n})$ *for all* k. *Moreover, as the bundles are globally generated, their products are elements of the Pliant cone. A good reference for positivity in higher codimension is [FL17]. Given a variety* X, *the Pliant cone* $\mathrm{Pl}^m(X) \subset \mathrm{Nef}^m(X)$, *is the closure of the cone generated by monomials in Schur classes of globally generated vector bundles on* X. *So in fact products of elements in* β_1 *generate full dimensional subcones of the Pliant cone* $\mathrm{Pl}^k(\overline{M}_{0,n})$ *for all* k.

Example 4.2.2. *The set of* S_n*-invariant, and level one, classes in*

$$\beta_2 = \{c_1(\mathbb{V}(\mathfrak{sl}_n, \omega_i^n, 1)) \mid 2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor\},$$

form a basis for $\mathrm{Pic}(\widetilde{M}_{0,n})$, *where* $\widetilde{M}_{0,n} = \overline{M}_{0,n}/S_n$. *Hence* β_2 *generates a full dimensional subcone of* $\mathrm{Nef}(\widetilde{M}_{0,n})$. *These divisors also span extremal rays in* $\mathrm{Nef}(\widetilde{M}_{0,n})$.

By [Gia13, GG12a] images of maps for bundles of type A *at level one parametrize configurations of points lying on Veronese curves. The images are constructed using GIT.*

Example 4.2.3. *Let* $n = 2(g+1)$. *The higher level classes*

$$\beta_3 = \{c_1(\mathbb{V}(\mathfrak{sl}_2, \omega_1^n, \ell)) \mid 1 \leqslant \ell \leqslant \lfloor g \rfloor\},$$

which are also S_n *invariant, form a basis for* $\mathrm{Pic}(\widetilde{M}_{0,n})$, *and the associated divisors are interesting. For instance For* $\ell = 1$, *the divisor defines a morphism from to the Satake compactification of the moduli space of abelian varieties of dimension g [AGS14, Theorem 7.2], and in general, images* *can be identified with GIT quotients parametrizing "generalized Veronese quotients" studied in [GJMS13]. These are moduli spaces parametrizing weighted configurations of points and Veronese curves where the points lie on the Veronese curves.*

4.3 Vanishing as the level grows

Conformal blocks divisors are quite often extremal in the nef cone, and the number of curves they contract increases as the level increases with respect to the pair $(\mathfrak{g}, \vec{\lambda})$.

4.3.1 The critical level

Definition 4.3.1. *Suppose* $r+1$ *divides* $\sum_{i=1}^{n}|\lambda_i|$, *and let* $\mathrm{cl}(\mathfrak{sl}_{r+1}, \vec{\lambda}) = -1 + \frac{\sum_{i=1}^{n}|\lambda_i|}{r+1}$, *be the critical level for the pair* $(\mathfrak{sl}_{r+1}, \vec{\lambda})$. *If* $\ell = \mathrm{cl}(\mathfrak{sl}_{r+1}, \vec{\lambda})$, *and* $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1})^n$, *then* $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ *is called a critical level bundle, and* $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = \mathbb{D}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ *is called a critical level divisor.*

Vanishing and identities

Note that if $\ell = \mathrm{cl}(\mathfrak{sl}_{r+1}, \vec{\lambda})$, *then* $r = \mathrm{cl}(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T)$, *where* $\vec{\lambda}^T = (\lambda_1^T, \dots, \lambda_n^T)$. *Here* λ_i^T *is the weight associated to the transpose of the Young diagram associated to the weight* λ_i. *In particular,* $|\lambda_i| = |\lambda_i^T|$, *and so*

$$\sum_{i=1}^{n}|\lambda_i| = (r+1)(\ell+1) = (\ell+1)(r+1) = \sum_{i=1}^{n}|\lambda_i^T|.$$

In particular, critical level bundles come in pairs, and as we shall prove:

Theorem 4.3.2. *[BGM15b] If* $\ell = \mathrm{cl}(\mathfrak{sl}_{r+1}, \vec{\lambda})$, *then*

1. $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell + c)) = 0$, *for* $c \geqslant 1$; *and*
2. $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \vec{\lambda}^T, r))$.

After giving an example and some applications, we will prove part (1) *of Theorem 4.3.2.*

Example 4.3.3. *The bundle* $\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)$ *is at the critical level for* $n = (r+1)(\ell+1)$. *In [BGM15b] we showed that the first Chern classes are all nonzero, and by Theorem 4.3.2, for* $n = (r+1)(\ell+1)$,

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r)); \text{ and}$$
$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \omega_1^n, \ell + c)) = c_1(\mathbb{V}(\mathfrak{sl}_{\ell+1}, \omega_1^n, r + c)) = 0 \text{ for all } c \geqslant 1.$$

More examples are in Section 4.7.2.

Applications

The main applications of vanishing above the critical level are extremality tests, which can be used to check that Chern classes of arbitrary codimension lie on various extremal faces of nef or Pliant cones (see [BGM16], [GM82]). Once we know how to check which F-Curves get contracted, one can for instance, give criteria for showing that maps given by certain conformal blocks divisors factor through contraction maps to Hassett spaces [BGM16].

Extremality test

Proposition 4.3.4. *Let $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{sl}_{r+1})^n$, and suppose that N_1, N_2, N_3, N_4 is a partition of $[n] = \{1, \ldots, n\}$ into nonempty subsets ordered so $\lambda(N_i) = \sum_{j \in N_i} |\lambda_j|$, then $\lambda(N_1) \leqslant \cdots \leqslant \lambda(N_4)$. If $\sum_{j \in \{1,2,3\}} \lambda(N_j) \leqslant r + \ell$, then*

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) \cdot F_{N_1,N_2,N_3,N_4} = 0,$$

and in particular, $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell))$ is extremal in the nef cone.

Proof. The intersection $c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)) \cdot F_{N_1,N_2,N_3,N_4}$ takes place in the boundary divisor

$$\Delta_{N_1 \cup N_2 \cup N_3} \cong \overline{M}_{0,|N_1 \cup N_2 \cup N_3|+1} \times \overline{M}_{0,|N_4|+1},$$

and in particular, in $\overline{M}_{0,|N_1 \cup N_2 \cup N_3|+1}$. We can use factorization to examine the first Chern class of the bundle $\mathbb{V}$ at points $(C, \vec{p}) \in \Delta_{N_1 \cup N_2 \cup N_3, N_4}$, we have $\mathbb{V}|_{(C,\vec{p})}$ is isomorphic to

$$\bigoplus_{\mu} \mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu, \ell)|_{(\tilde{C}, \vec{p}(N_1 \cup N_2 \cup N_3) \cup q_1)} \otimes \mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_4) \cup \mu^*, \ell))|_{(\tilde{C}, \vec{p}(N_4) \cup q_2)}.$$

We compute the critical level of $\mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu, \ell)$, which is

$$\begin{aligned} \mathrm{cl}(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu) &= -1 + \frac{\sum_{j \in N_1 \cup N_2 \cup N_3} |\lambda_j| + |\mu|}{r+1} \\ &\leqslant -1 + \frac{r + \ell + r\ell}{r+1} < -1 + \frac{r + \ell + r\ell + 1}{r+1} = -1 + \frac{(r+1)(\ell+1)}{r+1} = \ell. \end{aligned} \tag{4.1}$$

In particular, $\mathbb{V}(\mathfrak{sl}_{r+1}, \lambda(N_1 \cup N_2 \cup N_3) \cup \mu, \ell)$ is above the critical level, and so it has trivial first Chern class. □

Sketch of proof of vanishing above the critical level

To prove Part (1) of Theorem 4.3.2, we use the cohomological version of Witten's Dictionary, which follows from [Wit95] and the twisting procedure of [Bel08b], see Eq (3.10) from [Bel08b].

Theorem 4.3.5. *Let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$ such that $\sum_{i=1}^{n} |\lambda_i| = (r+1)(\ell+s)$ for some integer s.*

1. *For $s > 0$, let $\lambda = \ell\omega_1$. Then $\operatorname{Rank}(\mathbb{V})$ is the coefficient of $q^s\sigma_{\ell\omega_{r+1}}$ in the quantum product*
$$\sigma_{\lambda_1} \star \sigma_{\lambda_2} \star \cdots \star \sigma_{\lambda_n} \star \sigma_{\lambda}^{s} \in QH^*(\mathrm{Gr}(r+1, r+1+\ell)).$$
2. *For $s \leqslant 0$, then $\operatorname{Rank}(\mathbb{V})$ is the multiplicity of the class of a point $\sigma_{k\omega_{r+1}}$ in the product*
$$\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdot \cdots \cdot \sigma_{\lambda_n} \in H^*(\mathrm{Gr}(r+1, r+1+k)),$$
where $k = \ell + s$.

Examples of rank computations using Theorem 4.3.5 can be found in [BGM15b, BGM16, Kaz16, Hob15] and [BGK16].

Proof. Write $\tilde{\ell} = \mathrm{cl}(\mathfrak{sl}_{r+1}, \vec{\lambda}) + 1$. We'll consider the following two cases:

1. $\vec{\lambda} \in \mathcal{P}_{\tilde{\ell}}(\mathfrak{sl}_{r+1})^n$ so that $\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})$ makes sense, and there is a surjective map
$$\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda}) \twoheadrightarrow \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell}).$$
2. $\vec{\lambda} \notin \mathcal{P}_{\tilde{\ell}}(\mathfrak{sl}_{r+1})^n$.

In case $\vec{\lambda} \in \mathcal{P}_{\tilde{\ell}}(\mathfrak{sl}_{r+1})^n$, we know that by Beauville's quotient construction, as the level grows, the rank decreases:
$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})) \leqslant \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell) \leqslant \mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda})).$$
So it is enough to show in this case that
$$\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})) = \mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda})).$$
In the second case, we'll argue that $\mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda})) = 0$. Both follow from the Cohomological form of Witten's Dictionary, Theorem 4.3.5.

In the first case, since $\sum_{i=1}^{n} |\lambda_i| = (r+1)(\tilde{\ell})$, we have that $s = 0$ in Theorem 4.3.5, and so $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \tilde{\ell})) = \mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda}))$, as claimed.

In the second case, we know that $|\lambda_i| \leqslant \ell r$ for all i but that $|\lambda_i| > \tilde{\ell} r$ for some i. This means in particular that $\lambda_i^{(1)} > \tilde{\ell}$ for some i. We may relable so that $k = \lambda_1^{(1)} \geqslant \cdots \lambda_n^{(1)}$. Since $\sum_{i=1}^{n} |\lambda_i| < (r+1)k$, we write $\sum_{i=1}^{n} |\lambda_i| = (r+1)(k-p)$, for some $p > 0$. Setting $\mu_1 = \mu_2 = \cdots = \mu_p = \omega_{r+1} \cong \omega_0$, by Propagation of Vacua:
$$\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda} \cup \vec{\mu}, \ell) \cong \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell),$$
and since $\sum_{i=1}^{n} |\lambda_i| + \sum_{j=1}^{p} |\mu_j| = (r+1)(k-p) + (r+1)p = (r+1)k$, we can compute the rank by computing the intersection
$$\sigma_{\lambda_1} \star \cdots \star \sigma_{\lambda_n} \star \sigma_{\omega_{r+1}}^{p} \in \mathrm{Gr}(r+1, r+1+k).$$
By a calculation, this is zero. ∎

4.3.2 The theta level

The theta level (Def 4.3.6), comes from the interpretation of a vector space of conformal blocks as an explicit quotient [Bea96, Proposition 4.1], and holds in all types.

Definition 4.3.6. *[BGM15b] Given a pair $(\mathfrak{g}, \vec{\lambda})$, one refers to $\theta(\mathfrak{g}, \vec{\lambda}) = -1 + \frac{1}{2}\sum_{i=1}^{n}(\lambda_i, \theta) \in \frac{1}{2}\mathbb{Z}$ as the theta level of the pair $(\mathfrak{g}, \vec{\lambda})$.*

Remark 4.3.7. *As in Lecture 2.5.4, θ is the highest root, and $(\; , \;)$ is the normalized Killing form.*

Vanishing above the theta level

Proposition 4.3.8. *[BGM15b] Suppose that $\ell > \theta(\mathfrak{g}, \vec{\lambda})$, then $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)) = 0$.*

Proposition 4.3.9. *[BGM15b] Let $\vec{\lambda} \in \mathcal{P}_\ell(\mathfrak{g})^n$, and suppose that N_1, N_2, N_3, N_4 is a partition-partion of $[n] = \{1, \dots, n\}$ into four nonempty subsets ordered so that if $\lambda(N_i) = \sum_{j \in N_i} |\lambda_j|$, then $\lambda(N_1) \leqslant \cdots \leqslant \lambda(N_4)$. If $\sum_{j \in \{1,2,3\}} \lambda(N_j) \leqslant \ell + 1$, then*

$$c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)) \cdot F_{N_1,N_2,N_3} = 0,$$

and in particular, $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$ is extremal in the nef cone.

The proof of Proposition 4.3.9 is analogous to that of Proposition 4.3.4, using the stronger version of Proposition 3.2.1, which holds in $g = 0$ [Bea96, Proposition 4.1].

Remark 4.3.10. *It looks by comparing the criteria in Proposition 4.3.9 and Proposition 4.3.4 that the θ-level is stronger than the critical level, but there are plenty of examples of divisors that have lower CL than θ-level and are only known to be zero because of the theorem on CL vanishing. See the examples in Section 4.7.2 for instance. These propositions are simply not sharp.*

4.4 The problem of nonvanishing

Explicit formulas for Chern classes in all genera are known [Fak12, MOP15, MOP+17]. In case $g = 0$, the global generation of the bundles gives non-negativity of the classes, in the sense that they will nonnegatively intersect effective cycles of complementary codimension[1]. As we've seen, the classes tend to contract more cycles as the level or the rank of the underlying bundle grows, and they vanish if the level or the rank is high enough.

Interestingly, there are many cases in which the level and rank are low, but the cycles vanish anyway. Here is one:

[1]Incidentally, such classes are spanned by boundary cycles

Example 4.4.1. *The bundle* $\mathbb{V}(\mathfrak{sl}_4, \{\omega_2+\omega_3, \omega_1, \omega_1+2\omega_2, 2\omega_1+\omega_3\}, 3)$ *is at the critical level, and it is below the theta level (which is* 3.5*). The rank of* $\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}, 3)$ *is one, while the dimension of the vector space of coinvariants* $\mathbb{A}(\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\})$ *is* 2*. A calculation shows that* $\mathbb{D}(\mathfrak{sl}_4, \{\omega_1, (2\omega_1+\omega_3)^3\}, 3) = 0$.

It seems natural to want to know exactly when explicitly given, nonnegative classes are nonzero, and we asked the following in [BGM16]

Question 4.4.2. *[BGM16] What are necessary and sufficient conditions for a triple* $(\mathfrak{g}, \vec{\lambda}, \ell)$ *that guarantee that* $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$ *is nonzero?*

We also were able to answer this question for $\mathfrak{sl}_2$ *divisors:*

Theorem 4.4.3. *[BGM16]* $c_1(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) \neq 0$ *as long as*

$$1 \leqslant \ell \leqslant \mathrm{CL}(\mathfrak{sl}_2, \vec{\lambda}) = \theta\mathrm{L}(\mathfrak{sl}_2, \vec{\lambda}), \text{ and } \mathrm{rk}(\mathbb{V}(\mathfrak{sl}_2, \vec{\lambda}, \ell)) > 0.$$

The result as stated in Theorem 4.4.3 does not hold in general, as we saw for the $\mathfrak{sl}_4$*-divisor in Example 4.4.1, and as can be seen in many examples*[2]*. Recall that* $\mathrm{cl}(\mathfrak{sl}_2, \vec{\lambda}) = \theta(\mathfrak{sl}_2, \vec{\lambda})$ *for* $\mathfrak{sl}_2$*. In [BGM16], we proved a similar nonvanishing result holds for* $c_1(\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell))$ *in case* $\mathrm{CL}(\mathfrak{g}, \vec{\lambda}, \ell) = \theta\mathrm{L}(\mathfrak{g}, \vec{\lambda}, \ell)$*. One approach, to understand divisors like that given in Example 4.4.1 is to decompose the vector bundle into simpler ones, whose vanishing may be understood more readily.*

4.5 Additive identities dependent on ranks

I will explain the following criteria, given in [BGM16] for decomposing a divisor as an effective sum of simpler divisors.

Proposition 4.5.1. *Let* $\vec{\mu} \in P_\ell(\mathfrak{g})^n$*, and* $\vec{\nu} \in P_m(\mathfrak{g})^n$ *be two* n*-tuples of dominant weights such that* $\mathrm{rk}\,\mathbb{V}(\mathfrak{g}, \vec{\mu}, \ell) = 1$*, and* $\mathrm{rk}\,\mathbb{V}(\mathfrak{g}, \vec{\mu}+\vec{\nu}, \ell+m) = \mathrm{rk}\,\mathbb{V}(\mathfrak{g}, \vec{\nu}, m) = \delta$*. Then*

$$c_1(\mathbb{V}(\mathfrak{g}, \vec{\mu}+\vec{\nu}, \ell+m)) = \delta \cdot c_1(\mathbb{V}(\mathfrak{g}, \vec{\mu}, \ell)) + c_1(\mathbb{V}(\mathfrak{g}, \vec{\nu}, m)).$$

Remark 4.5.2. *We also have another type of identity in type* A*, where we decompose the Lie algebra and the weights. This gives a non-vanishing result in case the critical and theta levels coincide, such as the* $\mathfrak{sl}_2$ *result mentioned earlier. To prove the second identity one uses an interpretation of conformal blocks in terms of generalized theta functions.*

Below I give some of the applications of Proposition 4.5.1.

[2]Fakhruddin showed us a whole list of examples, including the one given in Example 4.4.1.

Explanation of vanishing

Taking the bundle from Example 4.4.1, we note it can be decomposed into as in Proposition 4.5.1, and we may write

$$\begin{aligned}(4.2)\quad & c_1(\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, 2\omega_1+\omega_3, 2\omega_1+\omega_3, 2\omega_1+\omega_3\}, 3)) \\ & = c_1(\mathbb{V}(\mathfrak{sl}_4, \{\omega_1, \ldots, \omega_1\}, 1)) + c_1(\mathbb{V}(\mathfrak{sl}_4, \{0, \omega_1+\omega_3, \omega_1+\omega_3, \omega_1+\omega_3\}, 2)).\end{aligned}$$

Both of the divisors on the right hand side turn out to be trivial: the first since it is above the critical level, and the second, because it is pulled back from $\overline{M}_{0,3}$.

Using Proposition 4.5.1 in conjunction with the quantum generalization of a conjecture of Fulton in invariant theory [Bel07] and [BK16, Remark 8.5], we show in Corollary 4.6.2 that if $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{r+1}, N\vec{\lambda}, N\ell) = 1$, *then*

$$c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, N\vec{\lambda}, N\ell)) = N \cdot c_1(\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)), \ \forall N \in \mathbb{N}.$$

As an application of this, one can identify images of the maps $\phi_{\mathbb{D}}$ for $\mathbb{D} = \mathbb{D}(\mathfrak{sl}_{r+1}, \ell\vec{\lambda}, \ell) = \ell\, \mathbb{D}(\mathfrak{sl}_{r+1}, \vec{\lambda}, 1)$, as the generalized Veronese quotients of [Gia13, GJM13].

Proposition 4.5.1 can be used to show that a divisor is nontrivial, by writing it as an effective sum of simpler divisors, and then showing one of the summands is nontrivial. One may also solve questions of mysterious vanishing, seeing for example a divisor as a sum of divisors whose vanishing can be explained by other means.

4.6 Identities, applications and generalizations

Fulton conjectured that if $\mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, \vec{\lambda})) = 1$ *then* $\mathrm{rk}(\mathbb{A}(\mathfrak{sl}_{r+1}, N\vec{\lambda})) = 1 \ \forall\ N \in \mathbb{Z}_{>0}$. *This was proved by Knutson, Tao and Woodward [KTW04].*

$$\textit{For } \ \mathbb{V} = \mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell), \ \textit{ set } \ \mathbb{V}[n] = \mathbb{V}(\mathfrak{sl}_{r+1}, N\vec{\lambda}, N\ell),$$

$$\textit{where } \ N\vec{\lambda} = (N\lambda_1, \ldots, N\lambda_n), \ \textit{ so that if } \ \lambda_i = \sum_j c_j\omega_j, \ \textit{ then we set } \ N\lambda_i = \sum_j Nc_j\omega_j.$$

The quantum generalization of Fulton's conjecture [Bel07, BK16] is the following:

Theorem 4.6.1. *Suppose* $\mathrm{rk}(\mathbb{V}) = 1$, *then* $\mathrm{rk}(\mathbb{V}[n]) = 1$ *for all positive integers* N.

Using Theorem 4.6.1 and Proposition 4.5.1, by induction we obtain:

Corollary 4.6.2. *If* $\mathrm{rk}(\mathbb{V}) = 1$, *then* $c_1(\mathbb{V}[n]) = N\, c_1(\mathbb{V}), \forall\ N \in \mathbb{Z}_{>0}$.

Corollary 4.6.2 appears in case $r = 1$ *and* $\vec{\lambda} = (\omega_1, \dots, \omega_1)$ *in [GJMS13, Proposition 5.2], and an analogous result for* $\mathfrak{g} = \mathfrak{so}_{2r+1}$ *appears in the work of Mukhopadhyay. As will be clear later when we talk about projective rank scaling, and more generally* δ *invariant zero rank scaling, we will refer to this as* **horizontal projective rank zero scaling***.*

There is another similar result, which says, that for $\mathbb{V}_{[n]} = \mathbb{V}(\mathfrak{sl}_{N(r+1)}, \vec{\lambda}_N, \ell)$*, such that* $\vec{\lambda}_N = ((\lambda_1)_N, \dots, (\lambda_n)_N)$*, where for* $\lambda_i = \sum_j c_j \omega_j$*, we set* $(\lambda_i)_N = \sum_j c_j \omega_{Nj}$*:*

Proposition 4.6.3. *If* $\mathrm{rk}(\mathbb{V}) = 1$*, then* $c_1(\mathbb{V}_{[n]}) = N\, c_1(\mathbb{V})$*,* $\forall\, N \in \mathbb{Z}_{>0}$*.*

We call Proposition 4.6.3 **vertical projective rank zero scaling***. A proof of Proposition 4.6.3 can be found in* `https://arxiv.org/pdf/1605.06184v1.pdf`*.*

Finite generation of cones

In ([Kaz16], Theorem 1.1) this result was used to prove that any S_n*-invariant divisor for* $\mathfrak{sl}_n$ *on* $\overline{M}_{0,n}$ *coming from a bundle of rank one was in fact a sum of level one divisors in type* A*. In particular, the cone generated by infinitely many such divisors is finitely generated.*

In **Quantum Kostka and the rank one problem for** $\mathfrak{sl}_{2m}$*,* `http://arxiv.org/abs/1508.06952`*, Hobson considers a family of all rank one bundles for* $\mathfrak{sl}_2$*, and for a large class of bundles for* $\mathfrak{sl}_{2m}$*. She described an infinite generating set of this cone explicitly, and using these results, also described a subcone of* $\mathrm{CB}_1(n, \mathfrak{sl}_{2m})$ *generated by an infinite number of rank one bundles with so-called rectangular weights. She then used the additive identities to decompose these generators into sums of first Chern classes of bundles of level one. It follows from work of [GG12a] showing the cone generated by level one bundles is finitely generated, that her cone is polyhedral.*

In **Conformal blocks in type C at level one***,* `https://arxiv.org/pdf/1605.06184v1.pdf`*, Hobson used an identity between first Chern classes of bundles In type A at level* ℓ *and for* $\mathfrak{sp}_{2\ell}$ *at level* 1 *together with additive identities to show finite generation of the cone spanned by such conformal blocks divisors in type C.*

Projective rank scaling

We'll see in Lecture 5 a generalization of the rank one horizontal scaling identity to what we call projective rank scaling (which as we shall see, generalizes further to what we call Δ*-invariant zero rank scaling).*

Definition 4.6.4. *We say that* $\mathbb{V}$ *has*

- *horizontal projective rank* d *scaling if* $\mathrm{rank}(\mathbb{V}[m]) = \binom{d+m}{m}$*; and*

- *vertical projective rank* d *scaling if* $\mathrm{rank}(\mathbb{V}_{[m]}) = \binom{d+m}{m}$.

Remark 4.6.5. *Note that rank one bundles have both horizontal and vertical projective rank* 0 *scaling.*

It turns out that if $\mathbb{V}$ *has horizontal projective rank* d *scaling, and if* $\mathbb{V}$ *has other good geometric properties that hold for bundles of rank one, then*

$$c_1(\mathbb{V}[m]) = \binom{d+m}{d+1} c_1(\mathbb{V}).$$

Note that if you specialize to the case $d = 0$*, this is the statement of Corollary 4.6.2.*

Problem Session 4.6.6. *There should be a vertical analogue to this (see Section 4.7.3).*

4.7 Second problem session

4.7.1 Background Reading: Chow rings using Chern classes

Definition 4.7.1. *Let* $A_k(X)$ *be the group of algebraic cycles of dimension* k *on* X.

In his book on Intersection theory, Fulton defines a Chern class as a linear operator:

Definition 4.7.2. *Let* X *be a proper variety, and* $\mathcal{E}$ *a vector bundle on* X*. The* r*-th Chern class of* $\mathcal{E}$ *is a linear operator*

$$c_r(\mathcal{E}) : A_k(X) \to A_{k-r}(X).$$

Definition 4.7.3. *Two cycles* Z_1 *and* Z_2 *on* X *are numerically equivalent if for every weight* k *monomial* p *in Chern classes of vector bundles, one has*

$$\deg(P \cdot Z_1) = \deg(P \cdot Z_2).$$

This defines a pairing between weight k*-Chern classes and cycles of dimension* k.

Definition 4.7.4. $N_k(X)_{\mathbb{Z}} = A_k(X)/$ *numerical equivalence .*

Definition 4.7.5. *The finitely generated Abelian group* $N_k(X)_{\mathbb{Z}}$ *is a lattice in the vector space* $N_k(X) = N_k(X)_{\mathbb{Z}} \otimes \mathbb{R}$.

Definition 4.7.6. *The pseudo effective cone* $\overline{\mathrm{Eff}}_k(X) \subset N_k(X)$ *is defined to be the closure of the cone generated by cycles with nonnegative coefficients.*

The cone $\overline{\mathrm{Eff}}_k(X)$ *is full dimensional, spanning the vector space* $N_k(X)$*. It is pointed (containing no lines), closed, and convex.*

Definition 4.7.7. *Its dual of the vector space* $N_k(X)$ *is:*

$$N^k(X) = \{\mathbb{R} \text{ polynomials in weight } k\text{-Chern classes } \}/\equiv,$$

where equivalence $\equiv$ *is given by intersection with cycles.*

Definition 4.7.8. *The Nef Cone* $\mathrm{Nef}^k(X) \subset N^k(X)$ *is the cone dual to* $\overline{\mathrm{Eff}}_k(X)$*.*

As the dual of $\overline{\mathrm{Eff}}_k(X)$*, the nef cone has all of the nice properties that* $\overline{\mathrm{Eff}}_k(X)$ *does.*

Example 4.7.9. *By the definition given above,* $N^1(X) = \{$ *first Chern classes* $\}/\equiv$*, where* $\equiv$ *is defined by intersection with* 1*-cycles. This is the same as what you are used to seeing because if* $\mathcal{E}$ *is any vector bundle, then* $c_1(\mathcal{E}) = c_1(\det(\mathcal{E}))$*, and* $\det(\mathcal{E})$ *is a line bundle.*

The cones $\overline{\mathrm{Eff}}_k(X)$*, and* $\overline{\mathrm{Eff}}^k(X)$ *are full dimensional, spanning the vector spaces* $N_k(X)$*, and* $N^k(X)$*. They are pointed (containing no lines), closed, and convex. Cones of positive cycles are combinatorial devices that encode geometric data about proper varieties. Such cones of divisors and curves are the customary, time-honored, long established, and even familiar tools of the minimal model program. As we're starting to learn, their higher codimension analogues can behave very differently. For instance, one has that for any proper variety* X*, cone of nef divisors* $\mathrm{Nef}^1(X)$ *is contained in the pseudo-effective* $\overline{\mathrm{Eff}}^1(X)$*. But, as was proved in DELV, if* E *is an elliptic curve with complex multiplication, then* $\overline{\mathrm{Eff}}^k(E^r) \subsetneq \mathrm{Nef}^k(E^r)$ *for* $1 < k < r-1$*. In [Ott], an example was given of a variety* X *of lines on a very general cubic fourfold where* $\overline{\mathrm{Eff}}^2(X) \subsetneq \overline{\mathrm{Nef}}^2(X)$*. Nef cycles of higher codimension fail to satisfy other nice properties of nef divisors: For instance, the product of two nef cycles is not necessarily nef.*

To more accurately capture the properties of cycles of higher codimension, Fulger and Lehmann have introduced three sub-cones: the Pliant cone, the base-point free cone, and the universally pseud-oeffective cone. A lot of work, and many open problems are emerging [FL17], [CC14], [Ott], [CC15], [LO16], [CLO16].

4.7.2 Exercises and Problems

4.7.3 Boundary Divisor Calculation Exercises

One of the open problems about the bundles of coinvariants is to give necessary and sufficient conditions that their Chern classes are nontrivial. On $\overline{M}_{0,n}$*, first Chern classes of the vector bundles of covacua are nef divisors: They nonnegatively intersect all curves. The problems I have chosen below are aimed at helping students to get used to working with divisor classes on* $\overline{M}_{0,n}$ *with the ultimate goal of identifying nontrivial nef divisors there.*

For $S \subset \{1,\dots,n\}$*, set* $\delta_{0,S} = \delta_S$*. Using facts from the lectures, and that for*

$$\delta_S \cap \delta_T \neq \emptyset \iff S \subset T, T \subset S, T \cap S = \emptyset, \text{ or } T \cup S = \{1,\dots,n\},$$

try the following exercises:

1. *Since* $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$*, one has that on* $\overline{M}_{0,4}$*, all boundary divisor classes are equivalent. So in particular,*

   $$\delta_{ij} \equiv \delta_{ik} \equiv \delta_{i\ell}, \text{ for } \{i,j,k,\ell\} = \{1,2,3,4\}.$$

   Show using the point dropping maps that for $n \geqslant 4$*, on* $\overline{M}_{0,n}$*,*

   $$\sum_{I \subset \{ijk\ell\}^c} \delta_{ij \cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{ik \cup I} \equiv \sum_{I \subset \{ijk\ell\}^c} \delta_{i\ell \cup I}, \text{ for any four indices } \{i,j,k,\ell\} \subset \{1,\dots,n\}.$$

2. *Consider the curve/divisor*

   $$D_1 = \delta_{12} + \delta_{13} + \delta_{123} = \delta_{12} + \delta_{13} + \delta_{45},$$

   on $\overline{M}_{0,5}$*. Find which the four boundary curves/divisors that it intersects in degree zero, and the remaining 6 that it positively intersects. Assuming this divisor is base-point free, can you make guesses about what the image of the map that it gives?*

3. *Fix three distinct indices* $\{i,j,k\} \subset \{1,\dots,n\}$*, and consider the divisor*

   $$D_i = \sum_{I \subset \{ijk\}^c} \delta_{i \cup I}$$

   on $\overline{M}_{0,n}$*. What boundary curves does* D_i *contract?*

4. *Assuming that there is a basis for* $\mathrm{Pic}(\overline{M}_{0,5})$ *given by the classes* $\{D_i : 1 \leqslant i \leqslant 5\}$*, so that one can write* $D = \sum_{1 \leqslant i \leqslant 5} b_i D_i$*, use the* 10 *boundary curves to come up with inequalities that must be satisfied by the coefficients* b_i *in order for* D *to nonnegatively intersect all effective curves (ie. to be nef). What further assumption(s) would guarantee that* D *is nontrivial?*

5. *More generally, assuming that there is a basis for* $\mathrm{Pic}(\overline{M}_{0,n})$ *given by the classes* $\{D_i : 1 \leqslant i \leqslant n\}$*, together with boundary divisors* $\{\delta_J : 3 \leqslant |J| \leqslant \lfloor \frac{n}{2} \rfloor\}$*, use the boundary curves to come up with necessary combinatorial conditions for* D *to be nontrivial and nef. Can you identify any sufficient conditions?*

CB Divisor Exercises

1. *Check that the bundle* $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_3, \omega_1^6, 1)$ *on* $\overline{M}_{0,6}$ *is at the critical level and so by Theorem 4.3.2, we have*

(a) $c_1\mathbb{V}(\mathfrak{sl}_3, \omega_1^6, 1) = c_1\mathbb{V}(\mathfrak{sl}_2, \omega_1^6, 2)$; *and*

(b) $c_1\mathbb{V}(\mathfrak{sl}_3, \omega_1^6, 1+c) = c_1\mathbb{V}(\mathfrak{sl}_2, \omega_1^6, 2+c) = 0$ *for all* $c \geqslant 1$.

Check that the critical and theta levels for these $\mathfrak{sl}_2$ *bundles are equal. What is the theta level for the bundle* $\mathbb{V}$*?. In this case, which is stronger CL-vanishing or* θ*-level vanishing?*

2. *Check that the bundle* $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_3, \{2\omega_1+\omega_2, \omega_2, 2\omega_1, 2\omega_2, 3\omega_2\}, 5)$ *on* $\overline{M}_{0,5}$ *is at the critical level and so by Theorem 4.3.2, we have*

   (a) $c_1(\mathbb{V}) = c_1(\mathbb{V}(\mathfrak{sl}_6, \{\omega_1+\omega_3, 2\omega_1, \omega_2, 2\omega_2, 2\omega_3\}, 2))$; *and*

   (b)

$$\text{(4.3)}\quad \text{for all} c \geqslant 1\text{: } c_1\mathbb{V}(\mathfrak{sl}_3, \{2\omega_1+\omega_2, \omega_2, 2\omega_1, 2\omega_2, 3\omega_2\}, 5+c)$$
$$= c_1\mathbb{V}(\mathfrak{sl}_6, \{\omega_1+\omega_3, 2\omega_1, \omega_2, 2\omega_2, 2\omega_3\}, 2+c) = 0$$

In this case, for the $\mathfrak{sl}_3$ *bundle* $\mathbb{V}$*, what is the theta level? Which is stronger in this case:* θ*-level vanishing or CL vanishing?*

3. *One can check that all the divisors in the table of extremal rays for* $\mathrm{Nef}(\overline{M}_{0,6})$ *below have at least one representative that is either at the critical level, or one below the critical level. Thinking back on the lecture, how does it make sense from the point of view of Beauville's quotient theorem to expect to see such divisors on this list?*

4. *In case* $\mathfrak{g} = \mathfrak{sl}_{r+1}$ *you can solve to show* $\mathrm{CL}(\mathfrak{sl}_{r+1}, \vec{\lambda}) + \mathrm{CL}(\mathfrak{sl}_{r+1}, \vec{\lambda}^*) = 2\theta\mathrm{L}(\mathfrak{sl}_{r+1}, \vec{\lambda})$*, and then show that* θ*-level vanishing follows from CL-vanishing. Note in particular that if* $\vec{\lambda} = \vec{\lambda_*}$*, then the critical and theta levels agree.*

Problems

Problem 4.7.10. *Products of elements of* β_2 *and* β_3 *give elements of* $\mathrm{Pl}^k(\widetilde{M}_{0,n})$ *for all* k*, but don't necessarily generate full dimensional subcones of* $\mathrm{Pl}^k(\widetilde{M}_{0,n})$ *since* $A^1(\widetilde{M}_{0,n})$ *does not generate* $A^k(\widetilde{M}_{0,n})$*. Can you find full dimensional subcones of* $\mathrm{Pl}^k(\widetilde{M}_{0,n})$*?*

Problem 4.7.11. *Are the first Chern classes of bundles of level 2 and rank 2 indecomposable?*

This question comes from looking at the table of extremal rays of the nef cone for $\overline{M}_{0,6}$ *in [Swi] (see also Section 4.7.4). The most interesting of those examples in the table that are known to be spanned by conformal blocks divisors seems to be the 18th ray* R_{18} *which is spanned by a level 2 rank 2 bundle with projective scaling (it has the largest equivalence class, and the smallest symmetry group in the list). The threshold* $r+1 = 8$*,* $\ell = 2$ *seems to be where Swinarski's M2* *program takes an unreasonable amount of time to finish, and so he may have missed other examples of rays spanned by conformal blocks divisors.*

Problem 4.7.12. *(Vertical Stretching) If* $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ *is a vector bundle of rank* R*, we say that* $\mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ *has vertical projective rank scaling if*

$$\mathrm{rk}(\mathbb{V}_{[m]}) = \binom{m+R-1}{R-1},$$

for all positive integers m.

Can you find conditions P *for which if* $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ *is a vector bundle of rank* R *with vertical projective rank scaling and satisfies property* P*, then there is a weight decomposition*

$$\vec{\lambda} = \vec{\mu} + \vec{\nu}, \quad \textit{with } \ \ell = \ell_\mu + \ell_\nu,$$

such that $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\mu}_{[m]}, \ell_\mu)) = 1$*, and* $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\nu}_{[m]}, \ell_\nu)) = \mathrm{rk}(\mathbb{V}_{[m]}) = \binom{m+R-1}{R-1}$*.*

If this is true, then using [BGM16, Proposition 19] one can prove the following identity.

Proposition 4.7.13. *Suppose that* $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ *is a vector bundle of rank* R *with vertical projective rank scaling down and satisfies property* P*. Then*

$$\binom{m+R-1}{R} c_1(\mathbb{V}) = c_1(\mathbb{V}_{[m]}).$$

Proof. We prove the identity by induction on the level ℓ with base case $\ell = 1$. This identity was proved for level one bundles in [GG12b]. Now let $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ be a bundle of rank R with projective rank scaling down such that $\ell > 1$ and suppose the result holds for all bundles of $k < \ell$, and rank R with projective rank scaling down. By Problem 4.7.12, there is a weight decomposition

$$\vec{\lambda} = \vec{\mu} + \vec{\nu}, \quad \text{with } \ \ell = \ell_\mu + \ell_\nu,$$

such that $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\mu}_{[m]}, \ell_\mu)) = 1$, and $\mathrm{rk}(\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\nu}_{[m]}, \ell_\nu)) = \mathrm{rk}(\mathbb{V}_{[m]}) = \binom{m+R-1}{R-1}$. So by the inductive hypothese, we have that

$$\binom{m+R-1}{R} c_1\mathbb{V}(\mathfrak{sl}_r, \nu, \ell_\nu) = c_1\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\nu}_{[m]}, \ell_\nu), \ \text{ and } \ m c_1\mathbb{V}(\mathfrak{sl}_r, \vec{\mu}, \ell_\mu) = c_1\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\mu}_{[m]}, \ell_\mu). \tag{4.4}$$

By [BGM16, Proposition 19]

$$c_1(\mathbb{V}_{[m]}) = \binom{m+R-1}{R-1} c_1\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\mu}_{[m]}, \ell_\mu) + c_1\mathbb{V}(\mathfrak{sl}_{rm}, \vec{\nu}_{[m]}, \ell_\nu) \tag{4.5}$$
$$\binom{m+R-1}{R-1} m\, c_1\mathbb{V}(\mathfrak{sl}_r, \vec{\mu}, \ell_\mu) + \binom{m+R-1}{R} c_1\mathbb{V}(\mathfrak{sl}_r, \nu, \ell_\nu)$$
$$= \binom{m+R-1}{R} \Big(R \cdot c_1\mathbb{V}(\mathfrak{sl}_r, \vec{\mu}, \ell_\mu) + c_1\mathbb{V}(\mathfrak{sl}_r, \vec{\nu}, \ell_\nu)\Big).$$

where the last line follows from the identity

$$m\binom{m+R-1}{R-1} = R\binom{m+R-1}{R}.$$

Finally, applying [BGM16, Proposition 19] again, which says in this case that:

$$\binom{m+R-1}{R} \Big(R \cdot c_1\mathbb{V}(\mathfrak{sl}_r, \vec{\mu}, \ell_\mu) + c_1\mathbb{V}(\mathfrak{sl}_r, \vec{\nu}, \ell_\nu)\Big) = \binom{m+R-1}{R} c_1\mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell), \tag{4.6}$$

we obtain the assertion. □

Problems 4.7.14. 1. *Is there a general principal underlying these families that form basis for* Pic *(and hence collections of generators of full dimensional subcones of the nef cone)?*

2. *Are there geometric interpretations of the full dimensional subcones of the nef cone?*

Problem 4.7.15. *From patterns found in the tables in Section 4.7.4, one might be able to prove the following divisors span extremal rays of the S_n-invariant nef cone, and perhaps there is some underlying patterns you can see that will produce more.*

1. *For $n = 2g+1$:*
   (a) $\{\mathbb{D}(\mathfrak{sl}_{2m}, \{\omega_m^{2g}, 0\}, 1) : m \geqslant 1\}$, $\mathbb{D}(\mathfrak{sl}_2, \{\omega_2^{2g-1}, \omega_1^2\}, 2)$,
   $\{\mathbb{D}(\mathfrak{sl}_{(2g-1)m}, \{\omega_{im}^{2g}\omega_{jm}\}, 1) : \{i,j\} = \{g, g-1\}, m \geqslant 1\}$,
   $\{\mathbb{D}(\mathfrak{sl}_{6m}, \{\omega_{2m}^3\omega_{3m}^{2(g-1)}\}, 1) : m \geqslant 1\}$.
   (b) $\mathbb{D}(\mathfrak{sl}_2, \{\omega_1^{2g}, 0\}, g-1)$, $\mathbb{D}(\mathfrak{sl}_2, \{\omega_1^{2g}, \omega_2\}, g)$, $\{\mathbb{D}(\mathfrak{sl}_{gm}, \{\omega_m^{2g}, 0\}, 1) : m \geqslant 1\}$,
   $\{\mathbb{D}(\mathfrak{sl}_{(2g+1)m}, \{\omega_{2m}^{2g}, 0\}, 1) : m \geqslant 1\}$, $\{\mathbb{D}(\mathfrak{sl}_{gm}, \{\omega_{(g-1)m}^{2g}, 0\}, 1) : m \geqslant 1\}$.
   (c) $\{\{\mathbb{D}(\mathfrak{sl}_{nm}, \{\omega_{im}^{n}\}, 1) : i \in \{j, n-j\}, m \geqslant 1\} : j \in \{2, \dots, n-2\}$
   (d) $\{\mathbb{D}(\mathfrak{sl}_{(2g-1)m}, \{\omega_{im}^{2g}, \omega_j\}, 1) : \{i,j\} = \{g-1, g\}, m \geqslant 1\}$,
   (e) $\mathbb{D}(\mathfrak{sl}_2, \{\omega_2^{2g+1}\}, 2g-1)$,
2. $n = 2(g+1)$,
   (a) $\{\mathbb{D}(\mathfrak{sl}_{2m}; 1; \omega_m^{2(g+1)}) : m \geqslant 1\}$,

(b) $\mathbb{D}(\mathfrak{sl}_2; 2; \omega_1^{2(g+1)})$, *rank* 2^g

(c) $\mathbb{D}(\mathfrak{sl}_2; g-1; \omega_1^{2(g+1)})$,

(d) $\{\mathbb{D}(\mathfrak{sl}_2; g; \omega_1^{2(g+1)})$, $\{\mathbb{D}(\mathfrak{sl}_{(g+1)m}; 1; \omega_m^{2(g+1)}) : m \geqslant 1\}$

(e) $\{\mathbb{D}(\mathfrak{sl}_{km}; 1; \omega_m^{2(g+1)}) : m \geqslant 1\}$,

(f) $\{\mathbb{D}(\mathfrak{sl}_{(g+1)m}; 1; \omega_{2m}^{2(g+1)}) : m \geqslant 1\}$,

(g) $\{\mathbb{D}(\mathfrak{sl}_{nm}; 1; \omega_{im}^{n}) : i \in \{j, n-j\}, m \geqslant 1\} : j \in \{2, \ldots, n-2\}$

(h) $\mathbb{D}(\mathfrak{sl}_3; g+1; \omega_1^{(g+1)}\omega_2^{(g+1)})$,

(i) $\mathbb{D}(\mathfrak{sl}_2; 2g+1; \omega_{2m}^{2(g+1)})$,

4.7.4 Tables of extremal rays

Extremal rays of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,6})$ spanned by CB divisors

Ray	$\mathbb{V}(\mathfrak{sl}_{r+1}, \vec{\lambda}, \ell)$	*constraint*	
R_1	$\mathbb{V}_{2m} = \mathbb{V}(\mathfrak{sl}_{2m}, \{\omega_m^6\}, 1)$	$m \geqslant 1$	
R_2	$\mathbb{V}_{3m} = \mathbb{V}(\mathfrak{sl}_{3m}, \{\omega_m^6\}, 1)$	$m \geqslant 1$	
R_2	$V_i^r = \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_i^3, \omega_{r+1-i}^3\}, 2)$	$i \overset{r\geqslant 1}{<} \frac{r+1}{2}$	$\overset{4(r=1)}{5(r \geqslant 2)}$
R_2	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^2, \omega_i, \omega_{r+1-i}, \omega_r^2\}, 2)$	$1 \leqslant i \leqslant \frac{(r+1)}{2}$	
R_3	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^3, \ell\omega_1, \omega_{r-2}, \ell\omega_r\}, \ell)$	$\ell \overset{r\geqslant 3}{\geqslant} 1$	
R_3	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^2, (\ell-1)\omega_1, \omega_2, \omega_{r-2}, \ell\omega_r\}, \ell)$	$\ell \overset{r\geqslant 2}{\geqslant} 2$	
R_5	$V_i = \mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_i^3, \omega_{r+1-i}^3\}, 1)$	$i \overset{r\geqslant 2}{<} \frac{r+1}{2}$	
R_6	$V_\ell^m = \mathbb{V}(\mathfrak{sl}_{r+1}, \{\ell\omega_1, m\omega_1, \ell\omega_r, m\omega_r, 0, 0\}, \ell)$	$r \geqslant 1$	
R_7	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^4, \omega_{r-1}^2\}, 1)$	$r \geqslant 2$	
R_9	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^3, (\ell-2)\omega_1 + \omega_r, \ell\omega_r, 0\}, \ell)$	$\ell \geqslant 3$	
R_9	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^3, \omega_2, 2\omega_r, 3\omega_r\}, 3)$	$r \geqslant 2$	
R_{10}	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^2, \omega_i, \omega_{r+1-i}, \omega_r^2\}, 1)$	$1 < i \leqslant \frac{(r+1)}{2}$	
$\overline{M}R_{11}$	$\mathbb{V}_{pm} = \mathbb{V}(\mathfrak{sl}_{pm}, \{\omega_m^3, \omega_{qm}^2, \omega_{(p-2)m}\}, 1)$	$m \overset{p=2q+1}{\geqslant} 1$	
R_{11}	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^2, (\ell-1)\omega_1, \ell\omega_1, \ell\omega_{r-1}, \omega_r\}, \ell)$	$\ell \overset{r\geqslant 2}{\geqslant} 2$	
R_{11}	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^2, \omega_2, 2\omega_r, \omega_r^2\}, 2)$	$r \geqslant 2$	
R_{12}	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^2, \omega_{i+j+1}, \omega_{r-i}, \omega_{r-j}, \omega_r\}, 1)$	$i \overset{r\geqslant 2i+j+1}{\geqslant j \geqslant} 1$	
R_{18}	$\mathbb{V}(\mathfrak{sl}_{r+1}, \{\omega_1^2, \omega_2^2, 2\omega_{r-1}, 2\omega_r\}, 2)$	$r \geqslant 2$	

Data on extremal rays in S_n invariant case

$n = 6 \quad g = 2 \quad m \in \mathbb{Z}_{\geqslant 1}$	
2,1	$\mathbb{D}(\mathfrak{sl}_{6m}; 1; \omega^6_{3m})$, $\mathbb{D}(\mathfrak{sl}_{2m}; 1; \omega^6_{m})$
1,3	$\{\mathbb{D}(\mathfrak{sl}_{6m}; 1; \omega^6_{im}) : i \in \{2,4\}\}$, $\{\mathbb{D}(\mathfrak{sl}_{3m}; 1; \omega^6_{im}) : i \in \{1,2\}\}$ $\mathbb{D}(\mathfrak{sl}_2; 2; \omega^6_1)$, $\mathbb{D}(\mathfrak{sl}_2; 5; \omega^6_2)$.

$n = 7 \quad g = 3 \quad m \in \mathbb{Z}_{\geqslant 1}$	
1,1	$\{\mathbb{D}(\mathfrak{sl}_{7m}; 1; \omega^7_{im}) : i \in \{3,4\}\}$
1,3	$\{\mathbb{D}(\mathfrak{sl}_{7m}; 1; \omega^7_{i}) : i \in \{2,5\}\}$, $\mathbb{D}(\mathfrak{sl}_2; 5; \omega^7_2)$

$n = 8 \quad g = 3 \quad m \in \mathbb{Z}_{\geqslant 1}$	
3,2,4	$\mathbb{D}(\mathfrak{sl}_{8m}; 1; \omega^8_4) = \{\mathbb{D}(\mathfrak{sl}_2; 1; \omega^8_i) : i \in \{1,3\}\}, \mathbb{D}(\mathfrak{sl}_4; 1; \omega^8_2)$
2,6,5	$\mathbb{D}(\mathfrak{sl}_2; 2; \omega^8_1)$, *rank* 2^g
1,3,6	$\{\mathbb{D}(\mathfrak{sl}_8; 1; \omega^8_i) : i \in \{2,6\}\}, \{\mathbb{D}(\mathfrak{sl}_4; 1; \omega^8_i) : i \in \{1,3\}\}$, $\mathbb{D}(\mathfrak{sl}_2; 3; \omega^8_1)$, $\mathbb{D}(\mathfrak{sl}; 7; \omega^8_2)$
6,11,8	$\{\mathbb{D}(\mathfrak{sl}_{8m}; 1; \omega^7_{im}) : i \in \{3,5\}\}$

$n = 9 \quad g = 4 \quad m \in \mathbb{Z}_{\geqslant 1}$	
3,3,4	$\{\mathbb{D}(\mathfrak{sl}_{9m}; 1; \omega^9_{im}) : i \in \{4,5\}, m \geqslant 1\}$
1,3,6	$\{\mathbb{D}(\mathfrak{sl}_{9m}; 1; \omega^9_{im}) : i \in \{2,7\}, m \geqslant 1\}$, $\mathbb{D}(\mathfrak{sl}_{2m}; 7; \omega^9_{2m})$, $\mathbb{D}(\mathfrak{sl}_{3m}; 2; \omega^9_{m})$
1,3,2	$\{\mathbb{D}(\mathfrak{sl}_{9m}; 1; \omega^9_{im}) : i \in \{3,6\}, m \geqslant 1\} = \{\mathbb{D}(\mathfrak{sl}_{3m}; 1; \omega^9_{im}) : i \in \{1,2\}, m \geqslant 1\}$
1,1,2	*???*

$n = 10 \quad g = 4 \quad m \in \mathbb{Z}_{\geqslant 1}$	
4,3,6,4	$\mathbb{D}(\mathfrak{sl}_{10m}; 1; \omega^{10}_{5m}) = \mathbb{D}(\mathfrak{sl}_{2m}; 1; \omega^{10}_{m})$
1,3,3,4	$\mathbb{D}(\mathfrak{sl}_2; 2; \omega^{10}_1)$, *rank* 2^g
2,6,12,11	$\mathbb{D}(\mathfrak{sl}_2; 3; \omega^{10}_1)$, *rank* $34 = 2 \cdot 17$
1,3,6,10	$\{\mathbb{D}(\mathfrak{sl}_{10}; 1; \omega^{10}_i) : i \in \{2,8\}\} = \{\mathbb{D}(\mathfrak{sl}_5; 1; \omega^{10}_i) : i \in \{1,4\}\}$, $\mathbb{D}(\mathfrak{sl}_{2m}; 4; \omega^{10}_{m})$, $\mathbb{D}(\mathfrak{sl}_{2m}; 7; \omega^{8}_{2m})\}$
2,6,6,5	$\{\mathbb{D}(\mathfrak{sl}_{10m}; 1; \omega^{10}_{im}) : i \in \{3,7\}\}$
2,3,3,5	$\{\mathbb{D}(\mathfrak{sl}_{10}; 1; \omega^{10}_i) : i \in \{4,6\}\} = \{\mathbb{D}(\mathfrak{sl}_5; 1; \omega^{10}_i) : i \in \{2,3\}\}$
4,6,6,7	*???*

$n = 11 \quad g = 5 \quad m \in \mathbb{Z}_{\geqslant 1}$	
2,2,3,3	$\{\mathbb{D}(\mathfrak{sl}_{11m}; 1; \omega^{11}_{im}) : i \in \{5, 6\}, m \geqslant 1\}$
1,3,6,10	$\{\mathbb{D}(\mathfrak{sl}_{11m}; 1; \omega^{11}_{im}) : i \in \{2, 9\}\}$
9,12,14,15	$V(1/9, 11)$
3,9,13,10	$\{\mathbb{D}(\mathfrak{sl}_{11m}; 1; \omega^{11}_{im}) : i \in \{3, 8\}\}$
6,13,11,15	$\{\mathbb{D}(\mathfrak{sl}_{11m}; 1; \omega^{11}_{im}) : i \in \{4, 7\}\}$
6,8,11,10	
3,9,8,10	
1,1,2,2	
1,3,6,5	
4,7,9,15	

$n = 12 \quad g = 5 \quad m \in \mathbb{Z}_{\geqslant 1}$	
5,4,8,6,9	$\mathbb{D}(\mathfrak{sl}_{12m}; 1; \omega^{12}_{6m}) = \mathbb{D}(\mathfrak{sl}_{2m}; 1; \omega^{12}_{m})$
4,12,13,18,16	$\mathbb{D}(\mathfrak{sl}_2; 2; \omega^{12}_1)$, *rank* 2^g
2,6,12,20,19	$\mathbb{D}(\mathfrak{sl}_2; 4; \omega^{12}_1)$,
1,3,6,10,15	$\{\mathbb{D}(\mathfrak{sl}_{12}; 1; \omega^{12}_i) : i \in \{2, 10\}\}\{\mathbb{D}(\mathfrak{sl}_6; 1; \omega^{12}_i) : i \in \{1, 5\}\}$, $\mathbb{D}(\mathfrak{sl}_2; 5; \omega^{12}_1)$
20,27,32,35,36	$\mathbb{D}(\mathfrak{sl}_2; 9; \omega^{11}_1\omega_1)$ *or* $V(1/10, 12)$
3,9,7,8,12	$\{\mathbb{D}(\mathfrak{sl}_{12}; 1; \omega^{12}_i) : i \in \{4, 8\}\}$, $\{\mathbb{D}(\mathfrak{sl}_3; 1; \omega^{12}_i) : i \in \{2, 4\}\}$
20,27,32,46,36	$\{\mathbb{D}(\mathfrak{sl}_{12m}; 1; \omega^{12}_{im}) : i \in \{5, 7\}\}$
8,13,15,25,21	
6,7,14,16,13	
2,6,12,9,8	$\{\mathbb{D}(\mathfrak{sl}_{12m}; 1; \omega^{12}_{im}) : i \in \{3, 9\}\}$, $\{\mathbb{D}(\mathfrak{sl}_{4m}; 1; \omega^{12}_{im}) : i \in \{1, 3\}\}$

Lecture 5

Geometric interpretations

5.1 Introduction

Today I will start by discussing two related questions, quite historical, about moduli spaces. In the last part of the lecture, I'll tie them to geometric interpretations of conformal blocks.

5.2 Two related problems about moduli spaces

Setup for Problem 1

Informally speaking, there is a family

$$\mathrm{U}_g(r,d) \longrightarrow \mathrm{M}_g, \;\; \mathrm{U}_C(r,d) \mapsto C,$$

whose fiber[1] *over a smooth curve* C *is* $\mathrm{U}_C(r,d)$*, the moduli space of semi-stable vector bundles on* C *of rank* r *and degree* d*. A vector bundle* ξ *of rank* r *on* C *has degree d if* $\Lambda^r\xi$ *has numerical degree* d*.*

Question 5.2.1. *Does this extend to a family over* $\overline{\mathrm{M}}_g$*?*

Work on Problem 1, for $r - 1$:

Many people[2] *have worked on this problem:*

- *Igusa [Igu56],*

[1]We really should restrict to automorphism free curves to describe this family

[2]The list below is not complete.

- *D'Souza, in his PhD Thesis [D'S79], considered a projective flat family* X *of irreducible and reduced curves parametrized by* $S = \text{Spec}(A)$*, for A a Noetherian, Henselian local ring with separably closed residue field. He showed that the Picard scheme* $P = \text{Pic}^0(X/S)$ *is quasi-projective, parametrizing families of invertible sheaves of degree zero on the fibers of* X *over* S*. He constructed a projective scheme over S containing P as an open set and which parametrizes families of torsion-free rank one sheaves of degree zero on the fibers of* X *over* S*. He shows if geometric fibers of* X *over* S *have at worst nodes or ordinary cusps as singularities, then the geometric fibers of this compactification are reduced, irreducible, and equi-dimensional. Hence the family is flat over* S *if S is reduced. Later work of Altman and Kleiman shows that in case singularities of the fibers of* X *over* S *are worse, these properties may fail to hold. His work uses GIT after Seshadri.*
- *In [OS79], Oda and Seshadri construct compactifications of the generalized Jacobian variety for connected (but possibly reducible) nodal curves, using GIT to give a number of different compactifications of the union of finitely many copies of the generalized Jacobian variety of a connected reduced but possibly reducible curve* X *over an algebraically closed field with at worst nodes as singularities. In particular, they show the compactified Picard scheme fails to be unique over families of curves with reducible fibers.*
- *M.-N. Ishida in [Ish78] generalizes the work of [OS79] to families.*
- *Altman and Kleiman in [AK80], and [AK79] generalized methods of Chow, Matsusaka and Grothendieck to construct the compactification of the Picard scheme of a family of higher-dimensional reduced irreducible varieties.*
- *Caporaso in [Cap93] considers for the universal Picard variety* $P_{d,g}$ *over* M_g^0*, the set of smooth, automorphism free curves, whose fibre over a point* C *can be identified with the Picard variety of line bundles of degree d on C. She constructs a projective variety* $\overline{P}_{d,g}$ *and a surjective morphism* $\phi : \overline{P}_{d,g} \longrightarrow \overline{M}_g$*, such that* $\phi^{-1}(M_g^0) = P_{d,g}$*. She studies many aspects of the construction including showing:*
  - *For any stable curve* C*, there is a bound* $\mu(C)$*, depending only on the dual graph of* C*, so that* $\phi^{-1}(C)$*, a connected scheme has at most* $\mu(C)$ *irreducible components.*
  - *If* C *has no automorphisms then* $\phi^{-1}(C)$ *is reduced, and its smooth locus is the disjoint union of a finite number of copies of the generalized Jacobian of the curve* C*.*

  Her construction was proposed by Gieseker and Morrison [GM84] for general r *(she follows their approach for* $r = 1$*).*

Work on Problem 1, for $r > 1$:

- *Seshadri, in [Ses82], defined $U_C(e, r)$, for C a reduced curve by using torsion free sheaves of uniform rank plus a semi-stability condition, depending on a parametrization of the curve.*
- *In [Pan96], Pandharipande generalizes this and forms a family $\overline{U}_g(e, r) \longrightarrow \overline{M}_g$.*
  - *Like Caporaso, Pandharipande uses Gieseker's construction of $\overline{M}_g$ as a GIT quotient;*
  - *He shows fibers are irreducible for $r \geqslant 1$, normal for $r = 1$, and*
  - *He shows his family is isomorphic to Caporaso's[3] for $r = 1$.*

Simpson's Approach

In [Sim94b] and [Sim94a], Simpson generalized all of the work above. He introduced a notion of stability for pure dimensional coherent sheaves on any projective variety.

Definition 5.2.2. *The support of a sheaf E on a Noetherian scheme X is the closed set* $\mathrm{Supp}(E) = \{x \in X : E_x \neq 0\}$.

Definition 5.2.3. *The dimension of a sheaf E is the dimension of the support of E and is denoted* $\dim(E)$.

Definition 5.2.4. *A coherent sheaf E is pure of dimension d if $\dim(F) = d$ for all non-trivial coherent subsheaves F of E.*

Definition 5.2.5. *A coherent sheaf E on an integral scheme X is torsion free if for each $x \in X$, and* $s \in \mathcal{O}_{X,x} \backslash \{0\}$*, multiplication by s is an injective homomorphism $E_x \longrightarrow E_x$.*

Remark 5.2.6. *Pure dimensional coherent sheaves on a curve are torsion free sheaves. So the pureness property is a generalization of the torsion freeness property.*

Simpson shows how to form a projective variety (using GIT) which is a moduli space of semi-stable torsion free sheaves on X. To embed the set of sheaves in a projective space (to form the GIT quotient), he uses that every torsion free sheaf on a projective variety can be expressed as a quotient (described below), and then uses Grothendieck's embedding to map the set he considers into a Grassmannian of quotients.

[3] Abramovich considers the more general family proposed by Gieseker and Morrison, and proves that it is not the same as Pandharipande's for larger r.

The second Problem

We consider the family

$$\mathcal{SU}(r) \longrightarrow M_g, \ \ \mathcal{SU}_C(r) \mapsto C,$$

whose fibers $\mathcal{SU}_C(r)$ *are moduli spaces of semistable vector bundles on* C *with trivializable determinant.*

Definition 5.2.7. *A vector bundle* ξ *on* C *of rank* r *has trivializable determinant if* $\Lambda^r \xi \cong \mathcal{O}_C$. *So* $\det(\xi)$ *is* **linearly** *equivalent to zero.*

Remark 5.2.8. *Linear equivalence implies numerical equivalence, this family* $\mathcal{SU}(r)$ *is a sublocus of* $\mathcal{U}_g(r, 0)$.

One can ask the question:

Problem 5.2.9. *Is there an extension of* $\mathcal{SU}(r) \to M_g$ *over* $\overline{M}_g$*?*

Work on Problem 2

- *Nagaraj and Seshadri gave a conjectural description of the closure of a 1-parameter family in this locus in Pandharipande's solution;*
- *Sun proved Nagaraj and Seshadri's Conjecture, finding that the central fiber over a such a 1-parameter family would sometimes be reducible and not normal.*
- *There is a new solution [BG16], stated below in Theorem 5.2.10.*

Theorem 5.2.10. *[BG16] There is a flat family* $p : \mathcal{X}(r) \to \overline{M}_g$, *with* $\mathcal{X}(r)$ *relatively projective over* $\overline{M}_g$, *such that*

1. $\mathcal{X}_C(r)$ *is integral, normal, and irreducible, for* $[C] \in \overline{M}_g$*; and*
2. $\mathcal{X}_C(r) \cong SU_C(r)$, *for* $[C] \in M_g$.

This family is constructed by setting

$$\mathcal{X}(r) = \mathrm{Proj}(\bigoplus_{\ell \in \mathbb{Z}_{>0}} \mathbb{V}(\mathfrak{sl}_r, \ell)^*),$$

where $\mathbb{V}(\mathfrak{sl}_r, \ell)^*$ *is the sheaf of conformal blocks on* $\overline{M}_g$. *It is a sheaf of* $\mathcal{O}_{\overline{M}_g}$*-algebras, and as taking fibers commutes with taking Proj:*

$$\mathcal{X}_C(r) = \mathrm{Proj}(\bigoplus_{\ell \in \mathbb{Z}_{>0}} \mathbb{V}(\mathfrak{sl}_r, \ell)|_C^*).$$

To take Proj*, this ring should be finitely generated.*

Finite generation is next described.

5.3 The section ring of the determinant bundle

Let G be a simple, simply connected, complex linear algebraic group, C a stable curve of arithmetic genus $g \geqslant 2$, and $\mathrm{Bun}_G(C)$ the smooth algebraic stack whose fiber over a scheme T is the groupoid of principal G-bundles on $C \times T$ (Def 2.5.2). Recall that the Determinant of Cohomology line bundle was defined in Section 2.2.2 for $G = SL(r)$. The following result, was proved in case of smooth curves in [BL94], and [Fal94], and for stable curves with singularities in [BG16].

Theorem 5.3.1. *For $G = SL(r)$, and the standard representation $SL(r) \to Gl(V)$,*

$$\mathcal{A}_{\bullet}^{C} = \bigoplus_{\ell \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^0(\mathrm{Bun}_{SL(r)}(C), \mathcal{D}(V)^{\otimes \ell})$$

is finitely generated.

To make the connection between Theorem 5.3.1 and conformal blocks, we recall the following results, mentioned in Lecture 2:

Theorem 5.3.2. $\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|^{*}_{(C;\vec{p})} \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^0(\mathrm{Bun}_{SL(r)}(C), \mathcal{D}^{\otimes \ell m})$.

The following generalization holds for pointed curves:

Theorem 5.3.3. $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)|^{*}_{(C;\vec{p})} \cong \mathcal{H}^0(\mathrm{Parbun}_G(C, \vec{p}), \mathcal{L}_G(C, \vec{p}, \vec{\lambda}))$.

The moduli stack $\mathrm{Parbun}_G(C, \vec{p})$ maps to $\mathrm{Bun}_G(C)$ and the line bundle $\mathcal{L}_G(C, \vec{p}, \vec{\lambda})$ on $\mathrm{Parbun}_G(C, \vec{p})$ is constructed from $\mathcal{D}(\mathcal{V})$ on $\mathrm{Bun}_G(C)$. Theorem 5.3.3 was proved for smooth curves by Laszlo and Sorger [LS97]. The result holds for families of stable curves by [BF15].

5.4 Geometric interpretations at smooth curves

If C is smooth, even more is true: stated in the case we are using now:

Theorem 5.4.1. *For $G = SL(r)$, and $SL(r) \to Gl(V)$,the standard representation,*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^0(\mathrm{Bun}_G(C), \mathcal{D}(V)^{\otimes \ell m}) \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^0(X, A^{\otimes \ell m}),$$

where $(X, A) = (M_C(r), \theta)$ is the projective polarized pair:

- *$X = M_C(r)$ is the moduli space parametrizing semi-stable vector bundles on C of rank r with trivializable determinant; and*
- *$A = \theta = \{\mathcal{E} \in M_C(r) \mid \mathcal{E} \otimes \mathcal{L}$ has a nonzero section$\}$, for $\mathcal{L}$ a fixed line bundle on C of rank $g-1$.*

Putting Theorems 5.3.1, 5.3.2, and 5.4.1 together, we say that for a point $[C] \in M_g$*, corresponding to a smooth curve* C*, there is a projective polarized pair* $(M_C(r), \theta)$ *such that*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^* \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^0(M_C(r), \theta^{\ell m}),$$

and so

$$\mathrm{Proj}(\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^*) \cong M_C(r).$$

In other words, there are geometric interpretations for conformal blocks at smooth curves. The same is true for conformal blocks at smooth pointed curves.

Example 5.4.2. *For* $[C] \in M_2$*, one has, that*

$$\mathbb{V}(\mathfrak{sl}_2, 1)|_C^* \cong \mathcal{H}^0(\mathrm{Bun}_{\mathrm{SL}(2)}(C), \mathcal{D}(V)) \cong \mathcal{H}^0(M_C(2), \theta) \cong \mathcal{H}^0(\mathbb{P}^3, \mathcal{O}(1)),$$

where the third isomorphism was proved in a 1960's Annals paper by Narasimhan and Ramanan. More generally, we write

$$\bigoplus_m \mathbb{V}(\mathfrak{sl}_2, m)|_C^* \cong \bigoplus_m \mathcal{H}^0(\mathbb{P}^3, \mathcal{O}(m)),$$

and

$$\mathrm{Proj}(\bigoplus_m \mathbb{V}(\mathfrak{sl}_2, m)|_C^*) \cong \mathbb{P}^3.$$

What can we say at stable curves of genus 2 with nodes?

5.5 Interpretations at stable curves with singularities

We consider whether such geometric interpretations for $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *exist at points* $(C; \vec{p}) \in \overline{M}_{g,n}$*, where* C *has singularities. We state this problem in the simplest case:*

Question 5.5.1. *Given a point* $[C] \in \overline{M}_g$*, corresponding to a curve* C *with singularities, is there is a projective polarized pair* (X, A) *such that*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^* \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathcal{H}^0(X, A^{\otimes m}),$$

and so

$$\mathrm{Proj}(\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^*) \cong X?$$

We showed in [BGK16] that for this question, and the analogous more general question for conformal blocks on pointed curves, while sometimes yes, the answer is no, not necessarily! Or said more correctly, one has to move to a weighted projective space.

5.5.1 How things can change for singular curves

Example 5.5.2. *[BGK16] Let* C *be a stable curve of genus 2 with a separating node. There is no projective polarized pair* (X, A) *such that*

$$\bigoplus_{m\in\mathbb{Z}} \mathbb{V}(\mathfrak{sl}_2, m)|_{[C]}^* \cong \bigoplus_{m\in\mathbb{Z}} H^0(X, A^m).$$

To show this we prove that if $\mathbb{V} = \mathbb{V}(\mathfrak{sl}_2, 1)$ *has geometric interpretations at such a curve* C*, then*

$$c_1(\mathbb{V}(\mathfrak{sl}_2, m)) = \binom{m+3}{4} c_1(\mathbb{V}(\mathfrak{sl}_2, 1)) \tag{5.1}$$

which one can show fails by intersecting with F*-curves. There are two types of* F*-curves on* $\overline{M}_2$*. The first is the image of a clutching map from* $\overline{M}_{0,4}$ *for which points are identified in pairs. The second is the image of a map from* $\overline{M}_{1,1}$ *given by attaching a point* $(E, p) \in M_{1,1}$*, gluing the curves at the marked points. One obtains a contradiction when we intersect with either type of* F*-curve, even just at* $m = 2$*.*

Example 5.5.3. *[BGK16] For* $(C, \vec{p}) \in \overline{M}_{2,n}$*, for* $n = 2k > 0$*, such that* C *has a single separating node, then here is no polarized pair* (X, A) *such that*

$$\bigoplus_{m\in\mathbb{Z}} \mathbb{V}(\mathfrak{sl}_2, \omega_1^n, m)|_{[C]}^* \cong \bigoplus_{m\in\mathbb{Z}} H^0(X, A^m).$$

So that $\mathbb{V}(\mathfrak{sl}_2, \omega_1^n, 1)$ *does not have geometric interpretations at such points* $(C, \vec{p}) \in \overline{M}_{2,n}$*.*

We do know that sometimes there are geometric interpretations.

Example 5.5.4. *By Theorem 5.5.5, the bundle* $\mathbb{V}(\mathfrak{sl}_2, 1)$ *has geometric interpretations at a point* $[C] \in \overline{M}_2$ *with only nonseparating nodes, even though it does not have if* C *has a separating node, while* $\mathbb{V}(\mathfrak{sl}_2, 2)$ *has geometric interpretations at all points* $[C] \in \overline{M}_2$*.*

Here are two types of results that generalize this example:

5.5.2 Positive results

Positive results for positive genus and $\mathfrak{sl}_r$

Theorem 5.5.5. *[BG16]Given $[C] \in \overline{\mathcal{M}}_g$, and a positive integer r, there exists a positive integer ℓ, and a projective polarized pair $(\mathcal{X}_C(r,\ell), \mathcal{L}_C(r,\ell))$, such that*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, m\ell)|^*_{[C]} \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} H^0(\mathcal{X}_C(r,\ell), \mathcal{L}_C(r,\ell)^{\otimes m}). \tag{5.2}$$

We can be more precise about ℓ in some cases:

1. *For general r if C has only nonseparating nodes, $\ell \geqslant 1$;*
2. *For $r = 2$, ℓ divisible by 2;*
3. *For general r, and C with separating nodes, we know such an ℓ exists.*

To prove Theorem 5.5.5

We use Theorems 5.3.1, and 5.3.2, together with the stratification of $\overline{\mathrm{M}}_g$ to prove that

$$\mathcal{A}_\bullet = \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)^*,$$

is finitely generated.

The sheaves of conformal blocks $\mathbb{V}(\mathfrak{sl}_r, \ell m)^*$ are locally free of finite rank. This sum forms the so-called algebra of conformal blocks, mentioned in Falting's work, and studied by Chris Manon mainly for SL(2) *and* SL(3)*. In these cases, Manon shows the algebra is finitely generated. Manon also takes* $\mathrm{Proj}(\mathcal{A}_\bullet)$ *in more general circumstances, without knowing or checking finite generation.*

For $\mathcal{A}_\bullet$ to be finitely generated, it means that the algebra is generated over $\mathcal{A}_0 \cong \mathcal{O}_{\overline{\mathrm{M}}_g}$ by finitely many elements $\{f_{d_i}\}_{i=1}^n$, with $f_{d_i} \in \mathcal{A}_{d_i} = \mathbb{V}(\mathfrak{sl}_r, d_i)^*$.

For $d = \prod_{i=1}^n d_i$, we let $\mathcal{S}_\bullet = \bigoplus_m \mathcal{S}_m$, where $\mathcal{S}_m = \mathcal{A}_{dm}$, be the d-th Veronese subring of $\mathcal{A}_\bullet$. Then $\mathcal{S}_\bullet$ is generated in degree 1 over $\mathcal{S}_0$, and

$$\mathcal{X} := \mathrm{Proj}(\mathcal{A}_\bullet) \cong \mathrm{Proj}(\mathcal{S}_\bullet) \xrightarrow{p} \overline{\mathrm{M}}_g$$

is a flat family.

Moreover, by definition, for $k >> 0$,

$$\mathbb{V}(\mathfrak{sl}_r, kd)^* = \mathcal{S}_k \longrightarrow p_*\mathcal{O}_X(k),$$

are isomorphisms. Since taking fibers commutes with taking Proj*,*

$$p^{-1}([C]) = \mathcal{X}_C \cong \mathrm{Proj}(\bigoplus_m \mathbb{V}(\mathfrak{sl}_r, \ell m)|^*_C) = \mathrm{Proj}(\mathcal{A}^{C,\ell}_\bullet),$$

where $\mathcal{A}_{\bullet}^{C,1} = \mathcal{A}_{\bullet}^{C}$.

By definition of pushforward,

$$\mathcal{S}_k|_{[C]} = \mathbb{V}(\mathfrak{sl}_r, kd)|_{[C}^{*} = (p|_{\mathcal{X}_C})_*(\mathcal{O}_{\mathcal{X}}(kd)|_{\mathcal{X}_C}) \cong H^0(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(kd)).$$

In other words, for $\ell = kd$*, and* $k >> 0$*, there is a projective polarized pair* $(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(\ell))$ *such that* $\mathbb{V}(\mathfrak{sl}_r, \ell)|_{[C]}^{*} \cong H^0(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(\ell))$*, and*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{sl}_r, \ell m)|_{[C]}^{*} \cong \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} H^0(\mathcal{X}_C, \mathcal{O}_{\mathcal{X}_C}(\ell m)).$$

So $\mathbb{V}(\mathfrak{sl}_r, \ell)$ *has geometric interpretations at* C *if* $\ell = kd$*, and* $k >> 0$*.*

Remark 5.5.6. *The flat family* $\mathcal{X} := \mathrm{Proj}(\mathcal{A}_{\bullet}) \cong \mathrm{Proj}(\mathcal{S}_{\bullet}) \xrightarrow{p} \overline{M}_g$ *is one way to complete the family* $\mathcal{X}^0 \xrightarrow{p} M_g$ *whose fibers over points* $[C]$ *are the moduli spaces* $M_C(r)$ *described earlier. There are other ways to complete this family and this problem is an old one with an interesting history.*

Positive result for bundles of rank one

Theorem 5.5.7. *[BGK16] Geometric interpretations hold at all points if* $\mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, \ell)$ *has rank one.*

More general results hold for bundles with restriction behavior that is similar to that for rank one bundles. We avoid stating these results here because they are involved.

While I don't know of any vector bundle of conformal blocks of rank one on $\overline{M}_{g,n}$ *for positive genus* g*, every bundle on* $\overline{M}_{0,n}$ *of the form* $\mathbb{V}(\mathfrak{sl}_r, \vec{\lambda}, 1)$ *has rank one, and by Theorem 5.5.7, all such bundles have geometric interpretations at all points of* $\overline{M}_{0,n}$*.*

Example 5.5.8. *For contrast with Example 5.5.3,* $\mathbb{V}(\mathfrak{sl}_2, \omega_1^{2k}, 1)$ *has rank one on* $\overline{M}_{0,2k}$*, and by Theorem 5.5.7, geometric interpretations at all points of* $\overline{M}_{0,2k}$*, whereas by [BGK16] the same bundle on* $\overline{M}_{2,2k}$ *will not have geometric interpretations at a point* $(C, \vec{p})$ *if* C *has a separating node.*

5.6 Appendix for Lecture five

5.6.1 Idea of proof of Theorem 5.3.1

The proof of Theorem 5.3.1 can be outlined in four steps:

1. *Define projective polarized pairs* $(X(\vec{a}), \mathcal{L}(\mathcal{G}))$*, where* $X(\vec{a})$ *is a compactification of a moduli space of* $\vec{a}$*-semistable vector bundles of rank* r *on* C *with trivializable determinant. The compactification is obtained as a GIT quotient of torsion free sheaves. The semi-stability condition is new; a generalization based on Seshadri and Simpson.*
2. *Show there are injections* $H^0(X(\vec{a}), \mathcal{L}(\mathcal{G})) \hookrightarrow H^0(\mathrm{Bun}_{SL(r)}(C), D(V)^m)$ *giving rise to a map*

$$F : \bigoplus_{(\vec{a},\mathcal{G})} H^0(X(\vec{a}), \mathcal{L}(G)) \longrightarrow \bigoplus_{m \in \mathbb{Z}} H^0(\mathrm{Bun}_{SL(r)}(C), D(V)^m).$$

3. *Using conformal blocks, show that* F *is surjective. For this we use Theorem 5.3.3 and Factorization. This involves a technical argument showing that certain sections extend across poles.*
4. *Show that the part of left hand side necessary for the surjection of* F *is finitely generated. This is achieved by noticing that the varieties* $X(\vec{a})$ *which are involved are all Geometric Invariant Theory (GIT) quotients of the same (master) space, and so one can use a variation of GIT argument to get the claim.*

5.6.2 Δ-invariant zero

In [BGK16] we prove the following:

Theorem 5.6.1. *There are points* $(C, \vec{p}) \in \overline{M}_{g,n}$ *and vector bundles of conformal blocks* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *on* $\overline{M}_{g,n}$ *for which there is no projective polarized pair for which*

$$\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)|^*_{(C,\vec{p})} \cong \bigoplus_{m \in \mathbb{Z}} H^0(X, A^m) \tag{5.3}$$

holds.

To prove Theorem 5.6.1, we give obstructions to geometric interpretations for those bundles where geometric interpretations at smooth curves are known to be varieties of minimal degree.

Given a projective polarized pair (X, A)*, there is a quantity called the* Δ*-invariant or* Δ*-genus, which is defined to be*

$$\Delta(X, A) = \dim(X) + A^{\dim(X)} - h^0(X, A).$$

Fujita (1990, Chapter 1 [Fuj90]) proved that $\Delta(X, A) \geqslant 0$*, and if* $\Delta(X, A) = 0$*, the section ring of* A*,* $\bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} H^0(X, A^{\otimes m})$ *is generated by its global sections* $H^0(X, A)$*, and so* A *is very ample.*

In this case, when A *is very ample, it gives an embedding of* X *into projective space*

$$X \hookrightarrow \mathrm{Proj}(B_\bullet) = \mathbb{P}^N, \ \ B_\bullet = \bigoplus_{m \in \mathbb{Z}_{\geqslant 0}} \mathrm{Sym}^m(H^0(X, A)).$$

The image of X *is a non-degenerate variety of degree*

$$A^{\dim X} = 1 + \mathrm{codim}(X).$$

A non-degenerate variety $X \hookrightarrow \mathbb{P}^N$ *is of minimal degree if* $\deg(X) = 1 + \mathrm{codim}(X)$.

So if (X, A) *is a projective polarized pair with* $\Delta(X, A) = 0$*, then the image of the variety* X *embedded by* A *is a variety of minimal degree.*

Varieties of minimal degree are classified. For instance $(X, A) \cong (\mathbb{P}^d, \mathcal{O}(1))$ *if and only if* $A^d = 1$.

What is crucial to our line of reasoning is that the Δ*-invariant is upper semi-continuous: If* $\mathbb{V}$ *is a vector bundle of conformal blocks on* $\overline{M}_{g,n}$ *that has geometric interpretation at some point* $(C, \vec{p}) \in M_{g,n}$ *such that the corresponding projective polarized pair has* Δ*-invariant zero, then if it has geometric interpretations at any other points, those corresponding pairs will also have* Δ*-invariant zero.*

We use this to prove the following result (paraphrased):

Theorem 5.6.2. *Suppose that* $\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)$ *has* Δ*-invariant zero rank scaling, and geometric interpretations exist for* $\mathbb{V}$ *at all points, then for all* m*,* $c_1(\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell))$ *can be expressed in terms of* $c_1(\mathbb{V}(\mathfrak{g}, k\vec{\lambda}, k\ell))$*, for* $k < m$.

There is an explicit statement for Theorem 5.6.2, which is rather long and technical. In Example 5.6.3, Theorem 5.6.2 is stated for the stronger case of projective rank scaling, where there is an if and only if result.

Example 5.6.3. $\mathrm{Rank}(\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)) = \binom{m+d}{d}$*, and geometric interpretations exist for* $\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$ *at all points* $(C, \vec{p}) \in \overline{M}_{g,n}$*, iff* $c_1(\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)) = \binom{m+d}{d+1} c_1\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell)$*. In particular, if* $d = 0$*, so that the rank is one, we know by [GG12b] for* $\mathfrak{sl}_r$ *and* $\ell = 1$*, and by [BGK16] for the general case,*

$$c_1\mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell) = mc_1\mathbb{V}(\mathfrak{g}, \vec{\lambda}, \ell).$$

Therefore for rank one bundles, geometric interpretations exist at all points.

We can outline the proof of Theorem 5.6.2 in two steps:

1. *Suppose that for every point* $x \in \overline{M}_{g,n}$*, there is a projective polarized pair* (X_x, A_x) *of* Δ*-invariant zero so that there is a canonical embedding as described above. One can then take the canonical resolution of the ideal sheaf* $\mathcal{I}_{X_x}$ *for* X_x.
2. *By "glueing" the resolutions, we show there is an exact sequence*

$$0 \to \mathcal{W}_D \otimes \mathrm{Sym}^{m-D}(\mathbb{V}) \to \cdots \to \mathcal{W} \otimes \mathrm{Sym}^{m-1}(\mathbb{V}) \to \mathrm{Sym}^m(\mathbb{V}) \to \mathbb{V}(\mathfrak{g}, m\vec{\lambda}, m\ell)^* \to 0,$$

(5.4)

where the $\mathcal{W}_i$ *are vector bundles on* $\overline{M}_{g,n}$.

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