ICTP Summer School on Dynamical Systems
Rotations of the circle and renormalization
Solutions 1

Solutions to Exercise 1.1

Part (a) An orbit of R_α for α for $p/q = 2/7$ and for $p/q = 5/8$ are in Figure 1. They are obtained dividing the circle in 7 or 8 equal arcs respectively and moving by steps of 2 or 5 respectively.

Figure 1: Orbits of R_α for α for $p/q = 2/7$ and for $p/q = 5/8$.

Assume now that $\alpha = p/q$ and $(p,q) = 1$ i.e. p and q are coprime. We have seen that for every $x \in \mathbb{R}/\mathbb{Z}$ we have $R^q_\alpha(x) = x$. Assume that $n \in \mathbb{N}^+$ is also a period, so that $R^n_\alpha(x) = x$. We want to show that $n \geq q$. We have

$$R^n_\alpha(x) = x + n\frac{p}{q} \mod 1 = x \quad \Leftrightarrow \quad x + \frac{np}{q} = x + k \quad \text{for some } k \in \mathbb{Z}.$$

Thus $np = qk$. Since $(q,p) = 1$, this shows that q divides n, so that $n \geq q$. We have shown moreover that all periods n are multiples of q.

We claim that $|p|$ gives the *winding number*, i.e. the number of "turns" that the orbit of any point does around the circle S^1 before closing up. Consider $z \in S^1$. In each iteration of R_α, $R^k_\alpha(z)$ is rotated by an arc of lenght $2\pi|\alpha| = 2\pi|p|/q$ (counterclockwise if $p > 0$, clockwise if $p < 0$). Thus, in q iterations, $R^q_\alpha(z)$ has been rotated by $q2\pi\alpha|p|/q = 2\pi|p|$. This shows that it has covered $|p|$ times the full circle lenght 2π, so the winding number is $|p|$.

Part (a) Assume that α is irrational. Let us first show that the each orbit consist of infinitely many distinct points, or, in other words, that for each $z_1 = e^{2\pi i x_1} \in S^1$, for all $m \neq n$, $R^n_\alpha(e^{2\pi i x_1}) \neq R^m_\alpha(e^{2\pi i x_1})$. Let us argue by contradiction. If $R^n_\alpha(e^{2\pi i x_1})$ and $R^m_\alpha(e^{2\pi i x_1})$ were equal,

$$\begin{aligned}
&e^{2\pi i(x_1 + m\alpha)} = e^{2\pi i(x_1 + n\alpha)}, \quad \text{thus} \\
&2\pi(x_1 + m\alpha) = 2\pi(x_1 + n\alpha) + 2\pi k \text{ for some integer } k \in \mathbb{N}, \text{ thus simplifying} \\
&m\alpha = n\alpha + k.
\end{aligned}$$

But since $m \neq n$, this shows that $\alpha = k/(m-n)$, contradicting the assumption that α is irrational.

Part (b) We want to show that for every $z_1 \in S^1$ the orbit of z_1 is dense in S^1, i.e. we have to show that for each $z_2 \in S^1$ and $\epsilon > 0$ there is a point of $\mathcal{O}^+_f(z_1)$ inside the ball $B(z_2, \epsilon)$. Let $\mathbb{N}$ be big enough so that $1/N < \epsilon$. We have already seen in class that if we consider the points $z_1, R_\alpha(z_1), \ldots, R^N_\alpha(z_1)$, since the rotation number is irrational, they are all distinct. Hence, by the *Pigeon Hole principle*, there exists n, m such that $0 \leq n < m \leq N$ and

$$d(R^n_\alpha(z_1), R^m_\alpha(z_1)) \leq \frac{1}{N} < \epsilon.$$

This means that for some θ with $|\theta| < 1/N$ we have

$$R^m_\alpha(z_1) = e^{2\pi i\theta} R^n_\alpha(z_1) \quad \Leftrightarrow \quad e^{2\pi i m\alpha} z_1 = e^{2\pi i\theta} e^{2\pi i n\alpha} z_1 \quad \Leftrightarrow \quad \frac{e^{2\pi i m\alpha}}{e^{2\pi i n\alpha}} = e^{2\pi i\theta} \tag{1}$$

Consider now R^{m-n}_α. We claim that it is again a rotation by an angle smaller than $2\pi\epsilon$. Indeed, from (1), we see that

$$R^{m-n}_\alpha(z_1) = e^{2\pi i(m-n)\alpha} z_1 = \frac{e^{2\pi i m\alpha}}{e^{2\pi i n\alpha}} z_1 = e^{2\pi i\theta} z_1 = R_\theta(z_1),$$

so that $R^{m-n}_\alpha = R_\theta$ is a rotation and that the rotation angle is $2\pi\theta$ with $|\theta| < 1/N$. Thus, if we consider the iteratates $R^{(m-n)}_\alpha(z_1), R^{2(m-n)}_\alpha(z_1), R^{3(m-n)}_\alpha(z_1), \ldots$, we see that the orbit contains the points

$$e^{2\pi i x_1}, e^{2\pi i(x_1+\theta)}, e^{2\pi i(x_1+2\theta)}, \ldots, e^{2\pi i(x_1+k\theta)}, \ldots$$

whose spacing on S^1 is less than $\pi\epsilon$, or in other words whose distance is less than ϵ (recall that the distance is the arc lenght divided by 2π). Thus, there will be a $j > 0$ such that $R^{j(m-n)}_\alpha(z_1)$ enters the ball $B(z_2, \epsilon)$. This concludes the proof that every orbit is dense.

Solutions to Exercise 1.2

Let us consider the orbit of the origin 0 under the rotation R_α (remark that $R_\alpha^n(0) = \{n\alpha\}$). Remark first that, reasoning as in class and using the pigeon hole principle, one can show that given $n \in \mathbb{N}$ there exists $0 < q \le n$ such that $q\alpha \mod 1 \le \frac{1}{n}$. Thus, there exists $p \in \mathbb{Z}$ such that

$$|q\alpha - p| \le \frac{1}{n} \qquad \Leftrightarrow \qquad \left|\alpha - \frac{p}{q}\right| \le \frac{1}{qn} \le \frac{1}{q^2},$$

where in the last inequality we used that $q \le n$. Assume now by contradiction that there are only finitely many fractions p/q where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and p, q coprime that solve the equation

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^2}, \qquad \text{say} \quad \left\{\frac{p_1}{q_1}, \dots, \frac{p_N}{q_N}\right\}.$$

Choose $n > 0$ such that

$$\frac{1}{n} < \min_{i=1,\dots,N} \left|\alpha - \frac{p_i}{q_i}\right|. \tag{2}$$

By the initial remark, we can find $0 \le q \le n$ and $p \in \mathbb{Z}$ such that p/q satisfies

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{nq} \le \frac{1}{q^2}.$$

Thus, it is a solution to our equation. We can assume that p/q has been simplifyed so that p, q are coprime, since if not we can simplify it and get a new solution p'/q' where still $q' \le q \le 1/\delta$. We claim that it is different than all the other solutions p_i/q_i, $i = 1, \dots, N$. This is because, since $q \ge 1$,

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{nq} \le \frac{1}{n} < \min_{i=1,\dots,N} \left|\alpha - \frac{p_i}{q_i}\right|,$$

where in the last inequality we used the choice of n, see (2). so that p/q is strictly closer than all the previous solutions. This gives a contradiction.